The Series Solution Method for Volterra Integral Equation of second kind

THE SERIES SOLUTION METHOD FOR VOLTERA INTEGRAL EQUATION OF FIRST KIND

BRIEF INTRODUCTION:

In Mathematics, the voltera integral equations are a special type of integral equations. The standard form of the volteral integral equations of the first kind is given by

$$f(x) = \lambda \int_a^x k(x,t)u(t)dt \quad \text{①}$$

where 'a' is a constant and x is variable;
$f(x)$ and $k(x,t)$ are known functions, while λ is non-zero parameter and the function $k(x,t)$ in the integral is called kernel

THE SERIES SOLUTION METHOD

A real function $u(x)$ is called analytic if it has derivatives of all orders such that the Taylor series at any point b in its domain

$$u(x) = \sum_{n=0}^{\infty} \frac{u^n(b)(x-b)^n}{n!} \quad (2)$$

Converges to $f(x)$ in a neighborhood of b. For simplicity, the generic form of Taylor Series at $x=0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad (3)$$

where the cofficients a_n will be determined recurrently.

Subsituting (3) in (1) gives Volterra integral Equation of first kind

$$T(f(x)) = \lambda \int_a^x k(x,t) \left(\sum_{n=0}^{\infty} a_n t^n \right) dt \quad (4)$$

or for simplicity we can use

$$T(f(x)) = \lambda \int_a^x k(x,t)(a_0 + a_1 t + a_2 t^2 + \cdots)\,dt \quad (5)$$

where $T(f(x))$ is the Taylor Series for $f(x)$.

The integral equation (1) will be converted to a traditional integral in (4) or (5) where instead of integrating the unknown function $u(x)$, terms of the form t^n, $n \geq 0$ will be integrated.

Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions etc, then Taylor expansions for functions involved in $f(x)$ should be used.

The method is identical to that presented before for the volterra integral equations

Date Day M T W T F S

of the second kind.

We first integrate the right side of the integral in (4) or (5), and collect the coefficients of like powers of x.

- We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrance relation in a_j, $j \geq 0$
- Solving the recurrance relation will lead to a complete determination of the cofficients a_j, $j \geq 0$
- Having determined the coefficients a_j, $j \geq 0$, the series solution follows immediately upon subsituting the derived coefficients into (3)
- The Solution may be obtained if such an exact solution exists. If an exact solution is not obtained, then the obtained serries can be used for numerical purposes.

Date Day M T W T F S

In this case, the more terms we evaluate, the higher accuracy level we achieve. This method will be illustrated by discussing the following examples.

EXAMPLE

Solve the volterra integral equation by using the series solution method

$$\sin x - x\cos x = \int_0^x t u(t)\,dt \quad \text{—— (1)}$$

Using Taylor series for $\sin x$ & $\cos x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{—— (2)}$$

Putting Taylor series of $\sin x$, $\cos x$ and $u(x) = \sum_{n=0}^{\infty} a_n x^n$ in (1)

$$(1) \Rightarrow \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7} + \cdots\right) - x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= \sum_{n=0}^{\infty} \int_0^x t a_n t^n\,dt$$

Date Day M T W T F S

$$\Rightarrow x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - x + \frac{x^3}{2!} - \frac{x^5}{4!} + \frac{x^7}{6!} + \cdots = \sum_{n=0}^{\infty} a_n \int_0^x t^{n+1}\,dt$$

$$-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^3}{2!} - \frac{x^5}{4!} + \frac{x^7}{6!} + \cdots = \sum_{n=0}^{\infty} a_n \left.\frac{t^{n+2}}{n+2}\right|_0^x$$

$$\Rightarrow x^3\left(\frac{-1}{6} + \frac{1}{2}\right) + x^5\left(\frac{1}{5!} - \frac{1}{4!}\right) + x^7\left(\frac{1}{6!} - \frac{1}{7!}\right) + \cdots = \sum_{n=0}^{\infty} a_n \frac{x^{n+2}}{n+2}$$

$$\Rightarrow x^3\left(\frac{-1+3}{6}\right) + x^5\left(\frac{1-5}{120}\right) + x^7\left(\frac{7-1}{5040}\right) + \cdots = \sum_{n=0}^{\infty} a_n \frac{x^{n+2}}{n+2}$$

$$x^3\left(\frac{2}{6}\right) + x^5\left(\frac{-4}{120}\right) + x^7\left(\frac{6}{5040}\right) + \cdots = \sum_{n=0}^{\infty} a_n \frac{x^{n+2}}{n+2}$$

Expanding R.H.S

$$\frac{x^3}{3} - \frac{x^5}{30} + \frac{x^7}{840} + \cdots = a_0\frac{x^2}{2} + a_1\frac{x^3}{3} + a_2\frac{x^4}{4} + a_3\frac{x^5}{5}$$

$$+ a_4\frac{x^6}{6} + a_5\frac{x^7}{7} + a_6\frac{x^8}{8} + a_7\frac{x^9}{9} + \cdots$$

Comparing cofficients of like power x as,

x^2; $a_0 = 0$

x^3; $\frac{1}{3} = \frac{a_1}{3} \Rightarrow a_1 = \frac{3}{3} \Rightarrow a_1 = 1$

x^4; $0 = \frac{a_2}{4} \Rightarrow a_2 = 0$

x^5; $\frac{-1}{30} = \frac{a_3}{5} \Rightarrow a_3 = \frac{-5}{30} \Rightarrow a_3 = \frac{-1}{3!}$

Date Day M T W T F S

$x^6;\quad 0 = \frac{a_4}{6} \Rightarrow a_4 = 0$

$x^7;\quad \frac{1}{840} = \frac{a_5}{7} \Rightarrow a_5 = \frac{7}{840} = \frac{1}{5!}$

$x^7 = \frac{1}{5!}$

(2) $\Rightarrow$

$u(x) = a_0 x + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$
$+ a_6 x^6 + a_7 x^7 + \cdots$

Putting the values of constant in above equation.

$u(x) = 0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 + \cdots$

$u(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$ which is Series of $\sin x$

$$\boxed{u(x) = \sin x}$$

EXAMPLE

$$2 + x - 2e^x + xe^x = \int_0^x (x-t)u(t)\,dt \quad (1)$$

Series of e^x is give by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$u(x)=\sum_{n=0}^{\infty} a_n x^n \quad \text{———} \quad (2)$$

equation (1) becomes;

$$2+x-2\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)+x\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)=\sum_{n=0}^{\infty}\int_0^x (x-t)a_n t^n\,dt$$

$$2+x-2\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)+x\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)=\int_0^x (x-t)\sum_{n=0}^{\infty} a_n t^n\,dt$$

$$\cancel{2}+\cancel{x}-\cancel{2}-2x-\frac{2x^2}{2}-\frac{2x^3}{6}-\frac{2x^4}{24}+x+x^2+\frac{x^3}{2!}+\frac{x^4}{3!}+\frac{x^5}{5!}+\cdots=\sum_{n=0}^{\infty} x a_n\int_0^x t^n\,dt-\sum_{n=0}^{\infty} a_n\int_0^x t^{n+1}\,dt$$

$$-\frac{x^3}{3}+\frac{x^3}{2}-\frac{x^4}{12}+\frac{x^4}{6}+\cdots=\sum_{n=0}^{\infty} x a_n \frac{t^{n+1}}{n+1}\Big|_0^x-\sum_{n=0}^{\infty} a_n\frac{t^{n+2}}{n+2}\Big|_0^x$$

$$\frac{x^3}{6}+\frac{x^4}{12}+\frac{x^5}{40}+\cdots=\sum_{n=0}^{\infty} a_n\frac{x^{n+2}}{n+1}-\sum_{n=0}^{\infty} a_n\frac{x^{n+2}}{n+2}$$

Expanding R.H.S of above equation

$$\frac{x^3}{6}+\frac{x^4}{12}+\frac{x^5}{40}+\cdots=\left[a_0x^2+\frac{a_1x^3}{2}+\frac{a_2x^4}{3}+\frac{a_3x^5}{4}+\cdots\right]-\left[\frac{a_0x^2}{2}+\frac{a_1x^3}{3}+\frac{a_2x^4}{4}+\frac{a_3x^5}{5}+\cdots\right]$$

Date Day M T W T F S

$$\frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{40} = a_0x^2 + a_1\frac{x^3}{2} + a_2\frac{x^4}{3} + a_3\frac{x^5}{4} + \cdots$$

$$- a_0\frac{x^2}{2} - a_1\frac{x^3}{3} - a_2\frac{x^4}{4} - a_3\frac{x^5}{5} \cdots$$

$$= a_0x^2\left(1-\frac{1}{2}\right) + a_1x^3\left(\frac{1}{2}-\frac{1}{3}\right) + a_2x^4\left(\frac{1}{3}-\frac{1}{4}\right) + \cdots$$

$$\frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{40} + \cdots = a_0\frac{x^2}{2} + a_1\frac{x^3}{6} + a_2\frac{x^4}{12} + a_3\frac{x^5}{20} + \cdots$$

Comparing co-efficients of like power x as,

x^2; $a_0 = 0$

x^3; $\frac{a_1}{6} = \frac{1}{6} \Rightarrow a_1 = 1$

x^4; $\frac{1}{12} = \frac{a_2}{12} \Rightarrow a_2 = 1$

x^5; $\frac{1}{40} = \frac{a_3}{20} \Rightarrow a_3 = \frac{20}{40} = \frac{1}{2}$

$a_3 = \frac{1}{2!}$

Similarly;

$a_4 = \frac{1}{3!}$

Putting these values

$$u(x) = x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$\boxed{u(x) = xe^x}$$

Date | Day M T W T F S

EXAMPLE

$$x-\frac{1}{2}x^2-\ln(1+x)+x^2\ln(1+x)=\int_0^x 2tu(t)\,dt \quad \text{——— (1)}$$

Using Taylor series for $\ln(1+x)$

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\cdots$$

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{——— (2)}$$

Putting Taylor series of $\ln(1+x)$ and eq (2) in (1)

$$x-\frac{x^2}{2}-\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{6}-\frac{x^6}{6}+\cdots\right)+x^2\left(x-\frac{x^2}{2}\right.$$

$$\left.+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}\right) = \sum_{n=0}^{\infty} 2a_n t\, t^n\, dt$$

$$x-\frac{x^2}{2}-x+\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}-\frac{x^5}{5}+\frac{x^6}{6}+x^3-\frac{x^4}{2}+\frac{x^5}{3}$$

$$-\frac{x^6}{4}+\frac{x^7}{5}-\frac{x^8}{6}+\cdots = 2\sum_{n=0}^{\infty} a_n\int_0^x t^{n+1}\,dt$$

$$x^3\left(1-\frac{1}{3}\right)+x^4\left(\frac{1}{4}-\frac{1}{2}\right)+x^5\left(\frac{1}{3}-\frac{1}{5}\right)+x^6\left(\frac{1}{6}-\frac{1}{4}\right)+\cdots$$

$$= 2\sum_{n=0}^{\infty} a_n \left.\frac{t^{n+2}}{n+2}\right|_0^x$$

$$x^3\left(\frac{2}{3}\right)+x^4\left(\frac{-1}{4}\right)+x^5\left(\frac{2}{15}\right)+x^6\left(-\frac{1}{12}\right)+\cdots$$

$$= 2\sum_{n=0}^{\infty} a_n \frac{x^{n+2}}{n+2}$$

Date: Day M T W T F S

$$\frac{2x^3}{3} - \frac{1}{4}x^4 + \frac{2}{15}x^5 - \frac{x^6}{12} + \cdots = 2\sum_{n=0}^{\infty} a_n \frac{x^{n+2}}{n+2}$$

On R.H.S Expanding

$$\frac{2x^3}{3} - \frac{1}{4}x^4 + \frac{2}{15}x^5 - \frac{x^6}{12} + \cdots = a_0x^2 + \frac{2}{3}a_1x^3 + \frac{a_2x^4}{4}$$

$$+ \frac{2a_3x^5}{5} + \frac{a_4x^6}{3} + \cdots$$

Comparing coefficients of like powers x as;

x^2; $a_0 = 0$

x^3; $\frac{2}{3} = \frac{2}{3}a_1 \Rightarrow a_1 = 1$

x^4; $\frac{-1}{4} = \frac{a_2}{2} \Rightarrow a_2 = \frac{-1}{2}$

x^5; $\frac{2}{15} = \frac{2}{5}a_3 \Rightarrow a_3 = \frac{5}{\cancel{2}} \times \frac{\cancel{2}}{15}$

$a_3 = \frac{1}{3}$

x^6; $-\frac{1}{12} = \frac{a_4}{3} \Rightarrow a_4 = -\frac{3}{12} = -\frac{1}{4}$

$a_4 = -\frac{1}{4}$

x^6; $a_4 = -\frac{1}{4}$

Similarly

$a_5 = \frac{1}{5}$, $a_6 = -\frac{1}{6}$, ---

Date Day M T W T F S

Now ② ⇒

$$u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^2 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

Putting the values of

$$= 0 + 1(x) + \left(\frac{-1}{2}\right)x^2 + \left(\frac{1}{3}\right)x^3 + \left(\frac{-1}{4}\right)x^4 + \left(\frac{1}{5}\right)x^5$$

$$u(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots$$

i.e $\boxed{u(x) = \ln(1+x)}$

Solving Some Questions Related To The Series Solution For Voltera Integral Equation of First Kind

Question No 1

$$e^x - 1 - x = \int_0^x (x - t + 1)\, u(t)\, dt \quad \text{—— ①}$$

Date Day M T W T F S

Using the Taylor series of e^x on the L.H.S of ① and put

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\therefore e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

① becomes

$$\cancel{1} + \cancel{x} + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots - \cancel{1} - \cancel{x} = \int_0^x (x-t+1) \sum_{n=0}^{\infty} a_n t^n dt$$

$$\frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} a_n x \int_0^x t^n dt - \sum_{n=0}^{\infty} a_n \int_0^x t^{n+1} dt + \sum_{n=0}^{\infty} a_n \int_0^x t^n dt$$

$$\frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} a_n x \left. \frac{t^{n+1}}{n+1} \right|_0^x - \sum_{n=0}^{\infty} a_n \left. \frac{t^{n+2}}{n+2} \right|_0^x + \sum_{n=0}^{\infty} a_n \left. \frac{t^{n+1}}{n+1} \right|_0^x$$

$$\frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} a_n \left[\frac{x^{n+2}}{n+1} - \frac{x^{n+2}}{n+2} + \frac{x^{n+1}}{n+1} \right]$$

$$\frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} a_n \left[\frac{x^{n+2}}{(n+1)(n+2)} + \frac{x^{n+1}}{n+1} \right]$$

Date Day M T W T F S

Expanding R.H.S of above equation,

$$\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots = a_0\frac{x^2}{2}+a_1\frac{x^3}{6}+a_2\frac{x^4}{12}$$

$$+\frac{a_3}{20}x^5+\cdots+a_0x+a_1\frac{x^2}{2}+a_2\frac{x^3}{3}+a_3\frac{x^4}{4}+\cdots$$

$$\frac{x^2}{2}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots = a_0x+\frac{1}{2}(a_0+a_1)x^2$$

$$+\frac{1}{6}(a_1+2a_2)x^3+\frac{1}{12}(a_2+3a_3)x^4+\frac{1}{20}(a_3+4a_4)x^5+\cdots$$

Comparing coefficient of 'x'

x: $\boxed{a_0=0}$

x^2: $\frac{1}{2}(a_0+a_1)=\frac{1}{2} \Rightarrow a_1=1$

x^3: $\frac{1}{6}(a_1+2a_2)=\frac{1}{6} \Rightarrow a_2=0$

x^4: $\frac{1}{12}(a_2+3a_3) \Rightarrow a_3=\frac{1}{6}$

$\frac{1}{20}(a_3+4a_4)=\frac{1}{120} \Rightarrow a_4=0$

equation 2

$\Rightarrow u(x) = a_0x^0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots$

So

$u(x) = 0+x+0+\frac{1}{6}x^3+0+\frac{1}{120}x^5$

$u(x) = x+\frac{1}{6}x^3+\frac{1}{120}x^5$

$\boxed{u(x) = \sinh x}$

Date Day M T W T F S

QUESTION No 2

$$x\cosh x - \sinh x = \int_0^x u(t)\,dt \quad \text{—— } ①$$

Using the Taylor series of $\cosh x$ and $\sinh x$ on the left hand side of ① and put

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{—— } ②$$

$$\because \cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots$$

$$\sinh x = x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots$$

① becomes

$$x\left[1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots\right] - \left[x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots\right] = \int_0^x \sum_{n=0}^{\infty} a_n t^n\,dt$$

$$\cancel{x} + \frac{x^3}{2} + \frac{x^5}{24} + \frac{x^7}{720} - \cancel{x} - \frac{x^3}{6} - \frac{x^5}{120} - \frac{x^7}{5040} + \cdots = \sum_{n=0}^{\infty} a_n \int_0^x t^n\,dt$$

$$\frac{1}{3}x^3 + \frac{1}{30}x^5 + \frac{1}{840}x^7 + \cdots = \sum_{n=0}^{\infty} a_n \left.\frac{t^{n+1}}{n+1}\right|_0^x$$

Date Day M T W T F S

$$\frac{1}{3}x^3 + \frac{1}{30}x^5 + \frac{1}{840}x^7 + \cdots = \sum_{n=0}^{\infty} a_n \left[\frac{x^{n+1}}{n+1}\right]$$

Expanding R.H.S of above equation

$$\frac{1}{3}x^3 + \frac{1}{30}x^5 + \frac{1}{840}x^7 + \cdots = a_0(x) + a_1\left(\frac{x^2}{2}\right)$$

$$+ a_2\left(\frac{x^3}{3}\right) + a_3\left(\frac{x^4}{4}\right) + a_5\left(\frac{x^5}{5}\right) + a_5\left(\frac{x^6}{6}\right) + \cdots$$

Comparing the coefficients of like power x, as

x^1; $a_0 = 0$

x^2; $a_1 = 0$

x^3; $\frac{1}{3} = \frac{a_2}{3} \Rightarrow a_2 = 1$

x^4; $a_3 = 0$

x^5; $\frac{1}{30} = \frac{a_4}{5} \Rightarrow a_4 = \frac{1}{6}$

x^6; $a_5 = 0$, $a_6 = \frac{1}{120}$ and so on

Thus the exact sol is given by:

(2) $\Rightarrow$ $u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

$$= 0 + 0 + x^2 + 0 + \frac{x^4}{6} + 0 + \frac{x^6}{120} + \cdots$$

$$= x^2 + \frac{x^4}{3!} + \frac{x^6}{5!} + \cdots$$

$$= x\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$\therefore$ $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ above equation becomes;

$$\boxed{u(x) = x\sinh x}$$

Que # 3

$$1 + e^x x - e^x = \int_0^x t\,u(t)\,dt \quad \text{—— (1)}$$

Using Taylor Series of e^x in eq (1) and put;

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{—— (2)}$$

$$\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$(1) \Rightarrow 1 + x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = \int_0^x \sum_{n=0}^{\infty} a_n t^{n+1}\,dt$$

$$\cancel{1} + \cancel{x} + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} \cancel{-1} \cancel{-x} - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} a_n \left|\frac{t^{n+2}}{n+2}\right|_0^x$$

Date Day M T W T F S

$$\frac{x^2}{2} + \frac{3x^3}{6} + \frac{3x^4}{12} + \cdots = \sum_{n=0}^{\infty} a_n \left(\frac{x^{n+2}}{n+2}\right)$$

$$\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \cdots = a_0\left(\frac{x^2}{2}\right) + a_1\left(\frac{x^3}{3}\right)$$

$$+ a_2\left(\frac{x^4}{4}\right) + \cdots$$

Comparing cofficient of like power

x as;

x^2; $a_0 = 1$

x^3; $\frac{a_1}{3} = \frac{1}{3} \Rightarrow a_1 = 1$

x^4; $\frac{a_2}{4} = \frac{1}{8} \Rightarrow a_2 = \frac{1}{2}$

(2) $\Rightarrow u(x) = a_0 + a_1 x + a_2 x^2$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$\therefore$ IN.K.T $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ so

above equation becomes

Hence

$$\boxed{u(x) = e^x}$$

Date Day M T W T F S

Que # 4

$$1+\frac{1}{3}x^3+xe^x-e^x=\int_0^x t\,u(t)\,dt \quad \text{---} (1)$$

Using Taylor series of e^x in (1) and put

$$u(x)=\sum_{n=0}^{\infty} a_n x^n \quad \text{---} (2)$$

$$\therefore e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\cdots$$

(1) $\Rightarrow$

$$1+\frac{x^3}{3}+x\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\cdots\right)$$

$$-\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\cdots\right)=\sum_{n=0}^{\infty}a_n\int_0^x t^{n+1}dt$$

$$\cancel{1}+\frac{x^3}{3}+\cancel{x}+x^2+\frac{x^3}{2}+\frac{x^4}{6}+\frac{x^5}{24}+\frac{x^6}{120}-\cancel{1}-\cancel{x}$$

$$-\frac{x^2}{2}-\frac{x^3}{6}-\frac{x^4}{24}-\frac{x^5}{120}+\cdots=\sum_{n=0}^{\infty}a_n\left(\frac{t^{n+2}}{n+2}\right)\Bigg|_0^x$$

$$x^2-\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^3}{2}-\frac{x^3}{6}+\frac{x^4}{6}-\frac{x^4}{24}+\frac{x^5}{24}-\frac{x^5}{120}$$

$$+\cdots = \sum_{n=0}^{\infty}a_n\frac{x^{n+2}}{n+2}$$

Date Day M T W T F S

$$\frac{x^2}{2} + \frac{2}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{30}x^5 + \cdots = a_0\frac{x^2}{2} + a_1\frac{x^3}{3}$$

$$+ a_2\frac{x^4}{4} + a_3\frac{x^5}{5} + \cdots$$

Comparing cofficient of like power x as;

x^2: $\frac{a_0}{2} = \frac{1}{2} \Rightarrow a_0 = 1$

x^3: $\frac{a_1}{3} = \frac{2}{3} \Rightarrow a_1 = 2$

x^4: $\frac{a_2}{4} = \frac{1}{8} \Rightarrow a_2 = \frac{1}{2}$

x^5: $\frac{a_3}{5} = \frac{1}{30} \Rightarrow a_3 = \frac{1}{6}$

equation (2)

$$\Rightarrow \quad u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

$$= 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$

$$= 1 + x + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$= x + \left\{1 + x + x^2 + \frac{x^3}{6} + \cdots\right)$$

$$\boxed{u(x) = x + e^x}$$

Date Day M T W T F S

QUES# 5

$$-1-x+\frac{1}{6}x^3+e^x=\int_0^x (x-t)u(t)\,dt \quad \text{—— (1)}$$

Using Taylor series of e^x in (1) and put $u(x)=\sum_{n=0}^{\infty} a_n x^n$ —— (2)

$$\therefore e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+---$$

eq (1) becomes

$$-1-x+\frac{1}{6}x^3+\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+---\right)$$

$$=\sum_{n=0}^{\infty} a_n \int_0^x (x-t)t^n\,dt$$

$$\cancel{-1}\cancel{-x}+\frac{1}{6}x^3+\cancel{1}+\cancel{x}+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+---=$$

$$\sum_{n=0}^{\infty} a_n \int_0^x (xt^n-t^{n+1})\,dt$$

$$\frac{1}{6}x^3+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+---=\sum_{n=0}^{\infty} a_n\left(\int_0^x xt^n\,dt - \int_0^x t^{n+1}\,dt\right)$$

$$\left(\frac{1}{6}+\frac{1}{6}\right)x^3+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^5}{120}+---=\sum_{n=0}^{\infty} a_n\left(x\frac{t^{n+1}}{n+1}\Big|_0^x - \frac{t^{n+2}}{n+2}\Big|_0^x\right)$$

Date Day M T W T F S

$$\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{24}+\frac{x^5}{120}+\cdots=\sum_{n=0}^{\infty}a_n\left(\frac{x^{n+2}}{n+1}-\frac{x^{n+2}}{n+2}\right)$$

$$\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{24}+\frac{x^5}{120}+\cdots=\sum_{n=0}^{\infty}a_n\left[\frac{(n+2)x^{n+2}-(n+1)x^{n+2}}{(n+1)(n+2)}\right]$$

$$\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{24}+\frac{x^5}{120}+\cdots=\sum_{n=0}^{\infty}a_n\left[\frac{x^{n+2}}{(n+1)(n+2)}\right]$$

Expanding $\sum_{n=0}^{\infty}\frac{a_n x^{n+2}}{(n+1)(n+2)}$ R.H.S of above equation

$$\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{24}+\frac{x^5}{120}+\cdots=\frac{a_0x^2}{2}+\frac{a_1x^3}{6}+\frac{a_2x^4}{12}$$

$$+\frac{a_3x^5}{20}+\cdots$$

Comparing cofficient of like power x as,

x^2: $\frac{a_0}{2}=\frac{1}{2} \Rightarrow a_0=1$

x^3: $\frac{a_1}{6}=\frac{1}{3} \Rightarrow a_1=2$

x^4: $\frac{a_2}{12}=\frac{1}{24} \Rightarrow a_2=\frac{1}{2}$

x^5: $\frac{a_3}{20}=\frac{1}{120} \Rightarrow a_3=\frac{1}{6}$

(2) $\Rightarrow u(x)=1+2x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots$

$$=1+x+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots$$

$$= x + \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right)$$

$$\because e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$

$$\boxed{u(x) = x + e^x}$$

QUE#6

$$-x + 2\sin x - x\cos x = \int_0^x (x-t)u(t)dt \quad \text{——— (1)}$$

Using Taylor Series of $\sin x$ & $\cos x$ in (1) and put

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{——— (2)}$$

$$\therefore \sin x = \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots\right)$$

$$\cos x = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots\right)$$

equation (1) becomes;

$$-x + 2\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots\right) - x\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots\right) = \sum_{n=0}^{\infty} a_n \int_0^x (x-t)t^n dt$$

$$-\cancel{x} + 2\cancel{x} - \frac{1}{3}x^3 + \frac{1}{60}x^5 - \frac{1}{2520}x^7 - \cancel{x} + \frac{1}{2}x^3 - \frac{1}{24}x^5 + \frac{1}{720}x^7 + \cdots = \sum_{n=0}^{\infty} a_n \left(x\int_0^x t^n dt - \int_0^x t^{n+1} dt\right)$$

Date Day M T W T F S

$$+\left(\frac{1}{2}-\frac{1}{3}\right)x^3+\left(\frac{1}{60}-\frac{1}{24}\right)x^5+\left(\frac{1}{720}-\frac{1}{2520}\right)x^7+\cdots$$

$$=\sum_{n=0}^{\infty} a_n\left(\frac{x^{n+2}}{(n+1)}-\frac{x^{n+2}}{(n+2)}\right)$$

$$\frac{1}{6}x^3+\left(\frac{2-5}{120}\right)x^5+\left(\frac{7-2}{5040}\right)x^7+\cdots$$

$$=\sum_{n=0}^{\infty} a_n\left(\frac{x^{n+2}}{(n+1)(n+2)}\right)$$

Expanding R.H.S

$$\frac{1}{6}x^3+\left(-\frac{1}{40}\right)x^5+\frac{1}{1008}x^7+\cdots=\frac{a_0x^2}{2}$$

$$+\frac{a_1x^3}{6}+\frac{a_2x^4}{12}+\frac{a_3x^5}{20}+\frac{a_4x^6}{30}+\frac{a_5x^7}{42}+\cdots$$

Comparing coefficient of like power x;

x^2: $\quad \frac{a_0}{2}=0 \Rightarrow a_0=0$

x^3: $\quad \frac{1}{6}=\frac{a_1}{6} \Rightarrow a_1=1$

x^4: $\quad \frac{a_2}{12}=0 \Rightarrow a_2=0$

x^5: $\quad \frac{a_3}{20}=\frac{-1}{40} \Rightarrow a_3=-1/2$

x^6: $\quad a_4=0$

x^7: $\quad \frac{a_5}{42}=\frac{1}{1008} \Rightarrow a_5=\frac{1}{24}$

Date Day M T W T F S

equation ②

$$\Rightarrow u(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + \ldots$$

$$= (0)x^0 + (1)x^1 + (0)x^2 + \left(\frac{-1}{2}\right)x^3 + (0)x^4 + \frac{1}{24}x^5 + \ldots$$

$$= x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \frac{1}{720}x^7 + \ldots$$

$$= x\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \ldots\right)$$

$$\therefore \cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \ldots$$

$$\boxed{u(x) = x\cos x}$$

Que # 7

$$-1 + \cosh x = \int_0^x (x-t)\,u(t)\,dt \quad \text{①}$$

Using Taylor Series of $\cosh x$ in ① and put

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{②}$$

$$\therefore \cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \ldots$$

equation ① become,

Date Day M T W T F S

$$-1+\left(1+\frac{1}{2}x^2+\frac{1}{24}x^4+\frac{1}{720}x^6+\cdots\right)=\sum_{n=0}^{\infty} a_n\int_0^x (x-t)t^n\,dt \quad (3)$$

$$\sum_{n=0}^{\infty} a_n\int_0^x (x-t)t^n\,dt = \sum_{n=0}^{\infty} a_n\int_0^x xt^n - t^{n+1}\,dt$$

$$= \sum_{n=0}^{\infty} a_n\left(x\int_0^x t^n\,dt - \int_0^x t^{n+1}\,dt\right)$$

$$= \sum_{n=0}^{\infty} a_n\left(x\cdot\frac{t^{n+1}}{n+1}\Big|_0^x - \frac{t^{n+2}}{n+2}\Big|_0^x\right)$$

$$= \sum_{n=0}^{\infty} a_n\left(\frac{x^{n+2}}{(n+1)} - \frac{x^{n+2}}{n+2}\right)$$

$$= \sum_{n=0}^{\infty} a_n\left(\frac{x^{n+2}}{(n+1)(n+2)}\right)$$

equation (3) becomes,

$$-\cancel{1}+\cancel{1}+\frac{1}{2}x^2+\frac{1}{24}x^4+\frac{1}{720}x^6+\cdots=\sum_{n=0}^{\infty} a_n\frac{x^{n+2}}{(n+1)(n+2)}$$

Expanding R.H.S

$$\frac{1}{2}x^2+\frac{1}{24}x^4+\frac{1}{720}x^6+\cdots = \frac{a_0}{2}x^2+\frac{a_1}{6}x^3+$$

$$\frac{a_2}{12}x^4+\frac{a_3}{20}x^5+\frac{a_4}{30}x^6+\cdots$$

Comparing coefficients of like power x as;

$$x^2: \quad \frac{a_0}{2}=\frac{1}{2} \Rightarrow a_0=1$$

Date | Day M T W T F S

x^3: $\frac{a_1}{6} = 0 \Rightarrow a_1 = 0$

x^4: $\frac{a_2}{12} = \frac{1}{24} \Rightarrow a_2 = \frac{1}{2}$

x^5: $\frac{a_3}{20} = 0 \Rightarrow a_3 = 0$

x^6: $\frac{a_4}{30} = \frac{1}{720} \Rightarrow a_4 = \frac{1}{24}$

equation (2)

$$\Rightarrow u(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

$$= (1)(1) + (0) + \frac{1}{2}x^2 + 0 + \frac{1}{24}x^4 + \cdots$$

$$= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots$$

$$\therefore \cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots$$

$$\boxed{u(x) = \cosh x}$$

Que # 8

$$x - 2\sinh x + x\cosh x = \int_0^x (x-t)u(t)\,dt \quad \text{—— (1)}$$

Using Taylor series of $\sinh x$, $\cosh x$ in (1) and put

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{—— (2)}$$

$\therefore$

$$\sinh x = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7$$

$$\cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \cdots$$

equation ① becomes,

$$x - 2\left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \cdots\right) + x\left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \cdots\right) = \sum_{n=0}^{\infty} a_n \int_0^x (x-t)t^n\,dt$$

$$\cancel{x} - 2\cancel{x} - \frac{1}{3}x^3 - \frac{1}{60}x^5 - \frac{1}{2520}x^7 + \cancel{x} + \frac{1}{2}x^3 + \frac{1}{24}x^5 + \frac{1}{720}x^7 + \cdots = \sum_{n=0}^{\infty} a_n\left(x\int_0^x t^n\,dt - \int_0^x t^{n+1}\,dt\right)$$

$$\left(\frac{1}{2} - \frac{1}{3}\right)x^3 + \left(\frac{1}{24} - \frac{1}{60}\right)x^5 + \left(\frac{1}{720} - \frac{1}{2520}\right)x^7 + \cdots$$

$$= \sum_{n=0}^{\infty} a_n\left(x\cdot\frac{t^{n+1}}{n+1}\Big|_0^x + \frac{t^{n+2}}{n+2}\Big|_0^x\right)$$

$$\frac{1}{6}x^3 + \left(\frac{5-2}{120}\right)x^5 + \left(\frac{7-2}{5040}\right)x^7 + \cdots =$$

$$\sum_{n=0}^{\infty} a_n\left(\frac{x^{n+2}}{(n+1)} - \frac{x^{n+2}}{(n+2)}\right)$$

$$\frac{1}{6}x^3 + \frac{1}{40}x^5 + \frac{1}{1008}x^7 + \cdots = \sum_{n=0}^{\infty} a_n\left(\frac{x^{n+2}}{(n+1)(n+2)}\right)$$

Date ______ Day M T W T F S

Expanding R.H.S of above equation

$$\frac{1}{6}x^3 + \frac{1}{40}x^5 + \frac{1}{1008}x^7 + \cdots = \frac{a_0}{2}x^2 + \frac{a_1}{6}x^3 + \frac{a_2}{12}x^4$$

$$+ \frac{a_3}{20}x^5 + \frac{a_4}{30}x^6 + \frac{a_5}{42}x^7 \cdots$$

Comparing coefficients of like power x as,

x^2: $\frac{a_0}{2} = 0 \Rightarrow a_0 = 0$

x^3: $\frac{a_1}{6} = \frac{1}{6} \Rightarrow a_1 = 1$

x^4: $\frac{a_2}{12} = 0 \Rightarrow a_2 = 0$

x^5: $\frac{a_3}{20} = \frac{1}{40} \Rightarrow a_3 = \frac{1}{2}$

x^6: $\frac{a_4}{30} = 0 \Rightarrow a_4 = 0$

x^7: $\frac{a_5}{42} = \frac{1}{1008} \Rightarrow a_5 = \frac{1}{24}$ and so on.

Equation ②

$$\Rightarrow u(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$$= (0) + (1)x + (0) + \frac{1}{2}x^3 + 0 + \frac{1}{24}x^5 + \cdots$$

$$u(x) = x + \frac{1}{2}x^3 + \frac{1}{24}x^5 + \cdots$$

$$u(x) = x\left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \cdots\right)$$

Date Day M T W T F S

$$\therefore \cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \cdots$$

i.e $\boxed{u(x) = x\cosh x}$

Que # 9

$$1 + x - \sin x - \cos x = \int_0^x (x-t)\,u(t)\,dt \quad \text{--- (1)}$$

Using Taylor Series of $\sin x$ and $\cos x$ in (1) and put

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{--- (2)}$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

(1) becomes

$$1 + x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) = \sum_{n=0}^{\infty} a_n \int_0^x (x-t)t^n\,dt$$

$$\cancel{1} + \cancel{x} - \cancel{x} + \frac{x^3}{6} - \frac{x^5}{120} + \frac{x^7}{5040} - \cancel{1} + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} + \cdots$$

$$= \sum_{n=0}^{\infty} a_n \int_0^x \left(xt^n - t^{n+1}\right) dt$$

Date　　　　　　　　　　　　　　　　　　　　　　　Day M T W T F S

$$\frac{x^3}{6} - \frac{x^5}{120} + \frac{x^7}{5040} + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} + \cdots = \sum_{n=0}^{\infty} a_n \left(x \cdot \frac{t^{n+1}}{n+1}\Big|_0^x - \frac{t^{n+2}}{n+2}\Big|_0^x \right)$$

$$\frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \cdots = \sum_{n=0}^{\infty} a_n \left(\frac{x^{n+2}}{(n+1)(n+2)} \right)$$

Expanding R.H.S

$$\frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \cdots = \frac{a_0}{2} x^2 + \frac{a_1}{6} x^3 + \frac{a_2}{12} x^4 + \frac{a_3}{20} x^5 + \frac{a_4}{30} x^6 + \frac{a_5}{42} x^7 + \cdots$$

Comparing coefficients of like power x as,

x^2: $\frac{a_0}{2} = \frac{1}{2} \Rightarrow a_0 = 1$

x^3: $\frac{a_1}{6} = \frac{1}{6} \Rightarrow a_1 = 1$

x^4: $\frac{a_2}{12} = \frac{-1}{24} \Rightarrow a_2 = \frac{-1}{2}$

x^5: $\frac{a_3}{20} = \frac{-1}{120} \Rightarrow a_3 = \frac{-1}{6}$

x^6: $\frac{a_4}{30} = \frac{1}{720} \Rightarrow a_4 = \frac{1}{24}$

Date Day M T W T F S

x^7: $\frac{a_5}{42} = \frac{1}{5040}$ $\Rightarrow a_5 = \frac{1}{120}$

equation ②

$\Rightarrow$ $u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a^5 x^5 + \cdots$

$= 1 + 1x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots$

$= 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$

$= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right)$

$$\boxed{u(x) = \cos x + \sin x}$$

Que# 10

$$1 - x - e^{-x} = \int_0^x (x-t)u(t)\,dt \quad \text{—— ①}$$

Using Taylor Series of e^{-x} in ①

and put

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{—— ②}$$

$\because e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \cdots$

equation ① becomes;

Date

Day M T W T F S

$$1-x-\left(1-x+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{120}x^5+\cdots\right)$$

$$=\sum_{n=0}^{\infty} a_n \int_0^x (x-t)t^n\,dt$$

$$\cancel{1}-\cancel{x}-\cancel{1}+\cancel{x}-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{1}{120}x^5+\cdots$$

$$=\sum_{n=0}^{\infty} a_n\left(\frac{x\cdot t^{n+1}}{n+1}\Big|_0^x-\frac{t^{n+2}}{n+2}\Big|_0^x\right)$$

$$-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{1}{120}x^5+\cdots=\sum_{n=0}^{\infty} a_n\left(\frac{x^{n+2}}{(n+1)(n+2)}\right)$$

Expanding R.H.S of above equation

$$-\frac{x^2}{2}+\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{1}{120}x^5+\cdots=\frac{a_0x^2}{2}+\frac{a_1x^3}{6}$$

$$+\frac{a_2}{12}x^4+\frac{a_3}{20}x^5+\frac{a_4}{30}x^6+\frac{a_5}{42}x^7+\cdots$$

Comparing coefficients of like powers x as;

x^2: $\frac{a_0}{2}=\frac{-1}{2} \Rightarrow a_0=-1$

x^3: $\frac{a_1}{6}=\frac{1}{6} \Rightarrow a_1=1$

x^4: $\frac{a_2}{12}=\frac{-1}{24} \Rightarrow a_2=\frac{-1}{2}$

x^5: $\frac{a_3}{20}=\frac{1}{120} \Rightarrow a_3=\frac{1}{6}$

Date Day M T W T F S

x^6: $\frac{a_4}{30} = -\frac{1}{720} \Rightarrow a_4 = -\frac{1}{24}$

x^7: $\frac{a_5}{42} = \frac{1}{5040} \Rightarrow a_5 = \frac{1}{120}$

and so on

equation ②

$$\Rightarrow u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$$= -1 + x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots$$

$$= -\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{120}x^5 + \cdots\right)$$

i.e $\boxed{u(x) = -e^{-x}}$

Que # 11

$$-x + \frac{1}{2}x^2 + \ln(1+x) + x\ln(1+x) = \int_0^x u(t)\,dt \quad \text{— ①}$$

Using Taylor Series of $\ln(1+x)$ in ① and put

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{— ②}$$

$$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

equ ① becomes;

$$-x+\frac{x^2}{2}+\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)+x\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)=\sum_{n=0}^{\infty}a_n\int_0^x t^n\,dt$$

$$\cancel{-x}+\cancel{\frac{x^2}{2}}+\cancel{x}-\cancel{\frac{x^2}{2}}+\frac{x^3}{3}-\frac{x^4}{4}+x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}+\cdots$$

$$=\sum_{n=0}^{\infty}a_n\left(\frac{t^{n+1}}{n+1}\right|_0^x$$

$$x^2+\left(\frac{1}{3}-\frac{1}{2}\right)x^3+\left(\frac{1}{3}-\frac{1}{4}\right)x^4-\frac{x^5}{20}+\cdots$$

$$=\sum_{n=0}^{\infty}a_n\frac{x^{n+1}}{n+1}$$

Expanding R.H.S of above equ

$$x^2-\frac{1}{6}x^3+\frac{1}{12}x^4-\frac{1}{20}x^5+\cdots=a_0x+a_1\frac{x^2}{2}+a_2\frac{x^3}{3}+a_3\left(\frac{x^4}{4}\right)+a_4\frac{x^5}{5}+\cdots$$

Comparing cofficients of like power x as;

x: $a_0=0$

x^2: $\frac{a_1}{2}=1 \Rightarrow a_1=2$

x^3: $\frac{a_2}{3}=-\frac{1}{6} \Rightarrow a_2=\frac{-1}{2}$

Date

x^4: $\frac{a_3}{4} = \frac{1}{12} \Rightarrow a_3 = \frac{1}{3}$

x^5: $\frac{a_4}{5} = \frac{-1}{20} \Rightarrow a_4 = \frac{-1}{4}$

and so on

equation ②

$$\Rightarrow u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= 0 + 2x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

$$= x + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

$$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\boxed{u(x) = x + \ln(1+x)}$$

QUE # 12.

$$\frac{1}{2}x^2 e^x = \int_0^x e^{x-t} u(t)\,dt \quad \text{——— ①}$$

Using the Taylor Series of exponential on both sides of ① and put

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{——— ②}$$

$$\therefore e^x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)$$

Date Day M T W T F S

$$\frac{1}{2}x^2\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right) = \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)$$

$$\int_0^x \left(1-t+\frac{t^2}{2}-\frac{t^3}{6}+\cdots\right)\sum_{n=0}^{\infty} a_n t^n dt$$

$$\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{1}{4}x^4+\frac{1}{12}x^5+\cdots = \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)$$

$$\sum_{n=0}^{\infty} a_n \int_0^x \left(t^n - t^{n+1} + \frac{t^{n+2}}{2} - \frac{t^{n+3}}{(n+3)}+\cdots\right)dt$$

$$\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{1}{4}x^4+\frac{1}{12}x^5+\cdots = \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)$$

$$\sum_{n=0}^{\infty} a_n \left[\frac{t^{n+1}}{(n+1)} - \frac{t^{n+2}}{(n+2)} + \frac{t^{n+3}}{2(n+3)}+\cdots\right]_0^x$$

$$\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{1}{4}x^4+\frac{1}{12}x^5+\cdots = \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)$$

$$\sum_{n=0}^{\infty} a_n \left[\frac{x^{n+1}}{(n+1)} - \frac{x^{n+2}}{(n+2)} + \frac{x^{n+3}}{2(n+3)}\cdots\right]$$

Expanding R.H.S of above equation

$$\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{1}{4}x^4+\frac{1}{12}x^5+\cdots = \left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)$$

$$\left[a_0\left(x-\frac{x^2}{2}+\frac{x^3}{6}\cdots\right)+a_1\left(\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{8}\cdots\right)\right.$$

$$\left.+a_2\left(\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{10}\cdots\right)+a_3\left(\frac{x^4}{4}-\frac{x^5}{5}+\frac{x^6}{12}\cdots\right)\cdots\right]$$

Date Day M T W T F S

Comparing coefficients of like power

x as;

x : $a_0 = 0$

x^2 : $\frac{a_1}{2} = \frac{1}{2} \Rightarrow a_1 = 1$ because $a_0 = 0$

So we not take as

x^3 : $\frac{a_1}{2} - \frac{a_1}{3} + \frac{a_2}{3} = \frac{1}{2}$

$\frac{1}{2} - \frac{1}{3} + \frac{a_2}{3} \Rightarrow \frac{a_2}{3} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

$a_2 = 1$

x^4 : $\frac{a_1}{4} - \frac{a_1}{3} + \frac{a_1}{8} + \frac{a_2}{3} - \frac{a_2}{4} + \frac{a_3}{4} = \frac{1}{4}$

$\frac{1}{4} - \frac{1}{3} + \frac{1}{8} + \frac{1}{3} - \frac{1}{4} + \frac{a_3}{4} = \frac{1}{4}$

$\frac{a_3}{4} = \frac{1}{4} - \frac{1}{8}$

$\frac{a_3}{4} = \frac{1}{8}$

$\boxed{a_3 = \frac{1}{2}}$ and so on

equation ② becomes,

② $\Rightarrow$ $u(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$

$= 0 + x + x^2 + \frac{1}{2}x^3 + \cdots$

$= x\left(1 + x + \frac{1}{2}x^2 + \cdots\right)$

$\boxed{u(x) = xe^x}$ exact Solution