INTRODUCTION

The Voltera integral equations were introduced by "Vito Volterra" and then studied by "Traian Lalescu" in his 1908 thesis, Sur les equations de Volterra, written under the direction of "Emile Picard". In 1911, Lalescu wrote the first book ever on integral equations.

Voltera integral equations find application in "demography", the study of "viscoelastic materials", and in "actuarial science" through the "renewal equation"

INTEGRAL EQUATIONS

Definition:—

An integral equation is an equation in which an unknown appears under the integral sign (or under one or more integral signs).

Example:—

$$g(x) = f(x) + \int_a^b k(x,t)\, g(t)\, dt$$

where, $g(x)$ is unknown function and

$f(x)$, $K(x,t)$ are known functions.

TYPES OF INTEGRAL EQUATIONS

There are two types of integral equation:

1. Fredholm Integral Equation
2. Voltera Integral Equation

Here, we are interested in "Voltera Integral Equations".

VOLTERA INTEGRAL EQUATION

Definition:—

In mathematics, "Voltera integral equations" are a special type of integral equation. A linear integral equation of the following form:

$$g(x)\,y(x) = f(x) + \lambda \int_a^x K(x,t)\,y(t)\,dt \quad \text{— (A)}$$

is known as Voltera integral equation of 3rd kind where,

(i) a is constant

(ii) $f(x)$, $g(x)$ and $K(x,t)$ are known functions.

(iii) λ is non-zero Parameter. and $K(x,t)$ is

Kernel.

KINDS OF VOLTERA INTEGRAL EQUATION

There are three kinds of Voltera integral equation name as 1^{st}, 2^{nd} and 3^{rd} kind but our concern is with Voltera integral eq of 2^{nd} kind.

VOLTERA INTEGRAL EQUATION OF 2^{ND} KIND

Definition:—

A linear integral equation of the form by setting $g(x) = 1$ in eq ①

$$y(x) = f(x) + \lambda \int_a^x k(x,t)\, y(t)\, dt \quad \text{—②}$$

is known as voltera integral equation of 2^{nd} kind.

SERIES SOLUTION METHOD FOR 2ND KIND OF VOLTERA INTEGRAL EQUATION

Explanation:-

A function $u(x)$ is analytic if its derivative of all order exists. Such that the Taylor series at any pt b in its domain

$$u(x) = \sum_{K=0}^{\infty} \frac{f^{(K)}(b)}{K!} (x-b)^{K} \quad \text{—— (1)}$$

converges to $f(x)$ in a neighborhood of b.
For simplicity, the generic form of series is

$$u(x) = \sum_{K=0}^{\infty} a_K (x-b)^{K}$$

at $x=0$ can be written as

$$u(x) = \sum_{N=0}^{\infty} a_N x^{N} \quad \text{—— (2)}$$

Now, we will discuss useful method to solve Voltera integral eq which stems mainly from Taylor series for analytic fn.

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Procedure:—

The series solution method for voltera integral eq of 2^{nd} Kind can be obtained from following procedure:

Voltera integral eq is:

$$u(x) = f(x) + \lambda \int_0^x K(x,t)\, u(t)\,dt \quad \text{—— ①}$$

Step: I

substitute $u(x)$ by series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{—— ②}$$

into both side of eq ①. Also, if $f(x)$ include elementary function such as trigonometry, exponential functions etc, then Taylor series for functions involved in $f(x)$ should be used. so,

$$① \Rightarrow \sum_{n=0}^{\infty} a_n x^n = T(f(x)) + \lambda \int_0^x K(x,t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

Step: II

Integrating the right side of the integral.

Step: III

collect the coefficients of like power of x.

Step : IV

Equate the coefficients of like power of x in both sides of the resulting eq to obtain a recurrence relation in $a_j, j \geq 0$.

Step : V

Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geq 0$.

Step : VI

Having determined the coefficients $a_j, j \geq 0$, the series solution follows immediately upon substituting the desired coefficients into eq (2).

Step : VII

The exact solution may be obtained if such an exact solution exist.

Step : VIII

If an exact solution is not obtainable then, obtained series can be used for numerical purposes.

REMARK:—

In this case, evaluating more terms we can achieve higher accuracy level.

Now, here is an example using this method I

Example: consider a Voltera integral eq $u(x) = 1 + \int_0^x u(t)\, dt.$ —— ①

SOL:— **Step-I**

substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{—— ②}$$

into both sides of eq ①

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n \right) dt$$

Step-II

Integrating the terms under integral, we have

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=0}^{\infty} a_n \int_0^x t^n\, dt$$

$$= 1 + \sum_{n=0}^{\infty} a_n \left| \frac{t^{n+1}}{n+1} \right|_0^x$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

Step-III

collect coefficients of like power of x. i.e

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots \quad (3)$$

Step-IV Comparing the coefficients of like power of x in both side of eq (3) gives the recurrence relation as follows

Step-V

$$a_0 = 1$$

$$a_1 = a_0 \Rightarrow a_1 = 1$$

$$a_2 = \frac{a_1}{2} \Rightarrow a_2 = \frac{1}{2} = \frac{1}{2!}$$

$$a_3 = \frac{a_2}{3} \Rightarrow a_3 = \frac{1}{6} = \frac{1}{3!}$$

$$\vdots$$

$$a_n = \frac{1}{n!} \quad ; \quad n \geq 0$$

Step-VI substitute this result in eq (2) gives the series solution. i.e

$$u(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Step-VII It converges to exact solution i.e

$$u(x) = e^x$$

Here, is another method discussed by an example in which there is no need to find n^{th} term for series solution.

(METHOD - II)

Exp# solve Volterra Integral eq by using series sol method.

$$u(x) = 1 - x\sin x + \int_0^x t\,u(t)\,dt \quad \text{—— ①}$$

SOL:— Since, Taylor series for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{—— (i)}$$

put geometric series in eq ① on both sides

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{—— (ii)}$$

$$① \Rightarrow \sum_{n=0}^{\infty} a_n x^n = 1 - x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) + \int_0^x t\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

on expanding, we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots =$$
$$1 - \frac{x^2}{2!} + \frac{x^4}{3!} - \frac{x^6}{6!} + \cdots + \int_0^x t(a_0 + a_1 t + a_2 t^2 + \cdots)\,dt \quad \text{—— ②}$$

by integrating and equating like powers of x

$$= 1 + \left[a_0 \frac{t^2}{2} + a_1 \frac{t^3}{3} + a_2 \frac{t^4}{4} + \cdots \right]_0^x$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \left(1 - x^2 + \frac{x^4}{3!} - \frac{x^6}{5!} + \cdots\right) + \left(a_0 \frac{x^2}{2} + a_1 \frac{x^3}{3} + a_2 \frac{x^4}{4} + \cdots\right)$$

$$\Rightarrow \quad 1+\left(\frac{a_0}{2}-1\right)x^2 + a_1\frac{x^3}{3} + \left(\frac{a^2}{4}+\frac{1}{6}\right)x^4 + \cdots$$

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = 1+\left(\frac{a_0}{2}-1\right)x^2 + a_1\frac{x^3}{3} + \left(\frac{a^2}{4}+\frac{1}{6}\right)x^4 + \cdots$$

Now, equating coefficients of like Power of x

x^0: $\quad a_0 = 1$

x: $\quad a_1 = 0$

x^2: $\quad a_2 = \frac{a_0}{2} - 1 \Rightarrow a_2 = \frac{-1}{2!}$

x^3: $\quad a_3 = \frac{a_1}{3!} \Rightarrow a_3 = 0$

x^4: $\quad a_4 = \frac{a_2}{4} + \frac{1}{6} \Rightarrow a_4 = \frac{1}{24}$

$\vdots$

put all values in eq (ii)

$$u(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

therefore, required sol is

$$u(x) = \cos x$$

Exp# Solve Voltera Integral Equation by using series solution method.

$$u(x) = 2e^{x} - 2 - x + \int_0^x (x-t)u(t)dt \quad \text{—— (1)}$$

SOL:— Since, Taylor series expansion for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{—— (i)}$$

substitute $u(x)$ by series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{—— (ii)}$$

put (i) and (ii) in eq (1)

$$\sum_{n=0}^{\infty} a_n x^n = 2\left(x + 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - 2 - x + \int_0^x (x-t)\sum_{n=0}^{\infty} a_n t^n dt$$

Since, series is convergent so,

$$= 2x + \cancel{2} + x^2 + \frac{x^3}{3} + \cdots - \cancel{2} - x + \sum_{n=0}^{\infty} a_n \int_0^x (x-t)t^n dt$$

$$= x + x^2 + \frac{x^3}{3} + \cdots + \sum_{n=0}^{\infty} a_n \int_0^x (x-t)t^n dt$$

integrating we have

$$= x + x^2 + \frac{x^3}{3} + \cdots + \sum_{n=0}^{\infty} a_n \left| \frac{xt^{n+1}}{n+1} - \frac{t^{n+2}}{n+2} \right|_0^x$$

$$= x + x^2 + \frac{x^3}{3} + \cdots + \sum_{n=0}^{\infty} a_n \left[\frac{x^{n+2}}{n+1} - \frac{x^{n+2}}{n+2} \right]$$

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$$= x + x^2 + \frac{x^3}{3} + \cdots + \sum_{n=0}^{\infty} a_n x^{n+2} \left[\frac{\cancel{n}+2-\cancel{n}-1}{(n+1)(n+2)} \right]$$

$$= x + x^2 + \frac{x^3}{3} + \cdots + \sum_{n=0}^{\infty} a_n \frac{x^{n+2}}{(n+1)(n+2)}$$

by expanding, we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = x + x^2 + \frac{x^3}{3} + \cdots +$$

$$\frac{a_0}{2} x^2 + \frac{a_1}{6} x^3 + \cdots$$

comparing like power of x to get a_i

x^0: $\quad a_0 = 0$

x: $\quad a_1 = 1$

x^2: $\quad a_2 = a_0/2 \Rightarrow a_2 = 0$

x^3: $\quad a_3 = a_1/6 \Rightarrow a_3 = 1/6$

x^4: $\quad a_4 = a_2/12 \Rightarrow 1/24 = a_4$

$\vdots$

$\quad a_n = 0 \quad ; \quad n \geq 2$

put all values in eq ②

$$u(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

that converges to exact sol

$$u(x) = \sinh x$$

Now, we are going to solve some questions through series solution method.

Question : 1

$$u(x) = x + \int_0^x \tan t\, u(t)\, dt \quad \text{——} \quad (1)$$

SOL:— since, Taylor series expansion for $\tan x$ is

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \quad \text{——} \quad (i)$$

put $u(x)$ by series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{——} \quad (ii)$$

put (i) and (ii) in eq (1)

$$\sum_{n=0}^{\infty} a_n x^n = x + \int_0^x \left(t + \frac{t^3}{3} + \frac{2t^5}{15} + \cdots\right)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

$$= x + \sum_{n=0}^{\infty} a_n \int_0^x t^n \left(t + \frac{t^3}{3} + \frac{2t^5}{15} + \cdots\right) dt$$

$$= x + \sum_{n=0}^{\infty} a_n \int_0^x \left[t^{n+1} + \frac{t^{n+3}}{3} + \frac{2t^{n+5}}{15} + \cdots\right] dt$$

by integrating, we have

$$= x + \sum_{n=0}^{\infty} a_n \left[\frac{t^{n+2}}{n+2} + \frac{t^{n+4}}{3(n+4)} + \frac{2t^{n+6}}{15(n+6)} + \cdots \Big|_0^x\right]$$

$$= x + \sum_{n=0}^{\infty} a_n \left[\frac{x^{n+2}}{n+2} + \frac{x^{n+4}}{3(n+4)} + \frac{2x^{n+6}}{15(n+6)} + \cdots\right]$$

$$= x + \sum_{n=0}^{\infty} \frac{a_n}{n+2} x^{n+2} + \sum_{n=0}^{\infty} \frac{a_n}{3(n+4)} x^{n+4} + \frac{2}{15} \sum_{n=0}^{\infty} \frac{a_n}{n+6} x^{n+6} + \cdots$$

Collect like power of 'x'

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6$$

$$= x + \left(\frac{a_0}{2} x^2 + \frac{a_1}{3} x^3 + \frac{a_2}{4} x^4 + \frac{a_3}{5} x^5 + \frac{a_4}{6} x^6 + \cdots \right) +$$

$$\frac{1}{3}\left(\frac{a_0}{4} x^4 + \frac{a_1}{5} x^5 + \frac{a_2}{6} x^6 + \cdots \right) + \frac{2}{15}\left(\frac{a_0}{6} x^6 + \cdots \right)$$

$$= x + \frac{a_0}{2} x^2 + \frac{a_1}{3} x^3 + \left(\frac{a_2}{4} + \frac{a_0}{12} \right) x^4 + \left(\frac{a_3}{5} + \frac{a_1}{15} \right) x^5 +$$

$$+ \left(\frac{a_4}{6} + \frac{a_2}{18} + \frac{a_0}{45} \right) x^6$$

Now, comparing the like power of 'x', we get

$$a_0 = 0$$

$$a_1 = 1$$

$$a_2 = a_0/2 \Rightarrow a_2 = 0$$

$$a_3 = a_1/3 \Rightarrow a_3 = 1/3$$

$$a_4 = a_2/4 + a_0/12 \Rightarrow a_4 = 0$$

$$a_5 = \frac{a_3}{5} + \frac{a_1}{15} \Rightarrow a_5 = 2/15$$

$$a_6 = a_4/6 + a_2/18 + a_0/45 \Rightarrow a_6 = 0$$

$$\vdots \qquad\qquad \vdots$$

So, sol^n in series form is

$$u(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

that converges to exact sol i.e

$$u(x) = \tan x$$

Question: 2

$$u(x) = 1 - \int_0^x u(t)\,dt \quad \text{——— ①}$$

SOL:— put $u(x)$ by series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{——— (i)}$$

on both side of eq ①

$$\sum_{n=0}^{\infty} a_n x^n = 1 - \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

Since, series is convergent so, we can replace Summation by integral.

$$= 1 - \sum_{n=0}^{\infty} a_n \int_0^x t^n\,dt$$

Now, integrating terms under integral

$$= 1 - \sum_{n=0}^{\infty} a_n \left| \frac{t^{n+1}}{n+1} \right|_0^x = 1 - \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

on expanding, we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots = 1 - a_0 x - \frac{a_1}{2} x^2 - \frac{a_2}{3} x^3 - \frac{a_3}{4} x^4 -$$

comparing like powers of 'x' to obtain a_i

$$a_0 = 1$$

$$a_1 = -a_0 \Rightarrow a_1 = -1$$

$$a_2 = -a_1/2 \Rightarrow a_2 = 1/2$$

$a_3 = -a_2/2 \Rightarrow a_3 = -1/6$

$a_4 = -a_3/4 \Rightarrow a_4 = 1/24$

and so on. Then, sol in series form is

$$u(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

So, required sol is

$$u(x) = e^{-x}$$

Question: 3

$$u(x) = 1 - \int_0^x (x-t)\, u(t)\, dt \quad \text{——— } ①$$

SOL:— put $u(x) = \sum_{n=0}^{\infty} a_n x^n$ in eq ①

$$\sum_{n=0}^{\infty} a_n x^n = 1 - \int_0^x (x-t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

since, series is convergent so,

$$= 1 - \sum_{n=0}^{\infty} a_n \int_0^x (x-t)\, t^n\, dt$$

integrating we have

$$= 1 - \sum_{n=0}^{\infty} a_n \left| \frac{x t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2} \right|_0^x .$$

$$= 1 - \sum_{n=0}^{\infty} a_n \left[\frac{x^{n+2}}{n+1} - \frac{x^{n+2}}{n+2} \right]$$

$$= 1 - \sum_{n=0}^{\infty} a_n x^{n+2} \left[\frac{\cancel{n}+2-\cancel{n}-1}{(n+1)(n+2)} \right] = 1 - \sum_{n=0}^{\infty} \frac{a_n x^{n+2}}{(n+1)(n+2)}$$

on expanding, we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$
$$= 1 - \frac{a_0}{2} x^2 - \frac{a_1}{6} x^3 - \frac{a_2}{12} x^4 - \frac{a_3}{20} x^5 - \frac{a_4}{30} x^6 - \cdots$$

comparing like power of x to get a_i

$$a_0 = 1 \quad , \quad a_1 = 0$$

$$a_2 = -a_0/2 \Rightarrow a_2 = -1/2$$

$$a_3 = -a_1/6 \Rightarrow a_3 = 0$$

$$a_4 = -a_2/12 \Rightarrow a_4 = 1/24$$

$$a_5 = -a_3/20 \Rightarrow a_5 = 0$$

$$a_6 = -a_4/30 \Rightarrow a_6 = -1/720$$

$$\vdots$$

put all values in eq ①

$$u(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$$

that converges to exact sol i.e

$$u(x) = \cos x$$

Question: 4

$$u(x) = x + \int_0^x (x-t)\, u(t)\, dt \quad \text{— ①}$$

SOL:— put $u(x) = \sum_{n=0}^{\infty} a_n x^n$ on both side of eq ①

$$\sum_{n=0}^{\infty} a_n x^n = x + \int_0^x (x-t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

since, series is convergent so,

$$= x + \sum_{n=0}^{\infty} a_n \int_0^x (x-t) t^n dt$$

integrating we have

$$= x + \sum_{n=0}^{\infty} a_n \left| \frac{x t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2} \right|_0^x$$

$$\sum_{n=0}^{\infty} a_n x^n = x + \sum_{n=0}^{\infty} a_n \left[\frac{x^{n+2}}{n+1} - \frac{x^{n+2}}{n+2} \right] = x + \sum_{n=0}^{\infty} \frac{a_n x^{n+2}(+1)}{(n+1)(n+2)}$$

on expanding we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \cdots$$

$$= x + \frac{a_0}{2} x^2 + \frac{a_1}{6} x^3 + \frac{a_2}{12} x^4 + \frac{a_3}{20} x^5 + \frac{a_4}{30} x^6 + \frac{a_5}{42} x^7 + \cdots$$

comparing like power of x to get a_i

$$a_0 = 0 \quad , \quad a_1 = 1$$

$$a_2 = a_0/2 \Rightarrow a_2 = 0$$

$$a_3 = a_1/6 \Rightarrow a_3 = 1/6$$

$$a_4 = a_2/12 \Rightarrow a_4 = 0$$

$$a_5 = a_3/20 \Rightarrow a_5 = 1/120$$

$$a_6 = a_4/30 \Rightarrow a_6 = 0$$

$a_7 = a_5/42 \Rightarrow a_7 = 1/5040$

$\vdots$

So, Sol in series form is

$$u(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots$$

that converges to exact sol

$$u(x) = \sinh x$$

Question : 5

$$u(x) = 1 + \frac{x}{2} + \frac{1}{2}\int_0^x (x-t+1)\,u(t)\,dt \quad \text{—— ①}$$

SOL:— substitute $u(x)$ by series in ①

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{—— ②}$$

$$① \Rightarrow \sum_{n=0}^{\infty} a_n x^n = 1 + \frac{x}{2} + \frac{1}{2}\int_0^x (x-t+1)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt$$

Since, series is convergent so,

$$= 1 + \frac{x}{2} + \frac{1}{2}\sum_{n=0}^{\infty} a_n \int_0^x (x-t+1)\,t^n\,dt$$

integrating we have

$$= 1 + \frac{x}{2} + \frac{1}{2}\sum_{n=0}^{\infty} a_n \left| \frac{xt^{n+1}}{n+1} - \frac{t^{n+2}}{n+2} + \frac{t^{n+1}}{n+1} \right|_0^x$$

$$= 1 + \frac{x}{2} + \frac{1}{2}\sum_{n=0}^{\infty} a_n \left[\frac{x^{n+2}}{n+1} - \frac{x^{n+2}}{n+2} + \frac{x^{n+1}}{n+1} \right]$$

$$= 1 + \frac{x}{2} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{a_n x^{n+2}}{(n+1)(n+2)} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

on expanding we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots = 1 + \frac{x}{2} + \frac{a_0 x^2}{4} + \frac{a_1 x^3}{12}$$

$$+ \frac{a_2 x^4}{24} + \cdots + \frac{a_0}{2} x + \frac{a_1 x^2}{4} + \frac{a_2}{6} x^3 + \frac{a_3}{8} x^4 + \cdots$$

comparing like powers of x to get a_i

$$a_0 = 1$$

$$a_1 = \frac{1}{2}(a_0 + 1) \Rightarrow a_1 = 1$$

$$a_2 = \frac{1}{4}(a_0 + a_1) \Rightarrow a_2 = 1/2$$

$$a_3 = \frac{1}{12}(a_1 + 2a_2) \Rightarrow a_3 = 1/6$$

$$a_4 = 1/24(a_2 + 3a_3) \Rightarrow a_4 = 1/24$$

$$\vdots$$

So, series sol is

$$u(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\Rightarrow \quad u(x) = e^x$$

Question: 6

$$u(x) = 1 + xe^x - \int_0^x t\,u(t)\,dt \quad \text{—— (1)}$$

SOL:— since, Taylor series of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{—— (i)}$$

Also, put $u(x) = \sum_{n=0}^{\infty} a_n x^n$ in eq ①

$$\sum_{n=0}^{\infty} a_n x^n = 1 + x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \int_0^x t\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

since, series is convergent so,

$$= 1 + x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \sum_{n=0}^{\infty} a_n \int_0^x t^{n+1} dt$$

on integrating we have

$$= 1 + x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \sum_{n=0}^{\infty} a_n \left| \frac{t^{n+2}}{n+2} \right|_0^x$$

$$= 1 + x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} - \sum_{n=0}^{\infty} a_n \frac{x^{n+2}}{n+2}$$

on expanding we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots = 1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} - \frac{a_0}{2} x^2 - \frac{a_1}{3} x^3 - \frac{a_2}{4} x^4 - \cdots$$

comparing like power of x, to get a_i

$a_0 = 1$ $\qquad a_1 = 1$

$a_2 = 1 - a_0/2 \Rightarrow a_2 = 1/2$

$a_3 = \frac{1}{2} - \frac{a_1}{3} \Rightarrow a_3 = 1/6$

$a_4 = \frac{1}{6} - \frac{a_2}{4} \Rightarrow a_4 = 1/24$

$\vdots$

then, sol in series is

$$u(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

so, $$u(x) = e^x$$

Question : 7

$$u(x) = 1 + 2x + 4\int_0^x (x-t)\,u(t)\,dt \quad \text{—— (1)}$$

SOL:— put $u(x) = \sum_{n=0}^{\infty} a_n x^n$ in (1)

$$\sum_{n=0}^{\infty} a_n x^n = 1 + 2x + 4\int_0^x (x-t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

since, series is convergent so,

$$= 1 + 2x + 4\sum_{n=0}^{\infty} a_n \int_0^x (x-t)\, t^n\, dt$$

integrating we have

$$= 1 + 2x + 4\sum_{n=0}^{\infty} a_n \left| \frac{x t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2} \right|_0^x$$

$$= 1 + 2x + 4\sum_{n=0}^{\infty} a_n \left[\frac{x^{n+2}}{n+1} - \frac{x^{n+2}}{n+2} \right]$$

$$= 1 + 2x + 4\sum_{n=0}^{\infty} \frac{a_n\, x^{n+2}}{(n+1)(n+2)}$$

on expanding we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \cdots = 1 + 2x$$
$$+ 4\frac{a_0}{2} x^2 + \frac{2a_1}{3} x^3 + \frac{a_2}{3} x^4 + \frac{a_3}{5} x^5 + \frac{2a_4}{15} x^6 + \frac{2}{21} x^7 + \cdots$$

Comparing like power of x to get a_i

$a_0 = 1 \qquad a_1 = 2$

$a_2 = 2a_0 \Rightarrow a_2 = 2$

$a_3 = \frac{2a_1}{3} \Rightarrow a_3 = 4/3$

$a_4 = a_2/3 \Rightarrow a_4 = 2/3$

$a_5 = a_3/5 \Rightarrow a_5 = 4/15$

$a_6 = 2a_4/15 \Rightarrow a_6 = 4/45$

$a_7 = 2a_5/21 \Rightarrow a_7 = 8/315$

$\vdots$

Then, sol in series is

$$u(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \frac{4}{45}x^6 + \frac{8}{315}x^7 + \cdots$$

$$= 1 + (2x) + \frac{1}{2}(2x)^2 + \frac{1}{6}(2x)^3 + \frac{1}{24}(2x)^4 + \frac{1}{120}(2x)^5 + \frac{1}{720}(2x)^6 + \cdots$$

$$\Rightarrow u(x) = e^{2x}$$

Question: 8

$$u(x) = 3 + x^2 - \int_0^x (x-t)\, u(t)\, dt \quad \text{——— (1)}$$

SOL:— put $u(x) = \sum_{n=0}^{\infty} a_n x^n$ in (1)

$$\sum_{n=0}^{\infty} a_n x^n = 3 + x^2 - \int_0^x (x-t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

since, series is convergent so,

$$= 3 + x^2 - \sum_{n=0}^{\infty} a_n \int_0^x (x-t)t^n\,dt$$

integrating we having

$$= 3 + x^2 - \sum_{n=0}^{\infty} a_n \left| \frac{xt^{n+1}}{n+1} - \frac{t^{n+2}}{n+2} \right|_0^x$$

$$= 3 + x^2 - \sum_{n=0}^{\infty} a_n \left[\frac{x^{n+2}}{n+1} - \frac{x^{n+2}}{n+2} \right] = 3 + x^2 - \sum_{n=0}^{\infty} \frac{a_n x^{n+2}}{(n+1)(n+2)}$$

on expanding we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots = 3 + x^2$$
$$- \frac{a_0}{2} x^2 - \frac{a_1}{6} x^3 - \frac{a_2}{12} x^4 - \frac{a_3}{20} x^5 - \frac{a_4}{30} x^6 - \cdots$$

comparing like power of x to get a_i

$a_0 = 3$. $\quad a_1 = 0$

$a_2 = 1 - a_0/2 \Rightarrow a_2 = -1/2$

$a_3 = -a_1/6 \Rightarrow a_3 = 0$

$a_4 = -a_2/12 \Rightarrow a_4 = 1/24$

$a_5 = -a_3/20 \Rightarrow a_5 = 0$

$a_6 = -a_4/30 \Rightarrow a_6 = -1/720$

$\vdots$

Then, sol in series form is

$$u(x) = 3 - x^2/2 + x^4/24 - x^6/720 + \cdots$$
$$= 2 + 1 - x^2/2 + x^4/24 - x^6/720 + \cdots$$

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$$= 2 + \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)$$

$$\Rightarrow \quad u(x) = 2 + \cos x$$

Question : 9

$$u(x) = 1 + 2\sin x - \int_0^x u(t)\,dt \quad \text{——} \; ①$$

SOL:— Since, Taylor series for $\sin x$ is

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \quad \text{——} \; (i)$$

put $u(x)$ by series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{——} \; (ii)$$

put (i) and (ii) in eq ①

$$\sum_{n=0}^{\infty} a_n x^n = 1 + 2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right) - \int_0^x \sum_{n=0}^{\infty} a_n t^n\,dt$$

Since, series is convergent so,

$$= 1 + 2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right) - \sum_{n=0}^{\infty} a_n \int_0^x t^n\,dt$$

integrating we have

$$= 1 + 2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right) - \sum_{n=0}^{\infty} a_n \left|\frac{t^{n+1}}{n+1}\right|_0^x$$

$$= 1 + 2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right) - \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$$

on expanding, we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \cdots$$

$$= 1 + 2x - \frac{x^3}{3} + \frac{x^5}{60} - \frac{x^7}{2520} + \cdots - a_0 x - \frac{a_1}{2} x^2 - \frac{a_2}{3} x^3 - \frac{a_4}{4} x^4 - \frac{a_5}{5} x^5$$

$$- \frac{a_6}{6} x^6 - \frac{a_7}{7} x^7 - \cdots$$

comparing like power of 'x' to get a_i

$$a_0 = 1$$

$$a_1 = 2 - a_0 \Rightarrow a_1 = 1$$

$$a_2 = -a_1/2 \Rightarrow a_2 = -1/2$$

$$a_3 = -\frac{1}{3}(a_2 + 1) \Rightarrow a_3 = -1/6$$

$$a_4 = -a_3/4 \Rightarrow a_4 = 1/24$$

$$a_5 = \frac{1}{60}(1 - 12a_4) \Rightarrow a_5 = 1/120$$

$$a_6 = -a_5/6 \Rightarrow a_6 = -1/720$$

$$a_7 = -\frac{1}{2520}(1 + 360\, a_6) \Rightarrow a_7 = -1/5040$$

$$\vdots$$

then, sol in series form is

$$u(x) = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} - \frac{x^6}{720} - \frac{x^7}{5040} + \cdots$$

$$= \left(1 - x^2/2 + x^4/24 - x^6/720 + \cdots\right) + \left(x - x^3/6 + x^5/120 - x^7/5040 + \cdots\right)$$

that converges to exact sol

$$\mathbf{u(x) = \cos x + \sin x}$$

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Question : 10

$$u(x) = x\cos x + \int_0^x t\,u(t)\,dt \quad \text{———} \quad ①$$

SOL:— since, Taylor series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots \quad \text{———} \quad (i)$$

put $u(x)$ by series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{———} \quad (ii)$$

put (i), (ii) in ①

$$\sum_{n=0}^{\infty} a_n x^n = x\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right) + \int_0^x t\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

$$= x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720} + \cdots + \sum_{n=0}^{\infty} a_n \int_0^x t^{n+1}\,dt$$

integrating we have

$$= x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720} + \cdots + \sum_{n=0}^{\infty} a_n \frac{x^{n+2}}{n+2}$$

on expanding we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \cdots = x - x^3/2 +$$

$$\frac{x^5}{24} - \frac{x^7}{720} + \cdots + \frac{a_0}{2} x^2 + \frac{a_1}{3} x^3 + \frac{a_2}{4} x^4 + \frac{a_3}{5} x^5 + \frac{a_4}{6} x^6 + \frac{a_5}{7} x^7 + \cdots$$

comparing like powers of 'x' to get a_i

$$a_0 = 0 \quad , \quad a_1 = 1$$

$$a_2 = a_0/2 \Rightarrow a_2 = 0$$

$$a_3 = -1/2 + a_1/3 \Rightarrow a_3 = -1/6$$

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$a_4 = a_2/4 \Rightarrow a_4 = 0$

$a_5 = 1/24 + a_3/5 \Rightarrow a_5 = 1/120$

$a_6 = a_4/6 \Rightarrow a_6 = 0$

$a_7 = -\frac{1}{720} + \frac{a_5}{7} \Rightarrow a_7 = -1/5040$

$\vdots$

So, Sol in series form is

$$u(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

that converges to exact sol

$$u(x) = \sin x$$

Question: 11

$$x\cosh x - \int_0^x t u(t)\,dt = u(x) \quad \text{—— } ①$$

SOL:— Since, Taylor series expansion for 'coshx is

$$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots \quad \text{—— (i)}$$

put $u(x)$ by series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{—— (ii)}$$

put (i), (ii) in eq ①

$$x\left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots\right) - \int_0^x t\left(\sum_{n=0}^{\infty} a_n t^n\right) dt =$$

$$= x + \frac{x^3}{2} + \frac{x^5}{24} + \frac{x^7}{720} + \cdots - \sum_{n=0}^{\infty} a_n \int_0^x t^{n+1}\,dt$$

integrating we have

$$= x + \frac{x^3}{2} + \frac{x^5}{24} + \frac{x^7}{720} + \cdots - \sum_{n=0}^{\infty} a_n \frac{x^{n+2}}{n+2}$$

on expanding we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \cdots$$

$$= x + \frac{x^3}{2} + \frac{x^5}{24} + \frac{x^7}{720} + \cdots - \frac{a_0}{2} x^2 - \frac{a_1}{3} x^3 - \frac{a_2}{4} x^4 - \frac{a_3}{5} x^5 - \frac{a_4}{6} x^6 - \frac{a_5}{7} x^7 - \cdots$$

comparing like powers of x to get a_i

$$a_0 = 0 \quad , \quad a_1 = 1$$

$$a_2 = -a_0/2 \Rightarrow a_2 = 0$$

$$a_3 = \frac{1}{2} - \frac{a_1}{3} \Rightarrow a_3 = 1/6$$

$$a_4 = -a_2/4 \Rightarrow a_4 = 0$$

$$a_5 = \frac{1}{24} - \frac{a_3}{3} \Rightarrow a_5 = 1/120$$

$$a_6 = -a_4/6 \Rightarrow a_6 = 0$$

$$a_7 = \frac{1}{720} - \frac{a_5}{7} \Rightarrow a_7 = 1/5040$$

$$\vdots$$

so, then sol in series form is

$$u(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots$$

that converges to exact sol

$$u(x) = \sinh x$$

Question : 12

$$u(x) = 2\cosh x - 2 + \int_0^x (x-t)\, u(t)\, dt \quad \text{--- (1)}$$

SOL:— since, $\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots$ — (2)

put $u(x) = \sum_{n=0}^{\infty} a_n x^n$ in eq (1)

$$\sum_{n=0}^{\infty} a_n x^n = 2\left(1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}+\cdots\right)+\int_0^x (x-t)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt$$

$$= 2+x^2+\frac{x^4}{12}+\frac{x^6}{360}+\cdots+\sum_{n=0}^{\infty} a_n \int_0^x (x-t)t^n dt$$

integrating we have

$$= 2+x^2+\frac{x^4}{12}+\frac{x^6}{360}+\cdots+\sum_{n=0}^{\infty} a_n \left[\frac{t^{n+1}x}{n+1}-\frac{t^{n+2}}{n+2}\Big|_0^x\right]$$

$$= 2+x^2+x^4/12+x^6/360+\cdots+\sum_{n=0}^{\infty} \frac{a_n}{(n+1)(n+2)} x^{n+2}$$

on expanding we have

$$a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+\cdots = 2+x^2+x^4/12$$
$$+x^6/360+\cdots+\frac{a_0}{2}x^2+\frac{a_1}{6}x^3+\frac{a_2}{12}x^4+\frac{a_3}{20}x^5+\frac{a_4}{30}x^6+\cdots$$

Comparing like power of x to get a_i

$a_0 = 0$ $\quad\quad a_1 = 0$

$a_2 = 1 + a_0/2 \Rightarrow a_2 = 1$

$a_3 = a_1/6 \Rightarrow a_3 = 0$

$a_4 = 1/12 + a_2/12 \Rightarrow a_4 = 1/6$

$a_5 = a_3/20 \Rightarrow a_5 = 0$

$a_6 = \frac{1}{360}+\frac{a_4}{30} \Rightarrow a_6 = 1/20$

$\vdots$

then, sol in series form is

$$u(x) = x^2 + x^4/6 + x^6/120 + x^8/5040 + \cdots$$
$$= x(x + x^3/6 + x^5/120 + x^7/5040 + \cdots)$$

that converges to exact sol

$$u(x) = x\sinh x$$

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Question : 13

$$u(x) = 1 - x - \int_0^x (x-t)\,u(t)\,dt \quad \text{—— (1)}$$

SOL:— substitute $u(x)$ by series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{—— (2)}$$

put (2) in eq (1)

$$\sum_{n=0}^{\infty} a_n x^n = 1 - x - \int_0^x (x-t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

$$= 1 - x - \sum_{n=0}^{\infty} a_n \int_0^x (x-t)\,t^n\,dt$$

integrating we have

$$\sum_{n=0}^{\infty} a_n x^n = 1 - x - \sum_{n=0}^{\infty} a_n \left[\frac{x^{n+2}}{n+1} - \frac{x^{n+2}}{n+2}\right] = 1 - x - \sum_{n=0}^{\infty} \frac{a_n x^{n+2}}{(n+1)(n+2)}$$

on expanding we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \cdots = 1 - x$$
$$- \frac{a_0}{2} x^2 - \frac{a_1}{6} x^3 - \frac{a_2}{12} x^4 - \frac{a_3}{20} x^5 - \frac{a_4}{30} x^6 - \frac{a_5}{42} x^7 - \cdots$$

comparing like powers of x to get a_i

$a_0 = 1 \qquad a_1 = -1$

$a_2 = -a_0/2 \Rightarrow a_2 = -1/2$

$a_3 = -a_1/6 \Rightarrow a_3 = 1/6$

$a_4 = -a_2/12 \Rightarrow a_4 = 1/24$

$a_5 = -a_3/20 \Rightarrow a_5 = -1/120$

$a_6 = -a_4/30 \Rightarrow a_6 = -1/720$

$a_7 = -a_5/42 \Rightarrow a_7 = 1/5040$

$\vdots$

put all values in eq ②

$$u(x) = 1 - x - x^2/2 + x^3/6 + x^4/24 - x^5/120 - x^6/720 + x^7/5040 + \cdots$$

$$= \left(1 - x^2/2 + x^4/24 - x^6/720 + \cdots\right) - \left(x - x^3/6 + x^5/120 - x^7/5040 + \cdots\right)$$

$$\Rightarrow \quad u(x) = \cos x - \sin x$$

Question: 14

$$u(x) = x - x\ln(1+x) + \int_0^x u(t)\,dt \quad \text{—— ①}$$

SOL:— since, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$

put $u(x) = \sum_{n=0}^{\infty} a_n x^n$ —— ② in ①

$$\sum_{n=0}^{\infty} a_n x_n = x - x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right) + \int_0^x \sum_{n=0}^{\infty} a_n t^n\,dt$$

$$= x - x^2 + \frac{x^3}{2} - \frac{x^4}{3} - \frac{x^5}{4} - \frac{x^6}{5} - \cdots + \sum_{n=0}^{\infty} a_n \int_0^x t^n\,dt$$

integrating we have

$$= x - x^2 + \frac{x^3}{2} - \frac{x^4}{3} - \frac{x^5}{4} - \frac{x^6}{5} - \cdots + \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$$

on expanding we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots = x - x^2 + x^3/2 - x^4/3$$
$$- x^5/4 - x^6/5 - \cdots + a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \frac{a_4}{5}x^5 + \frac{a_5}{6}x^6 + \cdots$$

comparing like powers of x, to get a_i

$a_0 = 0 \qquad a_1 = 1 + a_0 \Rightarrow a_1 = 1$

$a_2 = -1 + a_1/2 \Rightarrow a_2 = -1/2$

$a_3 = -1/2 + a_2/3 \Rightarrow a_3 = 1/3$

$a_4 = -1/3 + a_3/4 \Rightarrow a_4 = -1/4$

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$a_5 = 1/4 + a_4/5 \Rightarrow a_5 = 1/5$

$a_6 = -1/5 + a_5/6 \Rightarrow a_6 = -1/6$

$\vdots$

put all value in eq ②

$$u(x) = x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - x^6/6 + \cdots$$

$\Rightarrow$ $u(x) = \ln(1+x)$ required sol

Question: 15

$$u(x) = x^2 - x^3/2 + x^3 \ln(1+x) - \int_0^x 2x u(t)\,dt \quad — ①$$

SOL:— $\because \ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$

put $u(x) = \sum_{n=0}^{\infty} a_n x^n$ in eq ①

$$\sum_{n=0}^{\infty} a_n x^n = x^2 - x^3/2 + x^3(x - x^2/2 + x^3/3 - x^4/4 + \cdots) - 2\int_0^x x \sum_{n=0}^{\infty} a_n t^n\,dt$$

$$= x^2 - x^3/2 + x^4 - x^5/2 + x^6/3 - x^7/4 + \cdots - 2\sum_{n=0}^{\infty} a_n \int_0^x t^n\,dt$$

integrating we have

$$= x^2 - x^3/2 + x^4 - x^5/2 + x^6/3 - x^7/4 + \cdots - 2\sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$$

by expanding, we have

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + \cdots = x^2 - x^3/2 + x^4 - x^5/2 + x^6/3 - x^7/4 + \cdots - 2a_0 x - a_1 x^2 - \frac{2}{3}a_2 x^3 - \frac{a_3}{2}x^4 - \frac{2}{5}a_4 x^5 - \frac{a_5}{3}x^6 - \frac{2}{7}a_6 x^7 - \cdots$$

Comparing like powers of 'x' to get a_i

$a_0 = 0$ $\qquad a_1 = 0$

$a_2 = 1 - 2a_0 \Rightarrow a_2 = 1$

$a_3 = -1/2 - a_1 \Rightarrow a_3 = -1/2$

$a_4 = 1 - 2a_2/3 \Rightarrow a_4 = 1/3$

$a_5 = -1/2 - a_3/2 \Rightarrow a_5 = -1/4$

$a_6 = 1/3 - 2a_4/5 \Rightarrow a_6 = 1/5$

$a_7 = -1/4 - a_5/5 \Rightarrow a_7 = -1/6$

$\vdots$

put all values in eq ②

$$u(x) = x^2 - x^3/2 + x^4/3 - x^5/4 + x^6/5 - x^7/6 + \cdots$$
$$= x(x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - x^6/6 + \cdots)$$

$$u(x) = x \ln(1+x)$$

Question : 16

$$u(x) = \sec x + \tan x - \int_0^x \sec t \, u(t) dt \quad \text{—} ①$$

SOL:— put $u(x) = \sum_{n=0}^{\infty} a_n x^n$ in ①

$$\sum_{n=0}^{\infty} a_n x^n = \sec x + \tan x - \int_0^x \sec t \sum_{n=0}^{\infty} a_n t^n dt$$
$$= \sec x + \tan x - \sum_{n=0}^{\infty} a_n \int_0^x \sec t \, t^n dt$$
$$= \left(1 + x^2/2 + 5x^4/24 + 61x^6/720 + \cdots\right) + \left(x + x^3/3 + 2x^5/15 + \frac{17}{315}x^7 + \cdots\right) - \int_0^x t^n\left(1 + t^2/2 + \frac{5}{24}t^4 + \frac{61}{720}t^6 + \cdots\right) dt \sum_{n=0}^{\infty} a_n$$

integrating we have

$$= \left(1 + x^2/2 + 5x^4/24 + 61x^6/720 + \cdots\right) + \left(x + x^3/3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots\right)$$
$$- \sum_{n=0}^{\infty} a_n \left[\frac{x^{n+1}}{n+1} + \frac{x^{n+3}}{2(n+3)} + \frac{5x^{n+5}}{24(n+5)} + \frac{6x^{n+7}}{720(n+7)} + \cdots\right]$$

expanding terms on left side and right side

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots = 1 + x^2/2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots$$

$$+\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\frac{17}{315}x^7+\cdots\right)-\left(a_0x+\frac{a_1}{2}x^2+\frac{a_2}{3}x^3+\frac{a_3}{4}x^4\right.$$

$$\left.+\frac{a_4}{5}x^5+\frac{a_5}{6}x^6+\cdots\right)-\left(\frac{a_0}{6}x^3+\frac{a_1}{8}x^4+\frac{a_2}{10}x^5+\frac{a_3}{12}x^6+\cdots\right)-$$

$$\left(\frac{a_0}{24}x^5+\frac{5a_1}{144}x^6+\cdots\right)-\left(\frac{6}{7\times720}x^6+\cdots\right)$$

comparing like powers of 'x' to get a_i

$$a_0 = 1$$

$$a_1 = 1-a_0 \Rightarrow a_1 = 0$$

$$a_2 = \frac{1}{2}-\frac{a_1}{2} \Rightarrow a_2 = 1/2$$

$$a_3 = \frac{1}{3}-\frac{a_2}{3}-\frac{a_0}{6} \Rightarrow a_3 = 0$$

$$a_4 = \frac{5}{24}-\frac{a_3}{4}-\frac{a_1}{8} \Rightarrow a_4 = 5/24$$

$$a_5 = \frac{2}{15}-\frac{a_4}{5}-\frac{a_2}{10}-\frac{a_0}{24} \Rightarrow a_5 = 0$$

$$a_6 = \frac{61}{720}-\frac{a_5}{6}-\frac{a_3}{12}-\frac{5a_2}{144} \Rightarrow a_6 = \frac{61}{720}$$

$\vdots$

put all values in eq ②

$$u(x) = 1+\frac{x^2}{2}+\frac{5}{24}x^4+\frac{61}{720}x^6+\cdots$$

that converges to exact sol

$$u(x) = \sec x$$

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