Voltera Integral Equation of Second Kind

We will study voltera integral equation of second kind given by

$$U(x) = f(x) + \lambda \int_a^x K(x,t)\, u(t)\, dt$$

The unknown function $U(x)$, that we will determined, occurs inside and outside the integral sign the kernel $K(x,t)$ and the function $f(x)$ are given real valued functions. λ is a parameter. A variety of numerical methods have been used to solve voltera integral equations but here we will discuss the modified decomposition method.

The Modified Decomposition Method

As we see that, the Adomian decomposition method provides the solution in the form of infinite series of components and $f(x)$ consists of polynomial in volterra integral equation. The modified decomposition was develop by wazwaz. It is interesting to know that the modified decomposition method depends mainly on splitting the function $f(x)$ into two parts and it cannot be used when $f(x)$ consists of only one term.

To give a brief description, we finally recall the Adomian Decomposition method, where we use the recurrence relation as follow

$$U_0(x) = f(x)$$
$$U_{k+1}(x) = \lambda \int_0^x K(x,t) U_k(t)\, dt \quad ; k \geq 0 \qquad (1)$$

and where the solution is given as

$$U(x) = \sum_{n=0}^{\infty} U_n(x)$$

But in modified decomposition method, we introduce some variation in recurrence relation (in eq (i)) to determine U_x in easier way. Here we split $f(x)$ as:

$$f(x) = f_1(x) + f_2(x)$$

To reduce the calculation. we identified zeroth component by one part of $f(x)$ namely $f_1(x)$ or $f_2(x)$. The other part of $f(x)$ can be added to the component $U_1(x)$ among others. Here we introduce the modified recurrence relation as;

$$U_0(x) = f_1(x)$$
$$U_1(x) = f_2(x) + \lambda \int_0^x K(x,t)U_0(t)\,dt$$
$$U_{k+1}(x) = \lambda \int_0^x K(x,t)\,U_k(t)\,dt \quad , k \geq 1 \quad \longrightarrow (2)$$

This shows that the difference between the standard reccurence relation (eq ①) and the modified reccurence relation (eq ②) is in the formation of first two components $U_0(x)$ and $U_1(x)$. The other components U_j , $j \geq 2$ remain same for both reccurence relations

Remark:

1) The proper slection of the function $f_1(x)$ and $f_2(x)$, $U(x)$ can be obtained after few iteration, Sometimes by evaluating only two components. And the proper slection of the functions can be through trails only.

2) If $f(x)$ consists of one term only , then standard decomposition method can be used.

Now we have solved some examples to understand the concept of this method

Example # 01:

Solve the voltera integral equation by using the modified decomposition method

$$U(x) = \sin x + (e - e^{\cos x}) - \int_0^x e^{\cos t} u(t)\,dt \longrightarrow (1)$$

Solution:

we obtain

$$f(x) = \sin x + (e - e^{\cos x})$$

we first split $f(x)$ into two parts

$$f_1(x) = \sin x$$
$$f_2(x) = (e - e^{\cos x})$$

Now By using the modified recurrence formula

$$U_0(x) = f_1(x) = \sin x$$
$$U_1(x) = f_2(x) - \int_0^x e^{\cos t} u_0(t)\,dt$$
$$U_1(x) = e - e^{\cos x} - \int_0^x e^{\cos t} \sin t\,dt$$
$$U_1(x) = e - e^{\cos x} - \int_0^x e^{\cos t} \sin t\,dt \longrightarrow (a)$$

Now Take $I_1 = \int_0^x e^{\cos t} \sin t\,dt$

$$I_1 = -\int_0^x e^{\cos t}(-\sin t)\,dt$$
$$I_1 = -\left|e^{\cos t}\right|_0^x$$
$$\Rightarrow I_1 = -e^{\cos x} + e^{\cos 0} = -e^{\cos x} + e$$

Now using in eq (a) we have

$u_1(x) = e - e^{\cos x} - (-e^{\cos x} + e)$
$u_1(x) = e - e^{\cos x} + e^{\cos x} - e$
$u_1(x) = 0$

We get $u_1(x) = 0$

So the recurrence relation,

$$u_{k+1}(x) = -\int_0^x K(x,t)\, u_k(t)\, dt = 0 \quad , \quad k \geq 1$$

It is abvious that each component of u_j, $j \geq 1$ is zero. This in turn gives the exact solution

So we obtain;

$U(x) = \sin x$

$U(x) = \sin x$ is the exact solution of the given Voltera integral equation of the second kind.

Example # 2:

Solve voltera integral equation of second kind by using the modified decomposition method.

$$U(x) = \sec x \tan x + (e^{\sec x} - e) - \int_0^x e^{\sec t}\, u(t)\, dt \longrightarrow (1)$$

Solution:

We obtain $f(x) = \sec x \tan x + (e^{\sec x} - e)$

We first split $f(x) = \sec x \tan x + (e^{\sec x} - e)$ into two parts:

$f_1(x) = \sec x \tan x$
$f_2(x) = e^{\sec x} - e$

Now

By using modified recurrence formula

$$u_0(x) = f_1(x) = \sec x \tan x$$
$$u_1(x) = f_2(x) - \int_0^x e^{\sec t} u_0(t)\,dt$$
$$u_1(x) = e^{\sec x} - e - \int_0^x e^{\sec t} (\sec t \tan t)\,dt$$
$$u_1(x) = e^{\sec x} - e - \left| e^{\sec t} \right|_0^x$$
$$u_1(x) = e^{\sec x} - e - (e^{\sec x} - e^{\sec 0})$$
$$u_1(x) = \cancel{e^{\sec x}} - \cancel{e} - \cancel{e^{\sec x}} + \cancel{e}$$
$$u_1(x) = 0$$

we get $u_1(x) = 0$

So the recurrence relation:

$$u_{k+1}(x) = -\int_0^x k(x,t) u_k(t)\,dt = 0 \qquad , k \geq 1$$

It is obvious that each component of u_j, $j \geq 1$ is zero. This in turns give the exact solution by $u(x) = \sec x \tan x$.
Now $u(x) = \sec x \tan x$ is basically the exact solution of given voltera integral equation of second kind.

Example # 3:

Solve voltera integral equation of second kind by using modified decomposition method.

$$u(x) = 2x + \sin x + x^2 - \cos x + 1 - \int_0^x u(t)\,dt$$

Solution:

Here the function $f(x)$ consists of five terms. By trial we divide $f(x)$ given by $f(x) = 2x + \sin x + x^2 - \cos x + 1$

into two parts, first two terms we take as $f_1(x)$ and the next three terms we take as $f_2(x)$ given by

$$f_1(x) = 2x + \sin x$$
$$f_2(x) = x^2 - \cos x + 1$$

$$u_0(x) = f_1(x) = 2x + \sin x$$
$$u_1(x) = f_2(x) - \int_0^x u_0(t)\,dt$$
$$u_1(x) = x^2 - \cos x + 1 - \int_0^x (2t + \sin t)\,dt$$
$$u_1(x) = x^2 - \cos x + 1 - \left[2\left|\frac{t^2}{2}\right|_0^x + \left|-\cos t\right|_0^x\right]$$
$$u_1(x) = x^2 - \cos x + 1 - \left[(x^2 - 0) - (\cos x - \cos x)\right]$$
$$u_1(x) = \cancel{x^2} - \cos x + \cancel{1} - \cancel{x^2} + \cos x - \cancel{1}$$
$$u_1(x) = 0$$

we get $u_1(x) = 0$

So the recurrence relation

$$u_{k+1}(x) = -\int_0^x k(x,t)\,u_k(t)\,dt = 0 \quad , k \geq 1$$

It is obvious that each component of u_j, $j \geq 1$ is zero. This in turn gives the exact solution so we obtain $u(x) = 2x + \sin x$ is the exact solution of given voltera integral equation of second kind.

Example # 4:

Solve voltera integral equation of second kind by using modified decomposition method.

$$u(x) = 1+x^2+\cos x - x - \frac{1}{3}x^3 - \sin x + \int_0^x u(t)\,dt$$

Solution:

The function $f(x)$ consists of six terms by trail we split $f(x)$ given by
$f(x) = 1+x^2+\cos x - x - \frac{1}{3}x^3 - \sin x$ into two parts the first three terms as $f_1(x)$ and the next three terms as $f_2(x)$, hence we get

$$f_1(x) = 1+x^2+\cos x$$

$$f_2(x) = -x - \frac{1}{3}x^3 - \sin x$$

Now, using the modified recurrence relation

$$u_0(x) = f_1(x) = 1+x^2+\cos x$$

$$u_1(x) = f_2(x) + \int_0^x u_0(t)\,dt$$

$$u_1(x) = -x - \frac{1}{3}x^3 - \sin x + \int_0^x (1+t^2+\cos t)\,dt$$

$$u_1(x) = -x - \frac{1}{3}x^3 - \sin x + \int_0^x 1\,dt + \int_0^x t^2\,dt + \int_0^x \cos t\,dt$$

$$u_1(x) = -x - \frac{1}{3}x^3 - \sin x + |t|_0^x + \left|\frac{t^3}{3}\right|_0^x + |\sin t|_0^x$$

$$u_1(x) = -x - \frac{1}{3}x^3 - \sin x + (x-0) + \frac{1}{3}(x^3-0) + (\sin x - \sin 0)$$

$$u_1(x) = \cancel{-x} - \cancel{\frac{1}{3}x^3} - \cancel{\sin x} + \cancel{x} + \cancel{\frac{1}{3}x^3} + \cancel{\sin x} - 0$$

$$u_1(x) = 0$$

we get $u_1(x) = 0$

So the recurrence relation

$$u_{k+1}(x) = \int_0^x K(x,t)\,u_k(t)\,dt = 0 \quad , k \geq 1$$

It is obvious that each components of u_j, $j \geq 1$ is zero. This in turn gives the exact solution. So we obtain

$u(x) = 1 + x^2 + \cos x$ is the exact solutions of voltera integral equation of second kind.

Now we solve some quenstions of voltera integral equation of second kind by using modified decomposition method.

Question #1:

$$u(x) = \cos x + \sin x - \int_0^x u(t)\,dt$$

Solution:

we obtain $f(x) = \cos x + \sin x$

Now we first split $f(x) = \cos x + \sin x$ into two parts

$$f_1(x) = \cos x$$
$$f_2(x) = \sin x$$

Now

By using modified recurrence relation

$$u_0(x) = f_1(x) = \cos x$$
$$u_1(x) = f_2(x) - \int_0^x u_0(t)\,dt$$
$$u_1(x) = \sin x - \int_0^x \cos t\,dt$$
$$u_1(x) = \sin x - |\sin t|_0^x$$
$$u_1(x) = \sin x - (\sin x - \sin 0)$$
$$u_1(x) = \cancel{\sin x} - \cancel{\sin x} - 0$$
$$u_1(x) = 0$$

So, we obtain $u_1(x) = 0$

So, the recurrence relation

$$u_{K+1}(x) = -\int_0^x K(x,t)u_K(t)\,dt = 0 \quad , K \geq 1$$

It is obvious that all the components of the recurrence relation which are greater than or equal to 1 ((i.e, u_j , $j \geq 1$) is zero)
This in turn gives the exact solution
So, we obtain $u(x) = \cos x$ is the exact solution of given volterra integral equation of second kind.

Question #2:

$$u(x) = \sinh x + \cosh x - 1 - \int_0^x u(t)\,dt$$

Solution:

we obtain $f(x) = \sinh x + \cosh x - 1$
we first split $f(x) = \sinh x + \cosh x - 1$ into two parts

$$f_1(x) = \sinh x$$
$$f_2(x) = \cosh x - 1$$

Now
By using modified recurrence relation

$$u_0(x) = f_1(x) = \sinh x$$
$$u_1(x) = f_2(x) - \int_0^x u_0(t)\,dt$$
$$u_1(x) = \cosh x - 1 - \int_0^x \sinh t\,dt$$
$$u_1(x) = \cosh x - 1 - \left|\cosh t\right|_0^x$$
$$u_1(x) = \cosh x - 1 - (\cosh x - \cosh 0)$$
$$u_1(x) = \cosh x - \cancel{1} - \cosh x + \cancel{1}$$
$$u_1(x) = 0$$

Scanned with CamScanner

So we obtain $u_1(x) = 0$

So, the recurrence relation

$$u_{k+1}(x) = -\int_0^x K(x,t)u_k(t)dt = 0 \quad , k \geq 1$$

It is obvious that each component of u_j, $j \geq 1$ is zero. This in turn gives the exact solution. So we obtain $u(x) = \sinh x$ is the exact solution of given volterra integral equation of second kind.

Question # 3:

$$u(x) = 2x + 3x^2 + (e^{x^2+x^3} - 1) - \int_0^x e^{t^2+t^3} u(t)dt$$

Solution:

Let $f(x) = 2x + 3x^2 + (e^{x^2+x^3} - 1)$

we first split $f(x) = 2x + 3x^2 + (e^{x^2+x^3} - 1)$ Into two parts

$$f_1(x) = 2x + 3x^2$$
$$f_2(x) = (e^{x^2+x^3} - 1)$$

Now we use modified recurrence relation

$$f_1(x) = u_0(x) = 2x + 3x^2$$
$$u_1(x) = f_2(x) - \int_0^x e^{t^2+t^3} u_0(t)\,dt$$
$$u_1(x) = e^{x^2+x^3} - 1 - \int_0^{x-0} e^{t^2+t^3}(2t + 3t^2)\,dt \rightarrow (1)$$

Let $u = t^2 + t^3$

$du = (2t + 3t^2)\,dt$

when $t = x$, $u = x^2 + x^3$

and $t = 0$ then $u = 0$

So, (1) $\Rightarrow$

$$u_1(x) = e^{x^2+x^3} - 1 - \int_0^{x^2+x^3} e^u\,du$$

$$u_1(x) = e^{x^2+x^3} - 1 - \left| e^u \right|^{x^2+x^3}$$

$$u_1(x) = e^{x^2+x^3} - 1 - \left(e^{x^2+x^3} - e^0\right)$$

$$u_1(x) = e^{x^2+x^3} - \cancel{1} - e^{x^2+x^3} + \cancel{1}$$

$$u_1(x) = 0$$

we obtain $u(x) = 0$

So, the recurrence relation

$$u_{k+1}(x) = -\int_0^x e^{t^2+t^3} u_k(t)\, dt = 0 \quad , k \geq 1$$

It is obvious that each components of u_j, $j \geq 1$ is zero. This in turns gives the exact solution. So we obtain that $u(x) = 2x + 3x^2$ is the exact solution of given volterra integral equation of second kind.

Question # 4:

$$u(x) = 3x^2 + (1 - e^{-x^3}) - \int_0^x e^{-x^3+t^3} u(t)\, dt$$

Solution:

Let

$$f(x) = 3x^2 + (1 - e^{-x^3})$$

Now we first split $f(x) = 3x^2 + (1 - e^{-x^3})$ into two parts

$$f_1(x) = 3x^2$$

$$f_2(x) = 1 - e^{-x^3}$$

Now By using modified recurrence relation

$$u_0(x) = f_1(x) = 3x^2$$

$$u_1(x) = f_2(x) - \int_0^x e^{-x^3+t^3} u_0(t)\, dt$$

$$u_1(x) = 1 - e^{-x^3} - \int_0^x e^{-x^3} \cdot e^{t^3} (3t^2)\, dt$$

$$u_1(x) = 1 - e^{-x^3} - e^{-x^3} \int_0^x e^{t^3} (3t^2)\, dt$$

$$u_1(x) = 1 - e^{-x^3} - e^{-x^3}\left|e^{t^3}\right|_0^x$$
$$u_1(x) = 1 - e^{-x^3} - e^{-x^3}\left(e^{x^3} - e^{0}\right)$$
$$u_1(x) = 1 - e^{-x^3} - e^{0} + e^{-x^3}$$
$$u_1(x) = \cancel{1} - \cancel{e^{-x^3}} - \cancel{1} + \cancel{e^{-x^3}}$$
$$u_1(x) = 0$$

we get $u_1(x) = 0$

So, the recurrence relation,

$$u_{k+1}(x) = -\int_0^x e^{-x^3+t^3} u_k(t)\,dt = 0 \quad , k \geq 1$$

It is obvious that each components of u_j, $j \geq 1$ is zero. This in turn gives the exact solution. So we obtain that $u(x) = 3x^2$ is the exact solution of given Volterra integral equation of second kind

Question #5:

$$u(x) = 2x - (1 - e^{-x^2}) + \int_0^x e^{-x^2+t^2} u(t)\,dt$$

Solution:

Let $f(x) = -(1 - e^{-x^2}) + 2x$

Now we first split $f(x) = -1 + e^{-x^2} + 2x$ into two parts

$$f_1(x) = 2x$$
$$f_2(x) = -(1 - e^{-x^2})$$

Now we use modified recurrence relation to obtain

$$u_0(x) = f_1(x) = 2x$$
$$u_1(x) = f_2(x) + \int_0^x e^{-x^2+t^2} u_0(t)\,dt$$
$$u_1(x) = -(1 - e^{-x^2}) + \int_0^x e^{-x^2} \cdot e^{t^2} (2t)\,dt$$
$$u_1(x) = -1 + e^{-x^2} + e^{-x^2} \int_0^x e^{t^2}\, 2t\,dt$$

Scanned with CamScanner

$$u_1(x) = -1 + e^{-x^2} + e^{-x^2} \left| e^{t^2} \right|_0^x$$

$$u_1(x) = -1 + e^{-x^2} + e^{-x^2} (e^{x^2} - e^0)$$

$$u_1(x) = -1 + e^{-x^2} + e^0 - e^{-x^2}$$

$$u_1(x) = -\cancel{1} + e^{-x^2} + 1 - e^{-x^2}$$

$$u_1(x) = 0$$

we get $u_1(x) = 0$

So the recurrence relation

$$u_{k+1}(x) = \int_0^x e^{-x^2+t^2} u_k(t)\,dt = 0 \quad , k \geq 1$$

It is obvious that each components of u_j, $j \geq 1$ is zero this in turns gives the exact solution so we obtain $u(x) = 2x$ is the exact solution of given volterra integral equation of second kind.

Question #6

$$u(x) = e^{-x^2} + \frac{x}{2}(1 - e^{-x^2}) - \int_0^x x\,t\,u(t)\,dt$$

Let $f(x) = e^{-x^2} + \frac{x}{2}(1 - e^{-x^2})$

Now we first split $f(x) = e^{-x^2} + \frac{x}{2}(1 - e^{-x^2})$ into two parts

$$f_1(x) = e^{-x^2}$$

$$f_2(x) = +\frac{x}{2}(1 - e^{-x^2})$$

By using modified recurrence relation we have

$$u_0(x) = f_1(x) = e^{-x^2}$$

$$u_1(x) = f_2(x) - \int_0^x x\,t\,u(t)\,dt$$

$$u_1(x) = +\frac{x}{2}(1 - e^{-x^2}) - x\int_0^x t\,e^{-t^2}\,dt$$

$$u_1(x) = +\frac{x}{2}(1 - e^{-x^2}) + \frac{x}{2}\int_0^x e^{-t^2}(-2t)\,dt$$

$$u_1(x) = \frac{x}{2}\left(1-e^{-x^2}\right) + \frac{x}{2}\left| e^{-t^2} \right|_0^x$$

$$u_1(x) = \frac{x}{2}\left(1-e^{-x^2}\right) + \frac{x}{2}\left(e^{-x^2}-1\right)$$

$$u_1(x) = \frac{x}{2}\left(1-e^{-x^2}\right) - \frac{x}{2}\left(1-e^{-x^2}\right)$$

$$u_1(x) = 0$$

we get $u_1(x) = 0$

So the recurrence relation

$$u_{k+1}(x) = -x\int_0^x t\, u_k(t)\, dt = 0 \quad , k \geq 1$$

It is obvious that each components of u_j , $j \geq 1$ is zero. This in turn gives the exact solution. So we obtain that

$$u(x) = e^{-x^2}$$

is the exact solution of given voltera integral equation of second kind.

Question #7:

$$u(x) = \cosh x + x \sinh x - \int_0^x x\, u(t)\, dt$$

Solution:

Let $f(x) = \cosh x + x \sinh x$

First we split $f(x) = \cosh x + x \sinh x$ into two parts

$$f_1(x) = \cosh x$$
$$f_2(x) = x \sinh x$$

By using modified recursence relation

$$u_0(x) = f_1(x) = \cosh x$$
$$u_1(x) = f_2(x) - \int_0^x x\, u_0(t)\, dt$$
$$u_1(x) = x \sinh x - x \int_0^x \cosh t\, dt$$
$$u_1(x) = x \sinh x - x \left| \sinh t \right|_0^x$$
$$u_1(x) = x \sinh x - x(\sinh x - \sinh 0)$$
$$u_1(x) = x \sinh x - x \sinh x + 0$$
$$u_1(x) = 0$$

we get $u_1(x) = 0$

So, the recurrence relation

$$u_{k+1}(x) = -x \int_0^x u_k(t)\, dt = 0 \quad , k \geq 1$$

It is obvious that each components of u_j, $j \geq 1$ is zero. This in turns give the exact solution. So we obtain $u(x) = \cosh x$ is the exact solution of given volterra integral equation of second.

Question # 8:

$$u(x) = e^x + xe^x - x - \int_0^x x\, u(t)\,dt$$

Solution:

Let $f(x) = e^x + xe^x - x$

we first split $f(x)$ given by $f(x) = e^x + xe^x - x$ into two parts

$$f_1(x) = e^x$$
$$f_2(x) = xe^x - x$$

By using modified recurrence relation

$$u_0(x) = f_1(x) = e^x$$
$$u_1(x) = f_2(x) - x\int_0^x u_0(t)\,dt$$
$$u_1(x) = xe^x - x - x\int_0^x e^t\,dt$$
$$u_1(x) = xe^x - x - x\left|e^t\right|_0^x$$
$$u_1(x) = xe^x - x - x(e^x - e^0)$$
$$u_1(x) = xe^x - x - x(e^x - 1)$$
$$u_1(x) = \cancel{xe^x} - \cancel{x} - \cancel{xe^x} + \cancel{x}$$
$$u_1(x) = 0$$

we get $u_1(x) = 0$

So the recurrence relation

$$u_{k+1}(x) = -x\int_0^x u_k(t)\,dt = 0, \quad k \geq 1$$

It is obvious that each components of is zero. This in turn gives the exact solution. So we obtain $u(x) = e^x$ is the exact solution of given volterra integral equation of second kind

Question # 9:

$$u(x) = 1 + \sin x + x + x^2 - x\cos x - \int_0^x x\, u(t)\,dt$$

Solution:

The function $f(x)$ is given by
$f(x) = 1 + \sin x + x + x^2 - x\cos x$
By trail we split $f(x)$ into two parts,

$f_1(x) = 1 + \sin x$
$f_2(x) = x + x^2 - x\cos x$

we use modified recurrence relation to obtain

$u_0(x) = f_1(x) = 1 + \sin x$

$u_1(x) = f_2(x) - \int_0^x x\, u_0(t)\,dt$

$$u_1(x) = x + x^2 - x\cos x - x\int_0^x (1 + \sin t)\,dt$$

$$u_1(x) = x + x^2 - x\cos x - x\left[|t|_0^x + |-\cos x|_0^x \right]$$

$$u_1(x) = x + x^2 - x\cos x - x\left[(x - 0) - (\cos x - \cos 0)\right]$$

$$u_1(x) = \cancel{x} + \cancel{x^2} - \cancel{x\cos x} - \cancel{x^2} + \cancel{x\cos x} - \cancel{x}$$

$u_1(x) = 0$

we get $u_1(x) = 0$

So, the recurrence relation

$$u_{k+1}(x) = -x\int_0^x u_k(t)\,dt = 0 \quad ; k \geq 1$$

It is obvious that each components of u_j, $j \geq 1$ is zero. This in turn gives the exact solution. So we obtain $u(x) = 1 + \sin x$ is the exact solution of given volterra integral equation of second kind.

Question # 10:

$$u(x) = e^x - xe^x + \sin x + x\cos x + \int_0^x x\,u(t)\,dt$$

Solution:

The function $f(x)$ is given by

$$f(x) = e^x - xe^x + \sin x + x\cos x$$

By trail we split $f(x)$ into two parts

$$f_1(x) = e^x + \sin x$$

$$f_2(x) = -xe^x + x\cos x$$

we use modified recurrence relation to obtain

$$u_0(x) = f_1(x) = e^x + \sin x$$

$$u_1(x) = f_2(x) + \int_0^x x\,u_0(t)\,dt$$

$$u_1(x) = -xe^x + x\cos x + x\int_0^x (e^t + \sin t)\,dt$$

$$u_1(x) = -xe^x + x\cos x + x\left[\left|e^t\right|_0^x + \left|-\cos t\right|_0^x\right]$$

$$u_1(x) = -xe^x + x\cos x + x\left[(e^x - e^0) - (\cos x - \cos 0)\right]$$

$$u_1(x) = -xe^x + x\cos x + x\left(e^x - \cancel{1} - \cos x + \cancel{1}\right)$$

$$u_1(x) = \cancel{-xe^x} + \cancel{x\cos x} + \cancel{xe^x} - \cancel{x\cos x}$$

$$u_1(x) = 0$$

we get $u_1(x) = 0$

So The recurrence relation

$$u_{k+1}(x) = \int_0^x x\,u_k(t)\,dt = 0, \quad k \geq 1$$

It is obvious that each components of u_j, $j \geq 1$ is zero. This in turn gives the exact solution. So we obtain $u(x) = e^x + \sin x$ is the exact solution of given v.integral equation of second kind.

Question # 11:

$$u(x) = 1 + x + x^2 + \frac{1}{2}x^3 + \cosh x + x\sinh x - \int_0^x x u(t)\,dt$$

Solution:

The $f(x)$ consists of six terms. By trail we divide $f(x)$ given by

$$f(x) = 1 + x + x^2 + \frac{1}{2}x^3 + \cosh x + x\sinh x$$

into two parts

$$f_1(x) = 1 + x + \cosh x$$

$$f_2(x) = x^2 + \frac{1}{2}x^3 + x\sinh x$$

we use modified recurrence relation to obtain

$$u_0(x) = f_1(x) = 1 + x + \cosh x$$

$$u_1(x) = f_2(x) - x\int_0^x u_0(t)\,dt$$

$$u_1(x) = x^2 + \frac{1}{2}x^3 + x\sinh x - x\int_0^x (1 + t + \cosh t)\,dt$$

$$u_1(x) = x^2 + \frac{1}{2}x^3 + x\sinh x - x\left[\left|t\right|_0^x + \left|\frac{t^2}{2}\right|_0^x + \left|\sinh t\right|_0^x\right]$$

$$u_1(x) = x^2 + \frac{1}{2}x^3 + x\sinh x - x\left[(x-0) + \left(\frac{x^2}{2} - 0\right) + (\sinh x - 0)\right]$$

$$u_1(x) = \cancel{x^2} + \cancel{\frac{1}{2}x^3} + \cancel{x\sinh x} - \cancel{x^2} - \cancel{\frac{1}{2}x^3} - \cancel{x\sinh x}$$

$$u_1(x) = 0$$

we get $u_1(x) = 0$

So, the recurrence relation

$$u_{k+1}(x) = -x\int_0^x u_k(t)\,dt = 0 \quad , k \geq 1$$

It is obvious that each components of u_j, $j \geq 1$ is zero. This in turn gives the exact solution. So we obtain $u(x) = 1 + x + \cosh x$ is the exact solution of given volterra integral equation of second kind.

Question #12:

$$u(x) = \cos x - (1 - e^{\sin x})x - x\int_0^x e^{\sin t} u(t)\,dt$$

Solution:

The function $f(x)$ consist of three terms By that we divide $f(x)$ given by

$$f(x) = \cos x - (1 - e^{\sin x})x$$

into two parts;

$$f_1(x) = \cos x$$
$$f_2(x) = -(1 - e^{\sin x})x$$

we use modified recurrence relation to obtain

$$u_0(x) = f_1(x) = \cos x$$
$$u_1(x) = f_2(x) - x\int_0^x e^{\sin t} u_0(t)\,dt$$
$$u_1(x) = -(1 - e^{\sin x})x - x\int_0^x e^{\sin t}\cos t\,dt$$
$$u_1(x) = -(1 - e^{\sin x})x - x\left| e^{\sin t} \right|_0^x$$
$$u_1(x) = -(1 - e^{\sin x})x - x(e^{\sin x} - e^0)$$
$$u_1(x) = -(1 - e^{\sin x})x - x(e^{\sin x} - 1)$$
$$u_1(x) = -\cancel{x} + \cancel{xe^{\sin x}} - \cancel{xe^{\sin x}} + \cancel{x}$$
$$u_1(x) = 0$$

we get $u_1(x) = 0$

So, the Reccurrence relation

$$u_{k+1}(x) = -x\int_0^x e^{\sin t} u_k(t)\,dt = 0, \quad k \geq 1$$

It is obvious that each components of $u_j, j \geq 1$ is zero. This in turn gives the **exact** solution. So we obtain $u(x) = \cos x$ is the exact solution of given volterra integral equation of second kind.

Question # 13:

$$u(x) = \sec^2 x - (1 - e^{\tan x})x - x\int_0^x e^{\tan t} u(t)\,dt$$

Solution:

$f(x)$ is given by. $f(x) = \sec^2 x - (1 - e^{\tan x})x$

we divide $f(x)$ into two parts,

$$f_1(x) = \sec^2 x$$
$$f_2(x) = -(1 - e^{\tan x})x$$

we use modified recurrence relation to obtain

$$u_0(x) = f_1(x) = \sec^2 x$$
$$u_1(x) = f_2(x) - x\int_0^x e^{\tan t} u_0(t)\,dt$$

$$u_1(x) = -(1 - e^{\tan x})x - x\int_0^x e^{\tan t} \sec^2 t\,dt$$

$$u_1(x) = -(1 - e^{\tan x})x - x\left| e^{\tan t} \right|_0^x$$

$\because \frac{d}{dt}(\tan t) = \sec^2 t$

$$u_1(x) = -(1 - e^{\tan x})x - (e^{\tan x} - e^0)x$$
$$u_1(x) = -\cancel{x} + \cancel{xe^{\tan x}} - \cancel{xe^{\tan x}} + \cancel{x}$$
$$u_1(x) = 0$$

we get $u_1(x) = 0$

So, the recurrence relation

$$u_{k+1}(x) = -x\int_0^x e^{\tan t} u_k(t)\,dt = 0, \quad k \geq 1$$

It is obvious that each components of $u_j, j \geq 1$. This in turns gives the exact solution. So we obtain $u(x) = \sec^2 x$ is the exact solution of given volterra integral equation of second kind.

Question # 14:

$$u(x) = \cosh x + \frac{x}{2}(1 - e^{\sinh x}) + \frac{x}{2}\int_0^x e^{\sinh t} u(t)\,dt.$$

Solution:

$f(x)$ is given by $f(x) = \cosh x + \frac{x}{2}(1 - e^{\sinh x})$
First we split $f(x)$ into two parts

$$f_1(x) = \cosh x$$
$$f_2(x) = \frac{x}{2}(1 - e^{\sinh x})$$

we use modified recurrence relation to obtain

$$u_0(x) = \cosh x = f_1(x)$$
$$u_1(x) = f_2(x) + \frac{x}{2}\int_0^x e^{\sinh t} u_0(t)\,dt$$
$$u_1(x) = \frac{x}{2}(1 - e^{\sinh x}) + \frac{x}{2}\int_0^x e^{\sinh t}\cosh t\,dt$$
$$u_1(x) = \frac{x}{2}(1 - e^{\sinh x}) + \frac{x}{2}\left| e^{\sinh t}\right|_0^x$$
$$u_1(x) = \frac{x}{2}(1 - e^{\sinh x}) + \frac{x}{2}(e^{\sinh x} - e^0)$$
$$u_1(x) = \frac{x}{2}(1 - e^{\sinh x}) + \frac{x}{2}(e^{\sinh x} - 1)$$
$$u_1(x) = \frac{x}{2}(1 - \cancel{e^{\sinh x}}) - \frac{x}{2}(\cancel{1} - e^{\sinh x})$$
$$u_1(x) = 0$$

Scanned with CamScanner

we obtain $u_1(x) = 0$

So, the recurrence relation

$$u_{k+1}(x) = \frac{x}{2}\int_0^x e^{\sinh t} u_k(t)\,dt = 0 \quad , k \geq 1$$

It is abvious that each components of u_j, $j \geq 1$ is zero. This in turn gives the exact solution. so we obtain $u(x) = \cosh x$ is the exact solution of given volterra integral equation of second kind.

Question # 15:

$$u(x) = \sinh x + \frac{1}{10}\left(e - e^{\cosh x}\right) + \frac{1}{10}\int_0^x e^{\cosh t} u(t)\,dt$$

Solution:

we obtain $f(x) = \sinh x + \frac{1}{10}\left(e - e^{\cosh x}\right)$

First we divide $f(x)$ into two parts

$$f_1(x) = \sinh x$$
$$f_2(x) = \frac{1}{10}\left(e - e^{\cosh x}\right)$$

we use modified recurrence relation to obtain

$$u_0(x) = f_1(x) = \sinh x$$
$$u_1(x) = f_2(x) + \frac{1}{10}\int_0^x e^{\cosh t} u_0(t)\,dt$$

$$u_1(x) = \frac{1}{10}\left(e - e^{\cosh x}\right) + \frac{1}{10}\int_0^x e^{\cosh t}(\sinh t)\,dt$$

$$u_1(x) = \frac{1}{10}\left(e - e^{\cosh x}\right) + \frac{1}{10}\left|e^{\cosh t}\right|_0^x$$

$$u_1(x) = \frac{1}{10}\left(e - e^{\cosh x}\right) + \frac{1}{10}\left(e^{\cosh x} - e^{1}\right)$$

$$u_1(x) = \frac{1}{10}\left(e - e^{\cosh x}\right) + \frac{1}{10}\left(e^{\cosh x} - e\right)$$

$$u_1(x) = \frac{1}{10}\left(e - \cancel{e^{\cosh x}}\right) - \frac{1}{10}\left(e - \cancel{e^{\cosh x}}\right)$$

$$u_1(x) = 0$$

we get $u_1(x) = 0$

$$u_{k+1}(x) = \frac{1}{10}\int_0^x e^{\cosh t} u_k(t)\,dt = 0 \quad , k \geq 1$$

It is obvious that each components of u_j, $j \geq 1$ is zero. This in turns gives the exact solution. So we obtain

$$u(x) = \sinh x$$

is the exact solution of given volterra integral equation of second kind.

Question # 16:

$$u(x) = x^3 - \frac{x^5}{10} + \frac{5}{10}\int_0^x t\,u(t)\,dt$$

Solution:

$f(x)$ is given by;

$$f(x) = x^3 - \frac{x^5}{10}$$

First we split $f(x)$ into two parts

$$f_1(x) = x^3$$

$$f_2(x) = -\frac{x^5}{10}$$

We use recurrence relation to Obtain

$$u_0(x) = f_1(x) = x^3$$

$$u_1(x) = f_2(x) + \frac{5}{10}\int_0^x t\, u_0(t)\,dt$$

$$u_1(x) = -\frac{x^5}{10} + \frac{5}{10}\int_0^x t(t^3)\,dt$$

$$u_1(x) = -\frac{x^5}{10} + \frac{5}{10}\int_0^x t^4\,dt$$

$$u_1(x) = -\frac{x^5}{10} + \frac{5}{10}\left|\frac{t^5}{5}\right|_0^x$$

$$u_1(x) = -\frac{x^5}{10} + \frac{5}{10}\left(\frac{x^5}{5} - 0\right)$$

$$u_1(x) = -\frac{x^5}{10} + \frac{5}{10}\frac{x^5}{5}$$

$$u_1(x) = -\cancel{\frac{x^5}{10}} + \cancel{\frac{x^5}{10}}$$

$$u_1(x) = 0$$

we get $u_1(x) = 0$

So, recurrence relation

$$u_{k+1}(x) = \frac{5}{10}\int_0^x t\, u_k(t)\,dt = 0 \quad ; k \geq 1$$

It is obvious that each components of u_j, $j \geq 1$ is zero. This in turns gives the exact solution so we obtain $u(x) = x^3$ is the exact solution of given voltera Integral equation of second Kind