Outline

- The Successive Approximations Method for Volterra Integral Equation of second kind.
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The successive approximations Method for Volterra Integral Eq. of second kind:

The successive approximations method is also called **"Picard Iteration Method"** provides a scheme that can be used for solving initial value problems or integral equations.

In this section we deals with Volterra integral equation of second kind.

A linear Integral equation of the following form:

$$g(x)y(x) = f(x) + \lambda \int_a^x F(x,t)\, y(t)\, dt$$

by setting $g(x) = 1$ in above equation

$$y(x) = f(x) + \lambda \int_a^x F(x,t)\, y(t)\, dt.$$

The successive approximations method of Volterra Integral equation of second kind in much the same way as it has been done Fredholm

Integral equation of second kind.

This method solves any problem by finding successive approximations to the solution by starting with an initial guess, called the "**Zeroth approximation**". The zeroth approximation is any selective real valued function that will be used in a recurrence relation to (reduce) determine the other approximation.

We have given the linear Volterra integral equation of the second kind as

$$u(x) = f(x) + \lambda \int_0^x K(x,t)\, u(t)\, dt \longrightarrow (i)$$

where $u(x)$ is unknown function to be determined and $K(x,t)$ is the kernel and λ is a parameter.

The successive approximations method introduces the recurrence relation as

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t)\, u_{n-1}(t)\, dt \,, \; n \geq 1 \longrightarrow (ii)$$

where the zeroth approximation $u_0(x)$ can be any selective real valued function. We always starts with an initial guess for $u_0(x)$, mostly we select $0, 1, x$ for $u_0(x)$.

Hint: If $f(x) = x$ in given Volterra Integral equation then we choose $u_0(x) = x$ or if $f(x) = 1$, or any other constant then we will choose $u_0(x) = 1$ or also any other constant. Otherwise we take $u_0(x) = 0$ in most cases.

Of course other real values can be selected as well.

Thus, at the limit, the solution $u(x)$ of eq.(i) is obtained as

$$u(x) = \lim_{n\to\infty} u_{n+1}(x)$$

So that,

The resulting fun $u(x)$ is independent of the choice of zeroth approximations. And the question of convergence of $u_n(x)$ is justified by the Theorem:

"If $f(x)$ in eq. (1) is continuous for the interval $0 \leq x \leq a$, and the kernel $K(x,t)$ is also continuous in the triangle $0 \le x \le a$, $0 \le t \le x$, the sequence of successive approximation $u_n(x)$, $n \geq 0$ converges to the solution $u(x)$ of the integral equation."

This process of approximation is extremely simple. However, if we follow the successive approximations method, it needs to set $u_0(x)$ as

$$u_0(x) = (\text{any real valued function})$$

and determine $u_1(x)$ as

$$u_1(x) = f(x) + \lambda \int_a^x K(x,t)\, u_0(t)\, dt$$

$$u_2(x) = f(x) + \lambda \int_a^x K(x,t)\, u_1(t)\, dt$$

and

so on. continuing in the same manner, we have

$$u_n(x) = f(x) + \lambda \int_a^x K(x,t)\, u_{n-1}(t)\, dt$$

$$u_{n+1}(x) = f(x) + \lambda \int_a^x K(x,t)\, u_n(t)\, dt$$

Hence, the successive approximations method gives the exact sln, if it exist by

$$u(x) = \lim_{n \to \infty} u_{n+1}(x).$$

Conclusion:

Hence by above method we can easily findout the unknown functions $u(x)$.

Examples:-

- **Example # 1:**

Solve the Voltera Integral eq. by using the successive approximations method

$$u(x) = 1 - \int_0^x (x-t)u(t)dt.$$

- **Solution:**

We have

$$u(x) = 1 - \int_0^x (x-t)\,u(t)dt. \longrightarrow (i)$$

For zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 1 \longrightarrow (a)$$

By using successive approximations method, we have the iteration formulas as

$$u_{n+1}(x) = 1 - \int_0^x (x-t)\,u_n(t)\,dt \;;\; n \geq 0 \longrightarrow (ii)$$

For $n=0$, (ii) $\Rightarrow$

$$u_n(x) = 1 - \int_0^x (x-t)\,u_0(t)\,dt.$$

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Using (a), we have

$$u_1(x) = 1 - \int_0^x (x-t)(1)\,dt$$

$$= 1 - \int_0^x (x-t)\,dt$$

$$= 1 + \left|\frac{t^2}{2}\right|_0^x - x\,|t|_0^x$$

$$= 1 + \frac{x^2}{2} - x^2$$

$$u_1(x) = 1 - \frac{1}{2!}x^2$$

For $n = 1$, (ii) $\Rightarrow$

$$u_2(x) = 1 - \int_0^x (x-t)\,u_1(t)\,dt$$

(using $u_1(x)$)

$$= 1 - \int_0^x (x-t)\left(1 - \frac{1}{2}t^2\right)dt$$

$$= 1 - \int_0^x \left(x - \frac{x}{2}t^2 - t + \frac{t^3}{2}\right)dt$$

$$u_2(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$$

For $n = 2$, (ii) $\Rightarrow$

$$u_3(x) = 1 - \int_0^x (x-t)\,u_2(t)\,dt$$

$$= 1 - \int_0^x (x-t)\left(1 - \frac{1}{2}t^2 + \frac{1}{4}t^4\right)dt$$

(using $u_2(x)$)

$$u_3(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

For $n = 3$,

(ii) $\Rightarrow$

$$u_4(x) = 1 - \int_0^x (x-t)\, u_3(t)\, dt$$

$$= 1 - \int_0^x (x-t)\left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6\right) dt$$

(using $u_3(x)$)

$$\Rightarrow u_4(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8$$

$\vdots$

Consequently, we obtain

$$u_{n+1}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$$

Thus, the solution of equation (i) is:

$$\boxed{u(x) = \lim_{n\to\infty} u_{n+1}(x) = \cos x.}$$

Answer.

- **Example # 2.**

Solve the Volterra Integral eq. by using the successive approximations method.

$$u(x) = -1 + e^x + \frac{1}{2}x^2 e^x - \frac{1}{2}\int_0^x (t)^3 u(t)\, dt.$$

- **Solution:**

we have

$$u(x) = -1 + e^x + \frac{1}{2}x^2 e^x + \frac{1}{2}\int_0^x (t)^3 u(t)\, dt \rightarrow \text{(i)}$$

For zeroth approximation $u_0(x)$, we select

$$u_0(x) = 0 \longrightarrow (a)$$

By using successive approximations method, we have the iteration formula as

$$u_{n+1}(x) = -1 + e^x + \frac{1}{2}x^2e^x - \frac{1}{2}\int_0^x t\, u_n(t)\,dt \,,\ n \geq 0 \longrightarrow (iii)$$

For $n=0$,

(iii) $\Rightarrow$

$$u_1(x) = -1 + e^x + \frac{1}{2}x^2e^x - \frac{1}{2}\int_0^x t\, u_0(t)\,dt$$

$$= -1 + e^x + \frac{1}{2}x^2e^x - \frac{1}{2}\int_0^x t(0)\,dt$$

(using (a))

$$\Rightarrow u_1(x) = -1 + e^x + \frac{1}{2}x^2e^x.$$

or

$$u_1(x) = -1 + e^x + \frac{1}{2!}x^2e^x$$

For $n=1$, (iii) $\Rightarrow$

$$u_2(x) = -1 + e^x + \frac{1}{2}x^2e^x - \frac{1}{2}\int_0^x t\, u_1(t)\,dt$$

(using $u_1(x)$)

$$= -1 + e^x + \frac{1}{2}x^2e^x - \frac{1}{2}\int_0^x t\left(-1 + e^t + \frac{1}{2!}t^2e^t\right)dt$$

$$= -1 + e^x + \frac{1}{2}x^2e^x - \frac{1}{2}\int_0^x \left(-t + te^t + \frac{1}{2!}t^3e^t\right)dt$$

By solving, we get

$$u_2(x) = -3 + \frac{1}{4}x^2 + e^x\left(3 - 2x + \frac{5}{4}x^2 - \frac{1}{4}x^3\right).$$

For $n = 2$,

(ii) $\Rightarrow$

$$u_3(x) = -1 + e^x + \frac{1}{2}x^2 e^x - \frac{1}{2}\int_0^x t\, u_2(t)\, dt$$

$$= -1 + e^x + \frac{1}{2}x^2 e^x - \frac{1}{2}\int_0^x t\left(-3 + \frac{1}{4}t^2 + e^t\left(3 - 2t + \frac{5}{4}t^2 - \frac{1}{4}t^3\right)\right)dt$$

(By using $u_2(x)$)

$$= -1 + e^x + \frac{1}{2}x^2 e^x - \frac{1}{2}\int_0^x \left(-3t + \frac{1}{4}t^2 + e^t t\left(3 - 2t + \frac{5}{4}t^2 - \frac{1}{4}t^3\right)\right)dt$$

By solving, we get

$$u_3(x) = x\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \dots\right)$$

$$\vdots$$

$$u_{n+1}(x) = x\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \dots\right)$$

Notice that, we used the Taylor expansion for e^x to determine $u_3(x)$,

$u_4(x), \ldots$

Thus, The solution $u(x)$ of (i)

$$u(x) = \lim_{n \to \infty} u_{n+1}(x)$$

$$\boxed{u(x) = xe^x}$$ Answer:

- **Example # 3:**

Solve the Volterra Integral eq. by using successive approximations method

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt.$$

- **Solution:**

We have

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt \longrightarrow (i)$$

For The zeroth approximation $u_0(x)$, we can select

$$u_0(x) = 0 \longrightarrow (a)$$

By using successive approximations method, we have The iteration formula

$$u_{n+1}(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u_n(t)\,dt \,; \; n \geq 0 \longrightarrow (ii)$$

For $n = 0$,

(ii) $\Rightarrow$

$$u_1(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u_0(t)\,dt$$

$\because u_0(x) = 0$ (by (a))

$$\Rightarrow u_1(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 (0)\,dt$$

$\Rightarrow$

$$u_1(x) = 1 + x + \frac{1}{2!}x^2.$$

For $n=1$, (ii) $\Rightarrow$

$$u_2(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 (u_1(t))\,dt$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2\left(1 + t + \frac{1}{2!}t^2\right)dt$$

(By using $u_1(t)$)

$\Rightarrow$

$$u_2(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5$$

For $n=2$, (ii) $\Rightarrow$

$$u_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u_2(t)\,dt$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5\right)dt$$

(using $u_2(x)$)

$\Rightarrow$

$$u_3(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8.$$

$\vdots$

and so on, we get

$$u_{n+1} = e^x$$

Thus, the solution $u(x)$ of (i) is:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x)$$

$$\boxed{u(x) = e^x} \quad \text{(Answer).}$$

- **Example # 4:-**

Solve the Volterra Integral equations by using the successive approximations method

$$u(x) = 1 - x\sin x + x\cos x + \int_0^x t\,u(t)\,dt.$$

- **Solution:-**

we have

$$u(x) = 1 - x\sin x + x\cos x + \int_0^x t\,u(t)\,dt \longrightarrow (i)$$

For zeroth approximation $u_0(x)$, we may select

$$u_0(x) = x. \longrightarrow (a)$$

By using successive approximations method, we have the iteration formula

$$u_{n+1}(x) = 1 - x\sin x + x\cos x + \int_0^x t\,u_n(t)\,dt;\ n \geq 0 \longrightarrow (ii)$$

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For $n=0$, (iii) $\Rightarrow$

$$u_1(x) = 1 - x\sin x + x\cos x + \int_0^x t\, u_0(t)\, dt$$

$$u_1(x) = 1 - x\sin x + x\cos x + \int_0^x t^2\, dt$$

(using (a))

$$u_1(x) = 1 + \frac{1}{3}x^3 - x\sin x + x\cos x$$

For $n=1$, (iii) $\Rightarrow$

$$u_2(x) = 1 - x\sin x + x\cos x + \int_0^x t\, u_1(t)\, dt$$

$$= 1 - x\sin x + x\cos x + \int_0^x t\left(1 + \frac{1}{3}t^3 - t\sin t + t\cos t\right) dt$$

(using $u_1(x)$)

$$\Rightarrow u_2(x) = 3 + \frac{1}{2}x^2 + \frac{1}{15}x^3 - (2 + 3x - x^2)\sin x - (2 - 3x - x^2)\cos x.$$

For $n=2$, (iii) $\Rightarrow$

$$u_3(x) = 1 - x\sin x + x\cos x + \int_0^x t\, u_2(t)\, dt$$

$\Rightarrow$

$$u_3(x) = \left(+x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7\right) + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6\right)$$

$\vdots$

$$u_{n+1}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \sum_{k=0}^{n} (-1)^k \frac{(x)^{2k}}{(2k)!}$$

We used the Taylor expansions for $\sin x$ and $\cos x$ to determine the approximations $u_3(x), u_4(x), \ldots$

Thus, the solution $u(x)$ of (i) is

$$u(x) = \lim_{n \to \infty} u_{n+1}(x)$$

$$\boxed{u(x) = \sin x + \cos x}$$

Answer.

Exercise 3.2.5

Use the Successive approximation method to solve the following Voltera integral equations:-

Question # 1

$$u(x) = x + \int_0^x (x-t)u(t)dt$$

Solution

For zeroth approximation we select

$$u_0(x) = 0$$

The Successive approximation method admits the use of iteration method

$$u_{n+1}(x) = x + \int_0^x (x-t)u_n(t)dt \quad ; n \geq 0$$

$$u_1(x) = x + \int_0^x (x-t)u_0(t)dt$$

$$u_1(x) = x$$

$$u_2(x) = x + \int_0^x (x-t)u_1(t)dt$$

$$u_2(x) = x + \int_0^x (x-t)t\,dt$$

$$= x + x\left|\frac{t^2}{2}\right|_0^x - \left|\frac{t^3}{3}\right|_0^x$$

$$U_2(x) = x + \frac{x^3}{2} - \frac{x^3}{3}$$

$$U_2(x) = x + \frac{x^3}{6} = x + \frac{x^3}{3!}$$

$$U_3(x) = x + \int_0^x (x-t)U_2(t)\,dt$$

$$U_3(x) = x + \int_0^x (x-t)\left(t + \frac{t^3}{6}\right)dt$$

$$U_3(x) = x + \int_0^x \left(xt + \frac{xt^3}{6} - t^2 - \frac{t^4}{6}\right)dt$$

$$U_3(x) = x + x\left|\frac{t^2}{2}\right|_0^x + \frac{x}{6\,4}\left|t^4\right|_0^x - \left|\frac{t^3}{3}\right|_0^x - \frac{1}{6\,5}\left|t^5\right|_0^x$$

$$= x + \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^3}{3} - \frac{x^5}{30}$$

$$U_3(x) = x + \frac{x^3}{6} + \frac{x^5}{120}$$

$$U_3(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\vdots$$

$$U_{n+1}(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$U_{n+1}(x) = \operatorname{Sinh} x$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty} \operatorname{Sinh} x$$

$$\boxed{U(x) = \operatorname{Sinh} x}$$

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Question # 2

$$U(x) = x + \int_0^x U(t)dt$$

Solution

For the zeroth approximation we select

$$U_0(x) = 0$$

The successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = x + \int_0^x U_n(t)dt$$

$$U_1(x) = x + \int_0^x U_0(t)dt$$

$$U_1(x) = x$$

$$U_2(x) = x + \int_0^x U_1(t)dt$$

$$U_2(x) = x + \int_0^x t\,dt$$

$$U_2(x) = x + \frac{|t^2|_0^x}{2}$$

$$U_2(x) = x + \frac{x^2}{2}$$

$$U_3(x) = x + \int_0^x U_2(t)dt$$

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$$U_3(x) = x + \int_0^x \left(t + \frac{t^2}{2}\right) dt$$

$$U_3(x) = x + \left| \frac{t^2}{2} + \frac{t^3}{3} \right|_0^x$$

$$U_3(x) = x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$U_3(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$U_{n+1}(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!}$$

$$U_{n+1}(x) = \left(1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n+1}}{(n+1)!}\right) - 1$$

$$U_{n+1}(x) = \sum_{k=0}^{n+1} \frac{x^k}{k!} - 1$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty} \left(\sum_{k=0}^{n+1} \frac{x^k}{k!} - 1 \right)$$

$$U(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} - 1$$

$$\boxed{U(x) = e^x - 1}$$

Question #3

$$u(x) = \frac{1}{6}x^3 - \int_0^x (x-t)u(t)\,dt$$

Solution

For the zeroth approximation

we select

$$U_0(x) = 0$$

The successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = \frac{1}{6}x^3 - \int_0^x (x-t)U_n(t)\,dt$$

$$U_1(x) = \frac{1}{6}x^3 - \int_0^x (x-t)U_0(t)\,dt$$

$$U_1(x) = \frac{1}{6}x^3 = \frac{x^3}{3!}$$

$$U_2(x) = \frac{1}{6}x^3 - \int_0^x (x-t)U_1(t)\,dt$$

$$U_2(x) = \frac{1}{6}x^3 - \int_0^x (x-t)\frac{1}{6}t^3\,dt$$

$$U_2(x) = \frac{1}{6}x^3 - \int_0^x \left(\frac{xt^3}{6} - \frac{t^4}{6}\right)dt$$

$$U_2(x) = \frac{1}{6}x^3 - \frac{x}{6\cdot 4}\left[t^4\right]_0^x + \frac{1}{6\cdot 5}\left[t^5\right]_0^x$$

$$U_2(x) = \frac{1}{6}x^3 - \frac{x^5}{24} + \frac{x^5}{30}$$

$$= \frac{x^3}{6} - \frac{x^5}{120}$$

$$U_2(x) = \frac{x^3}{3!} - \frac{x^5}{5!}$$

$$U_3(x) = \frac{1}{6}x^3 - \int_0^x (x-t)U_2(t)\,dt$$

(4)

$$U_3(x) = \frac{1}{6}x^3 - \int_0^x (x-t)\left(\frac{t^3}{6} - \frac{t^5}{120}\right)dt$$

$$U_3(x) = \frac{1}{6}x^3 - \int_0^x \left(\frac{xt^3}{6} - \frac{xt^5}{120} - \frac{t^4}{6} + \frac{t^6}{120}\right)dt$$

$$U_3(x) = \frac{x^3}{6} - \frac{x}{6.4}|t^4|_0^x + \frac{x}{120.6}|t^6|_0^x + \frac{|t^5|_0^x}{6.5} - \frac{|t^7|_0^x}{120.7}$$

$$U_3(x) = \frac{x^3}{6} - \frac{x^5}{24} + \frac{x^7}{720} + \frac{x^5}{30} - \frac{x^7}{840}$$

$$U_3(x) = \frac{x^3}{6} - \frac{x^5}{120} + \frac{x^7}{5040}$$

$$U_3(x) = \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!}$$

$$U_{n+1}(x) = \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}$$

$$U_{n+1}(x) = x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}$$

$$U_{n+1}(x) = x - \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}\right]$$

$$U_{n+1}(x) = x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty}\left[x - \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}\right]$$

$$U(x) = x - \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\boxed{U(x) = x - \sin x}$$

Question # 4

$$U(x) = 1 + 2x + 4\int_0^x (x-t)u(t)\,dt$$

Solution:

For the zeroth approximation we select

$$U_0(x) = 1$$

The Successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = 1 + 2x + 4\int_0^x (x-t)U_n(t)\,dt$$

$$U_1(x) = 1 + 2x + 4\int_0^x (x-t)U_0(t)\,dt$$

$$U_1(x) = 1 + 2x + 4\int_0^x (x-t).1\,dt$$

$$U_1(x) = 1 + 2x + 4\left| xt - \frac{t^2}{2} \right|_0^x$$

$$U_1(x) = 1 + 2x + 4x^2 + 2x^2$$

$$U_1(x) = 1 + 2x + 2x^2$$

$$U_2(x) = 1 + 2x + 4\int_0^x (x-t)U_1(t)\,dt$$

$$U_2(x) = 1 + 2x + 4\int_0^x (x-t)(1+2t+2t^2)\,dt$$

$$= 1 + 2x + 4\int_0^x (x + 2xt + 2xt^2 - t - 2t^2 - 2t^3)\,dt$$

$$U_2(x) = 1+2x+4\left[x|t|_0^x + \frac{2x}{2}|t^2|_0^x + \frac{2x}{3}|t^3|_0^x - \frac{|t^2|_0^x}{2} - \frac{2}{3}|t^3|_0^x - \frac{2}{4}|t^4|_0^x\right]$$

$$U_2(x) = 1+2x+4\left[x^2 + x^3 + \frac{2x^4}{3} - \frac{x^2}{2} - \frac{2x^3}{3} - \frac{2x^4}{4}\right]$$

$$U_2(x) = 1+2x+2x^2+\frac{4x^3}{3}+\frac{2}{3}x^4$$

$$U_3(x) = 1+2x+4\int_0^x (x-t)U_2(t)\,dt$$

$$U_3(x) = 1+2x+4\int_0^x (x-t)\left(1+2t+2t^2+\frac{4t^3}{3}+\frac{2}{3}t^4\right)dt$$

$$= 1+2x+4\int_0^x \left(x+2xt+2xt^2+\frac{4}{3}xt^3+\frac{2}{3}xt^4\right)dt$$

$$-4\int_0^x \left(t+2t^2+2t^3+\frac{4}{3}t^4+\frac{2}{3}t^5\right)dt$$

$$U_3(x) = 1+2x+4\left|xt+\frac{2xt^2}{2}+\frac{2xt^3}{3}+\frac{4xt^4}{12}+\frac{2xt^5}{15}\right|_0^x$$

$$-4\left|\frac{t^2}{2}+\frac{2t^3}{3}+\frac{2t^4}{4}+\frac{4t^5}{15}+\frac{2t^6}{18}\right|_0^x$$

$$U_3(x) = 1+2x+4\left[x^2+x^3+\frac{2}{3}x^4+\frac{x^5}{3}+\frac{2x^6}{15}\right]$$

$$-4\left[\frac{x^2}{2}+\frac{2x^3}{3}+\frac{2x^4}{4}+\frac{4x^5}{15}+\frac{x^6}{9}\right]$$

$$U_3(x) = 1+2x+2x^2+\frac{4x^3}{3}+\frac{2x^4}{3}+\frac{4x^5}{15}+\frac{4x^6}{45}$$

$$U_{n+1}(x) = 1+2x+2x^2+\frac{4x^3}{3}+\cdots+\frac{(2x)^{2n+2}}{(2n+2)!}$$

$$U_{n+1}(x) = \sum_{k=0}^{2n+2} \frac{(2x)^k}{k!}$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty} \sum_{k=0}^{2n+2} \frac{(2x)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(2x)^k}{k!}$$

$$\boxed{U(x) = e^{2x}}$$

Question #5

$$U(x) = \frac{1}{6}x^3 + \int_0^x (x-t)U(t)\,dt$$

Solution:

For the zeroth approximation we select

$$U_0(x) = 0$$

The successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = \frac{1}{6}x^3 + \int_0^x (x-t)U_n(t)\,dt$$

$$U_1(x) = \frac{1}{6}x^3 + \int_0^x (x-t)U_0(t)\,dt$$

$$U_1(x) = \frac{1}{6}x^3$$

$$U_2(x) = \frac{1}{6}x^3 + \int_0^x (x-t)\,U_1(t)\,dt$$

$$U_2(x) = \frac{1}{6}x^3 + \int_0^x (x-t)\,\frac{1}{6}t^3\,dt$$

$$= \frac{1}{6}x^3 + \frac{1}{6}\left| \frac{xt^4}{4} - \frac{t^5}{5} \right|_0^x$$

$$U_2(x) = \frac{1}{6}x^3 + \frac{1}{6}\left(\frac{x^5}{4} - \frac{x^5}{5} \right)$$

$$U_2(x) = \frac{x^3}{6} + \frac{x^5}{120}$$

$$U_2(x) = \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$U_3(x) = \frac{1}{6}x^3 + \int_0^x (x-t)\,U_2(t)\,dt$$

$$U_3(x) = \frac{1}{6}x^3 + \int_0^x (x-t)\left(\frac{t^3}{3!} + \frac{t^5}{5!} \right)dt$$

$$U_3(x) = \frac{x^3}{6} + \int_0^x \left(\frac{xt^3}{3!} - \frac{t^4}{3!} + \frac{xt^5}{5!} - \frac{t^6}{5!} \right)dt$$

$$= \frac{x^3}{6} + \frac{x}{3!\,4}\left|t^4\right|_0^x - \frac{\left|t^5\right|_0^x}{3!\,5} + \frac{x}{5!\,6}\left|t^6\right|_0^x - \frac{\left|t^7\right|_0^x}{5!\,7}$$

$$= \frac{x^3}{6} + \frac{x^5}{24} - \frac{x^5}{30} + \frac{x^7}{720} - \frac{x^7}{840}$$

$$U_3(x) = \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}$$

$$\vdots$$

$$U_{n+1}(x) = \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\cdots \frac{x^{2n+1}}{(2n+1)!}$$

$$U_{n+1}(x) = \left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{2n+1}\right] - x$$

$$U_{n+1}(x) = \sum_{k=0}^{n+1} \frac{x^{2(k+1)}}{(2K+1)!} - x$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty} \sum_{k=0}^{2n+1} \frac{x^{2(k+1)}}{(2K+1)} - x$$

$$U(x) = \sum_{k=0}^{\infty} \frac{x^{2(k+1)}}{(2K+1)} - x$$

$$\boxed{U(x) = \sinh x - x}$$

Question #6

$$U(x) = 1 + x^2 - \int_0^x (x-t+1)^2 \, dt$$

Solution

For the zeroth approximation we select

$$U_0(x) = 1$$

The Successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = 1 + x^2 - \int_0^x (x-t+1)^2 U_n(t) \, dt$$

$$U_1(x) = 1 + x^2 - \int_0^x \left((x-t)^2 + 2(x-t) + 1\right) \cdot 1 \, dt$$

$$= 1 + x^2 - \left[-\frac{(x-t)^3}{3} - 2\frac{(x-t)^2}{2} + t \right]_0^x$$

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$$U_1(x) = 1+x^2 - 1\left[x + \frac{x^3}{3} + x^2\right]$$

$$= 1+x^2 - x - \frac{x^3}{3} - x^2$$

$$U_1(x) = 1 - x - \frac{x^3}{3}$$

$$U_2(x) = 1+x^2 - \int_0^x (x^2+t^2-2xt+2x-2t+1)\, U_1(t)\,dt$$

$$U_2(x) = 1+x^2 - \int_0^x (x^2+t^2-2xt+2x-2t+1)\left(1-t-\frac{t^3}{3}\right)dt$$

$$= 1+x^2 - \int_0^x (x^2+t^2-2xt+2x-2t+1)\,dt$$

$$+ \int_0^x (x^2t+t^3-2xt^2+2xt-2t^2+t)\,dt$$

$$+ \frac{1}{3}\int_0^x (x^2t^3+t^5-2xt^4+2xt^3-2t^4+t^3)\,dt$$

$$U_2(x) = 1+x^2 - \left[x^2t + \frac{t^3}{3} - \frac{2xt^2}{2} + 2xt - \frac{2t^2}{2} + t\right]_0^x$$

$$+ \left|\frac{x^2t^2}{2} + \frac{t^4}{4} - \frac{2xt^3}{3} + \frac{2xt^2}{2} - \frac{2t^3}{3} + \frac{t^2}{2}\right|_0^x$$

$$+ \frac{1}{3}\left|\frac{x^2t^4}{4} + \frac{t^6}{6} - \frac{2xt^5}{5} + \frac{2xt^4}{4} - \frac{2t^5}{5} + \frac{t^4}{4}\right|_0^x$$

$$U_2(x) = 1 - x + \frac{1}{2}x^2 + \frac{1}{6}x^4 + \frac{1}{30}x^5 + \frac{x^6}{180}$$

$$= 1 - x + \frac{x^2}{2!} + \frac{x^4}{3!} + \frac{x^5}{30} + \frac{x^6}{180}$$

$$U_{n+1}(x) = \sum_{k=0}^{n+1} \frac{(-1)^k x^k}{k!} + \text{terms that vanish as } n \to \infty$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty} \sum_{k=0}^{n+1} \frac{(-1)^k x^k}{k!}$$

$$U(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}$$

$$\boxed{U(x) = e^{-x}}$$

Question#7

$$U(x) = \frac{1}{2}x^2 - \int_0^x (x-t)u(t)dt$$

Solution

For the zeroth approximation we select

$$U_0(x) = 0$$

The Successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = \frac{1}{2}x^2 - \int_0^x (x-t)U_n(t)dt$$

$$U_1(x) = \frac{1}{2}x^2 - \int_0^x (x-t)U_0(t)dt$$

$$U_1(x) = \frac{1}{2}x^2$$

$$U_2(x) = \frac{1}{2}x^2 - \int_0^x (x-t)\,U_1(t)\,dt$$

$$= \frac{1}{2}x^2 - \int_0^x (x-t)\frac{t^2}{2}\,dt$$

$$U_2(x) = \frac{1}{2}x^2 - \left| x\frac{t^3}{6} - \frac{t^4}{8} \right|_0^x$$

$$= \frac{x^2}{2} - \frac{x^4}{6} + \frac{x^4}{8}$$

$$U_2(x) = \frac{x^2}{2} - \frac{x^4}{24} = \frac{x^2}{2!} - \frac{x^4}{4!}$$

$$U_3(x) = \frac{x^2}{2} - \int_0^x (x-t)\left(\frac{t^2}{2} - \frac{t^4}{24}\right)dt$$

$$U_3(x) = \frac{x^2}{2} - \int_0^x \left(\frac{xt^2}{2} - \frac{xt^4}{24} - \frac{t^3}{2} + \frac{t^5}{24}\right)dt$$

$$U_3(x) = \frac{x^2}{2} - \left| \frac{xt^3}{6} - \frac{xt^5}{24.5} - \frac{t^4}{8} + \frac{t^6}{24.6} \right|_0^x$$

$$U_3(x) = \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720}$$

$$U_3(x) = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!}$$

$$\vdots$$

$$U_{n+1}(x) = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2(n+1)}}{2(n+1)!}$$

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$$U_{n+1}(x) = 1 - \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \frac{(-1)^n x^{2(n+1)}}{2(n+1)!}\right]$$

$$= 1 - \sum_{k=0}^{n+1} \frac{(-1)^{k+1} x^{2k}}{(2k)!}$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty}\left[1 - \sum_{k=0}^{n+1} \frac{(-1)^{k+1} x^{2k}}{(2k)!}\right]$$

$$\lim_{n\to\infty} U_{n+1}(x) = 1 - \lim_{n\to\infty}\sum_{k=0}^{n+1} \frac{(-1)^{k+1} x^{2k}}{(2k)!}$$

$$U(x) = 1 - \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k}}{(2k)!}$$

$$\boxed{U(x) = 1 - \cos x}$$

Question # 8

$$u(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)\,dt$$

Sol:

For zeroth approximation we select

$$u_0(x) = 0$$

The successive approximation method admits the use of iteration formula

$$u_{n+1}(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u_n(t)\,dt \quad ; n \geq 0$$

$$u_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u_0(t)\,dt$$

$$u_1(x) = 1 - \frac{1}{2}x^2$$

$$u_2(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u_1(t)\,dt$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3\left(1 - \frac{1}{2}t^2\right)dt$$

$$\because (x-t)^3 = x^3 - 3x^2t + 3xt^2 - t^3$$

$$u_2(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x \left(x^3 - 3x^2t + 3xt^2 - t^3 - \frac{x^3t^2}{2} + \frac{3}{2}x^2t^3 - \frac{3}{2}xt^4 + \frac{t^5}{2}\right)dt$$

$$U_2(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\left[x^3|t|_0^x - \frac{3}{2}x^2|t^2|_0^x + 3x\left|\frac{t^3}{3}\right|_0^x - \left|\frac{t^4}{4}\right|_0^x\right.$$

$$\left. - \frac{x^3}{2\cdot 3}|t^3|_0^x + \frac{3}{2\cdot 4}|t^4|_0^x - \frac{3x}{2\cdot 5}|t^5|_0^x + \frac{1}{2\cdot 6}|t^6|_0^x\right]$$

$$U_2(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\left[x^4 - \frac{3}{2}x^4 + x^4 - \frac{x^4}{4} - \frac{x^6}{6} + \frac{3x^6}{8} - \frac{3x^6}{10} + \frac{x^6}{12}\right]$$

$$U_2(x) = 1 - \frac{1}{2}x^2 + \frac{x^4}{24} - \frac{x^6}{720}$$

$$U_2(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\vdots$$

$$U_{n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \cos x$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty}[\cos x]$$

$$\boxed{u(x) = \cos(x)}$$ Answer

Question#9

$$u(x) = 1 + 3\int_0^x u(t)\,dt$$

Solution:-

For zeroth approximation we select

$$U_0(x) = 0$$

The successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = 1 + 3\int_0^x U_n(t)\,dt \quad ; \quad n \geq 0$$

$$U_1(x) = 1 + 3\int_0^x U_0(t)\,dt$$

$$= 1 + 3\int_0^x 0\,dt$$

$$U_1(x) = 1$$

$$U_2(x) = 1 + 3\int_0^x U_1(t)\,dt$$

$$= 1 + 3\int_0^x 1\,dt$$

$$U_2(x) = 1 + 3x$$

$$U_3(x) = 1 + 3\int_0^x U_2(t)\,dt$$

$$= 1 + 3\int_0^x (1+3t)\,dt$$

$$= 1 + 3\int_0^x 1\,dt + 9\int_0^x t\,dt$$

$$U_3(x) = 1 + 3x + \frac{9x^2}{2}$$

$$U_4(x) = 1 + 3\int_0^x U_3(t)\,dt$$

$$U_4(x) = 1+3\int_0^x \left(1+3t+\frac{9}{2}t^2\right)dt$$

$$= 1+3\int_0^x 1dt + 9\int_0^x t\,dt + \frac{27}{2}\int_0^x t^2 dt$$

$$= 1+3|t|_0^x + \frac{9}{2}|t^2|_0^x + \frac{27}{6}|t^3|_0^x$$

$$U_4(x) = 1+3x+\frac{9}{2}x^2+\frac{27}{6}x^3$$

$$\vdots$$

$$U_{n+1}(x) = 1+3x+\frac{9}{2}x^2+\frac{27}{6}x^3+\cdots$$

$$\because e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$U_{n+1}(x) = 1+\frac{3x}{1!}+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\cdots$$

$$U_{n+1}(x) = e^{3x}$$

Question # 10

$$U(x) = 1-2\sinh x+\int_0^x (x+2-t)U(t)\,dt$$

Sol:-

For zeroth approximation we select

$$U_0(x) = \cosh x$$

The successive approximation

method admits the use of iteration formula.

$$u_{n+1}(x) = 1 - 2\sinh x + \int_0^x (x+2-t)\,u_n(t)\,dt \quad ; n \geq 0$$

$$u_1(x) = 1 - 2\sinh x + \int_0^x (x+2-t)\,u_0(t)\,dt$$

$$u_1(x) = 1 - 2\sinh x + \int_0^x (x+2)\cosh t\,dt - \int_0^x t\cosh t\,dt$$

$$u_1(x) = 1 - 2\sinh x + \frac{(x+2)}{2}\int_0^x (e^t + e^{-t})\,dt - \frac{1}{2}\int_0^x t(e^t + e^{-t})\,dt$$

$$u_1(x) = 1 - 2\sinh x + \frac{(x+2)}{2}\left[|e^t|_0^x - |e^{-t}|_0^x \right] - \frac{1}{2}|te^t|_0^x$$

$$+ \frac{|e^t|_0^x}{2} + \frac{1}{2}|te^{-t}|_0^x + \frac{1}{2}|e^{-t}|_0^x$$

$$u_1(x) = 1 - 2\sinh x + \frac{(x+2)}{2}\left[e^x - \cancel{1} - e^{-x} + \cancel{1} \right]$$

$$- \frac{xe^x}{2} + \frac{e^x}{2} - \frac{1}{2} + \frac{xe^{-x}}{2} + \frac{e^{-x}}{2} - \frac{1}{2}$$

$$u_1(x) = \cancel{1} - 2\sinh x + (x+2)\frac{(e^x - e^{-x})}{2} + \frac{(e^x + e^{-x})}{2}$$

$$- x\left(\frac{e^x - e^{-x}}{2}\right) - \cancel{1}$$

$$u_1(x) = -2\sinh x + x\sinh x + 2\sinh x$$
$$+ \cosh x - x\sinh x$$

$$u_1(x) = \cosh x$$

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$$U_2(x) = 1 - 2\sinh x + \int_0^x (x+2-t)\,u_1(t)\,dt$$

$$U_2(x) = 1 - 2\sinh x + \int_0^x (x+2-t)\cosh t\,dt$$

$$= 1 - 2\sinh x + (x+2)\int_0^x \cosh t\,dt$$

$$- \int_0^x t\cosh t\,dt$$

$$U_2(x) = 1 - 2\sinh x + \frac{(x+2)}{2}\int_0^x (e^t + e^{-t})\,dt - \frac{1}{2}\int_0^x t(e^t + e^{-t})\,dt$$

$$U_2(x) = 1 - 2\sinh x + \frac{(x+2)}{2}\left[\left|e^t\right|_0^x - \left|e^{-t}\right|_0^x\right]$$

$$- \frac{1}{2}\left|te^t - e^t\right|_0^x - \frac{1}{2}\left|-te^{-t} - e^{-t}\right|_0^x$$

$$U_2(x) = 1 - 2\sinh x + (x+2)\left(\frac{e^x - e^{-x}}{2}\right)$$

$$- \frac{xe^x}{2} + \frac{e^x}{2} - \frac{1}{2} + \frac{xe^{-x}}{2} + \frac{e^{-x}}{2} - \frac{1}{2}$$

$$U_2(x) = 1 - 2\sinh x + (x+2)\left(\frac{e^x - e^{-x}}{2}\right)$$

$$+ \left(\frac{e^x + e^{-x}}{2}\right) - x\left(\frac{e^x - e^{-x}}{2}\right) - 1$$

$$U_2(x) = \cancel{1} - \cancel{2\sinh x} + \cancel{x\sinh x} + \cancel{2\sinh x}$$

$$+ \cosh x - \cancel{x\sinh x} - \cancel{1}$$

$$U_2(x) = \cosh x$$

$$U_{n+1}(x) = \cosh x$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty} \cosh x$$

$$\boxed{U(x) = \cosh x}$$

Question # 11

$$U(x) = 3 + x^2 - \int_0^x (x-t)U(t)\,dt$$

Solution:

For zeroth approximation we select

$$U_0(x) = 1$$

The Sccessive approximation method admits the use of iteration formula-

$$U_{n+1}(x) = 3 + x^2 - \int_0^x (x-t)U_n(t)\,dt \quad ; \; n \geq 0$$

$$U_1(x) = 3 + x^2 - \int_0^x (x-t)\,U_0(t)\,dt$$

$$U_1(x) = 3 + x^2 - \int_0^x (x-t)(1)\,dt$$

$$U_1(x) = 3 + x^2 - x|t|_0^x + \left|\frac{t^2}{2}\right|_0^x$$

$$U_1(x) = 3 + \cancel{x^2} - \cancel{x^2} + \frac{x^2}{2}$$

$$U_1(x) = 3 + \frac{x^2}{2!}$$

$$U_2(x) = 3 + x^2 - \int_0^x (x-t)U_1(t)\,dt$$

$$U_2(x) = 3 + x^2 - \int_0^x (x-t)\left(3 + \frac{t^2}{2}\right)dt$$

$$U_2(x) = 3 + x^2 - \int_0^x \left(3x + \frac{xt^2}{2} - 3t - \frac{t^3}{2}\right)dt$$

$$U_2(x) = 3 + x^2 - 3x|t|_0^x - \frac{x}{2\cdot3}|t^3|_0^x + \frac{3}{2}|t^2|_0^x + \frac{1}{2\cdot4}|t^4|_0^x$$

$$U_2(x) = 3 + x^2 - 3x^2 - \frac{x^4}{6} + \frac{3x^2}{2} + \frac{x^4}{8}$$

$$U_2(x) = 3 - \frac{x^2}{2!} - \frac{x^4}{4!}$$

$$U_3(x) = 3 + x^2 - \int_0^x (x-t)U_2(t)\,dt$$

$$= 3 + x^2 - \int_0^x (x-t)\left(3 - \frac{t^2}{2} - \frac{t^4}{24}\right)dt$$

$$= 3 + x^2 - \int_0^x \left(3x - \frac{xt^2}{2} - \frac{xt^4}{24} - 3t + \frac{t^3}{2} + \frac{t^5}{24}\right)dt$$

$$U_3(x) = 3 + x^2 - 3x|t|_0^x + \frac{x}{2\cdot3}|t^3|_0^x + \frac{x}{24\cdot5}|t^5|_0^x$$

$$+ \frac{3}{2}|t^2|_0^x - \frac{1}{8}|t^4|_0^x - \frac{1}{24\cdot6}|t^6|_0^x$$

$$U_3(x) = 3 + x^2 - 3x^2 + \frac{x^4}{6} + \frac{x^6}{120} + \frac{3x^2}{2} - \frac{x^4}{8} - \frac{x^6}{144}$$

$$U_3(x) = 3 - \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720}$$

$$U_3(x) = 3 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}$$

$$U_{n+1}(x) = 2 + 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$U_{n+1}(x) = 2 + \cos x$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty} (2 + \cos x)$$

$$\boxed{U(x) = 2 + \cos x.}$$

Question # 12

$$U(x) = 1 - x\sin x + \int_0^x t\,u(t)\,dt$$

Solution

For zeroth approximation we select

$$U_0(x) = \cos x$$

The Successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = 1 - x\sin x + \int_0^x t\,U_n(t)\,dt$$

$$U_1(x) = 1 - x\sin x + \int_0^x t\,U_0(t)\,dt$$

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$$U_1(x) = 1 - x\sin x + \int_0^x t\cos t\,dt$$

$$U_1(x) = 1 - x\sin x + |t\sin t|_0^x + |\cos t|_0^x$$

$$U_1(x) = \cancel{1} - \cancel{x\sin x} + \cancel{x\sin x} + \cos x - \cancel{1}$$

$$U_1(x) = \cos x$$

$$U_2(x) = 1 - x\sin x + \int_0^x t\,U_1(t)dt$$

$$U_2(x) = 1 - x\sin x + \int_0^x t\cos t\,dt$$

$$U_2(x) = 1 - x\sin x + |t\sin t|_0^x + |\cos t|_0^x$$

$$= \cancel{1} - \cancel{x\sin x} + \cancel{x\sin x} + \cos x - \cancel{1}$$

$$U_2(x) = \cos x$$

$$\vdots$$

$$U_{n+1}(x) = \cos x$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty} \cos x$$

$$\boxed{U(x) = \cos x}$$

Question # 13

$$U(x) = x\cosh x - \int_0^x t\,U(t)dt$$

Solution:-

For zeroth approximation we select

$$U_0(x) = \sinh x$$

The Successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = x\cosh x - \int_0^x t\,U_n(t)\,dt$$

$$U_1(x) = x\cosh x - \int_0^x t\,U_0(t)\,dt$$

$$U_1(x) = x\cosh x - \int_0^x t\sinh t\,dt$$

$$U_1(x) = x\cosh x - |t\cosh t|_0^x + |\sinh t|_0^x$$

$$U_1(x) = \cancel{x\cosh x} - \cancel{x\cosh x} + \sinh x$$

$$U_1(x) = \sinh x$$

$$U_2(x) = x\cosh x - \int_0^x t\,U_1(t)\,dt$$

$$U_2(x) = x\cosh x - \int_0^x t\sinh t\,dt$$

$$U_2(x) = x\cosh x - |t\cosh t|_0^x + |\sinh t|_0^x$$

$$U_2(x) = \cancel{x\cosh x} - \cancel{x\cosh x} + \sinh x$$

$$U_2(x) = \sinh x$$

$$\vdots$$

$$U_{n+1}(x) = \sinh x$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty} \sinh x$$

$$\boxed{U(x) = \sinh x}$$

Question # 14

$$U(x) = 1 - x - \int_0^x (x-t)U(t)\,dt$$

Solution

For zeroth approximation we select

$$U_0(x) = 0$$

The Successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = 1 - x - \int_0^x (x-t)U_n(t)\,dt \quad , n \geq 0$$

$$U_1(x) = 1 - x - \int_0^x (x-t)U_0(t)\,dt$$

$$U_1(x) = 1 - x$$

$$U_2(x) = 1 - x - \int_0^x (x-t)U_1(t)\,dt$$

$$U_2(x) = 1 - x - \int_0^x (x-t)(1-t)\,dt$$

$$U_2(x) = 1 - x - \int_0^x (x - xt - t + t^2)\,dt$$

$$U_2(x) = 1 - x - \int_0^x x\,dt + \int_0^x xt\,dt + \int_0^x t\,dt - \int_0^x t^2\,dt$$

$$U_2(x) = 1 - x - x|t|_0^x + \frac{x}{2}|t^2|_0^x + \left|\frac{t^2}{2}\right|_0^x - \left|\frac{t^3}{3}\right|_0^x$$

$$= 1 - x - x^2 + \frac{x^3}{2} + \frac{x^2}{2} - \frac{x^3}{3}$$

$$U_2(x) = 1 - x - \frac{x^2}{2} + \frac{x^3}{6}$$

$$U_2(x) = 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$U_3(x) = 1 - x - \int_0^x (x-t)\,U_2(t)\,dt$$

$$U_3(x) = 1 - x - \int_0^x (x-t)\left(1 - t - \frac{t^2}{2} + \frac{t^3}{6}\right)dt$$

$$U_3(x) = 1 - x - \int_0^x \left(x - xt - \frac{xt^2}{2} + \frac{xt^3}{6} - t + t^2 + \frac{t^3}{2} - \frac{t^4}{6}\right)dt$$

$$U_3(x) = 1 - x - x|t|_0^x + \frac{x}{2}|t^2|_0^x + \frac{x}{2 \cdot 3}|t^3|_0^x + \frac{|t^2|_0^x}{2} - \frac{x}{6 \cdot 4}|t^4|_0^x$$

$$- \frac{|t^3|_0^x}{3} - \frac{|t^4|_0^x}{24} + \frac{|t^5|_0^x}{6 \cdot 5}$$

$$U_3(x) = 1 - x - x^2 + \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^5}{24} + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{8} + \frac{x^5}{30}$$

$$U_3(x) = 1 - x - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$$

$$U_3(x) = 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}$$

$$\vdots$$

$$U_{n+1}(x) = 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \cdots$$

$$U_{n+1}(x) = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$U_{n+1}(x) = \cos x - \sin x$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty} \cos x - \sin x$$

$$\boxed{U(x) = \cos x - \sin x}$$

Question #15

$$U(x) = 1 - \int_0^x 3t^2 u(t)\,dt$$

Solution

For zeroth approximation we select

$$U_0(x) = 0$$

The successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = 1 - \int_0^x 3t^2 U_n(t)\,dt \quad ; \; n \geq 0$$

$$U_1(x) = 1 - \int_0^x 3t^2 U_0(t)\,dt$$

$$U_1(x) = 1$$

$$U_2(x) = 1 - \int_0^x 3t^2 U_1(t)\,dt$$

$$U_2(x) = 1 - \int_0^x 3t^2 \cdot 1\,dt$$

$$U_2(x) = 1 - 3\frac{t^3}{3}\Big|_0^x$$

$$U_2(x) = 1 - x^3$$

$$U_3(x) = 1 - \int_0^x 3t^2 U_2(t)\,dt$$

$$U_3(x) = 1 - 3\int_0^x t^2(1-t^3)\,dt$$

$$U_3(x) = 1 - 3\int_0^x (t^2 - t^5)\,dt$$

$$U_3(x) = 1 - 3\left| \frac{t^3}{3} - \frac{t^6}{6} \right|_0$$

$$U_3(x) = 1 - x^3 + \frac{x^6}{2}$$

$$U_4(x) = 1 - 3\int_0^x t^2 U_3(t)\,dt$$

$$U_4(x) = 1 - 3\int_0^x t^2\left(1 - t^3 + \frac{t^6}{2}\right)dt$$

$$U_4(x) = 1 - 3\int_0^x \left(t^2 - t^5 + \frac{t^8}{2}\right)dt$$

$$U_4(x) = 1 - 3\left| \frac{t^3}{3} - \frac{t^6}{6} + \frac{t^9}{18} \right|_0^x$$

$$U_4(x) = 1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6}$$

$$\vdots$$

$$U_{n+1}(x) = 1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + \cdots\cdots$$

$$U_{n+1}(x) = 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \cdots\cdots$$

$$U_{n+1}(x) = 1 - x^3 + \frac{(x^3)^2}{2!} - \frac{(x^3)^3}{3!} + \cdots$$

$$U_{n+1}(x) = e^{-x^3}$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty} e^{-x^3}$$

$$\boxed{U(x) = e^{-x^3}}$$

Question #17

$$U(x) = 1 + \sinh x - \sin x - \cosh x + \cos x + \int_0^x u(t)\,dt$$

Solution

For the zeroth approximation we select

$$U_0(x) = \sinh x + \cos x$$

The Successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = 1 + \sinh x - \sin x + \cos x - \cosh x + \int_0^x u_n(t)\,dt$$

$$U_1(x) = 1 + \sinh x - \sin x + \cos x - \cosh x + \int_0^x U_0(t)\,dt$$

$$U_1(x) = 1 + \sinh x - \sin x + \cos x - \cosh x + \int_0^x (\sinh t + \cos t)\,dt$$

$$U_1(x) = 1 + \sinh x - \sin x + \cos x - \cosh x + \left[\cosh t + \sin t\right]_0^x$$

$$U_1(x) = 1 + \sinh x - \sin x + \cos x - \cosh x + \cosh x + \sin x - 1$$

$$U_1(x) = \sinh x + \cos x$$

$$U_2(x) = 1 + \sinh x - \sin x + \cos x - \cosh x + \int_0^x (\sinh t + \cos t)\,dt$$

$$U_2(x) = 1 + \sinh x - \sin x + \cos x - \cosh x + \left| \cosh t + \sin t \right|_0^x$$

$$U_2(x) = 1 + \sinh x - \sin x + \cos x - \cosh x + \cosh x + \sin x - 1$$

$$U_2(x) = \sinh x + \cos x$$

$$\vdots$$

$$U_{n+1}(x) = \sinh x + \cos x$$

$$\lim_{n \to \infty} U_{n+1}(x) = \lim_{n \to \infty} (\sinh x + \cos x)$$

$$\boxed{U(x) = \sinh x + \cos x}$$

Question # 18

$$u(x) = 1 + \sinh x + \sin x - \cos x + \cosh x - \int_0^x u(t)\,dt$$

Solution

For zeroth approximation we select

$$U_0(x) = \sin x + \cosh x$$

The Successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = 1+\sinh x+\sin x-\cos x+\cosh x - \int_0^x U_n(t)\,dt$$

$$U_1(x) = 1+\sinh x+\sin x-\cos x+\cosh x - \int_0^x (\sin t+\cosh t)\,dt$$

$$U_1(x) = 1+\sinh x+\sin x-\cos x+\cosh x - \left|-\cos t+\sinh t\right|_0^x$$

$$U_1(x) = \cancel{1}+\cancel{\sinh x}+\sin x-\cancel{\cos x}+\cosh x+\cancel{\cos x} - \cancel{\sinh x} - \cancel{1}$$

$$U_1(x) = \sin x+\cosh x$$

$$\vdots$$

$$U_{n+1}(x) = \sin x+\cosh x$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty}(\sin x+\cosh x)$$

$$\boxed{U(x) = \sin x+\cosh x}$$

Question # 19

$$u(x) = 2 - 2\cos x - \int_0^x (x-t)u(t)\,dt$$

Solution:

For the zeroth approximation we select

$$u_0(x) = 0$$

The successive approximation method admits the use of iteration formula

$$u_{n+1}(x) = 2 - 2\cos x - \int_0^x (x-t)u_n(t)\,dt$$

$$u_1(x) = 2 - 2\cos x - \int_0^x (x-t)u_0(t)\,dt$$

$$u_1(x) = 2 - 2\cos x$$

$$u_1(x) = 2 - 2\cos x.$$

$$u_1(x) = 2 - 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

$$u_1(x) = 2 - 2 + 2\frac{x^2}{2} - 2\left(\frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$u_1(x) = x^2, \text{ Neglecting higher order terms}$$

$$u_2(x) = 2 - 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) - \int_0^x (x-t)u_1(t)\,dt$$

$$u_2(x) = 2 - 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) - \int_0^x (x-t)t^2\,dt$$

$$U_2(x) = 2 - 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) - \left[\frac{xt^3}{3} - \frac{t^4}{4}\right]_0^x$$

$$U_2(x) = 2 - 2 + \frac{2x^2}{2} - \frac{2x^4}{4!} + \cdots - \left[\frac{x^4}{3} - \frac{x^4}{4}\right]$$

$$U_2(x) = x^2 - \frac{x^4}{12} - \frac{x^4}{12}$$

$$U_2(x) = x^2 - \frac{x^4}{6} = x^2 - \frac{x^4}{3!}$$

$$U_3(x) = 2 - 2\cos x - \int_0^x (x-t)U_2(t)\,dt$$

$$U_3(x) = 2 - 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) - \int_0^x (x-t)\left(t^2 - \frac{t^4}{6}\right)dt$$

$$U_3(x) = 2 - 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) - \int_0^x \left(xt^2 - \frac{xt^4}{6} - t^3 + \frac{t^5}{6}\right)dt$$

$$U_3(x) = 2 - 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) - \left[\frac{xt^3}{3} - \frac{xt^5}{30} - \frac{t^4}{4} + \frac{t^6}{36}\right]_0^x$$

$$U_3(x) = 2 - 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - \left[\frac{x^4}{3} - \frac{x^6}{30} - \frac{x^4}{4} + \frac{x^6}{36}\right]$$

$$U_3(x) = 2 - 2 + x^2 - \frac{x^4}{12} + \frac{x^6}{360} - \frac{x^4}{12} + \frac{x^6}{180}$$

$$= x^2 - \frac{x^4}{6} + \frac{x^6}{120}$$

$$U_3(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}$$

$$\vdots$$

$$U_{n+1}(x) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots$$

$$U_{n+1}(x) = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\right)$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty} x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\right)$$

$$\boxed{U(x) = x\sin x}$$

Question# 20

$$U(x) = -x + 2\sinh x + \int_0^x (x-t)\,U(t)\,dt$$

Solution:

For the zeroth approximation we select

$$U_0(x) = x$$

The successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = -x + 2\sinh x + \int_0^x (x-t)\,U_n(t)\,dt$$

$$U_1(x) = -x + 2\sinh x + \int_0^x (x-t)\,U_0(t)\,dt$$

$$U_1(x) = -x + 2\sinh x + \int_0^x (x-t)\,t\,dt$$

$$U_1(x) = -x + 2\sinh x + \left|\frac{xt^2}{2} - \frac{t^3}{3}\right|_0^x$$

$$U_1(x) = -x + 2\sinh x + \frac{x^3}{2} - \frac{x^3}{3}$$

$$U_1(x) = -x + 2\sinh x + \frac{x^3}{6}$$

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$$U_1(x) = -x + 2\sinh x + \frac{x^3}{3!}$$

Since $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$$\Rightarrow U_1(x) = -x + 2\left[x + \frac{x^3}{3!} + \cdots\right] + \frac{x^3}{3!}$$

$$U_1(x) = -x + 2x + \frac{2x^3}{3!} + \cdots + \frac{x^3}{3!}$$

$$U_1(x) = x + \frac{3x^3}{3!} + \text{Neglecting Higher order Terms}$$

$$U_1(x) = x + \frac{x^3}{2!} + \text{Neglecting Higher order term}$$

$$U_2(x) = -x + 2\sinh x + \int_0^x (x-t)\, U_1(t)\,dt$$

$$U_2(x) = -x + 2\sinh x + \int_0^x (x-t)\left(t + \frac{t^3}{2} + \text{Neglecting Higher order}\right)dt$$

$$U_2(x) = -x + 2\sinh x + \int_0^x \left(xt + \frac{xt^3}{2} - t^2 - \frac{t^4}{2} + \text{N.H.O.T}\right)dt$$

$$U_2(x) = -x + 2\sinh x + \left[\frac{xt^2}{2} + \frac{xt^4}{8} - \frac{t^3}{3} - \frac{t^5}{10} + \cdots\right]_0^x$$

$$U_2(x) = -x + 2\sinh x + \left(\frac{x^3}{2} + \frac{x^5}{8} - \frac{x^3}{3} - \frac{x^5}{10} + \cdots\right)$$

$$U_2(x) = -x + 2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$+ \frac{x^3}{6} + \frac{x^5}{40} + \cdots$$

$$U_2(x) = -x + 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \cdots + \frac{x^3}{6} + \frac{x^5}{40} + \cdots$$

$$U_2(x) = x + \frac{2x^3}{6} + \frac{x^3}{6} + \frac{2x^5}{120} + \frac{x^5}{40} + \cdots$$

$$U_2(x) = x + \frac{x^3}{2} + \frac{x^5}{24} + \text{Neglecting Higher O.T}$$

$$U_2(x) = x + \frac{x^3}{2!} + \frac{x^5}{4!} + \text{Neglecting H.O.T}$$

$\vdots$

$$U_{n+1}(x) = x + \frac{x^3}{2!} + \frac{x^5}{4!} + \cdots \frac{x^{2n+1}}{2n!}$$

$$U_{n+1}(x) = x\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{2n!}\right)$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty} x\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{2n!}\right)$$

$$U(x) = x\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

$$\boxed{U(x) = x\cosh x}$$

Question # 16

$$U(x) = 2x\cosh 2x - 4\int_0^x t\,u(t)dt$$

Solution:

For the zeroth approximation we select

$$U_0(x) = \sinh 2x$$

The successive approximation method admits the use of iteration formula

$$U_{n+1}(x) = 2x\cosh 2x - 4\int_0^x t\,U_n(t)dt$$

$$U_1(x) = 2x\cosh 2x - 4\int_0^x t\,U_0(t)dt$$

$$U_1(x) = 2x\cosh 2x - 4\int_0^x t\sinh 2t\,dt$$

$$U_1(x) = 2x\cosh 2x - 4\left[\frac{t\cosh 2t}{2} - \frac{\sinh 2t}{4}\right]_0^x$$

$$U_1(x) = 2x\cosh 2x - 2x\cosh 2x + \sinh 2x$$

$$U_1(x) = \sinh 2x$$

$$U_2(x) = 2x\cosh 2x - 4\int_0^x t\,U_1(t)dt$$

$$U_2(x) = 2x\cosh 2x - 4\int_0^x t\sinh 2t\,dt$$

$$U_2(x) = \sinh 2x$$

$$\vdots$$

$$U_{n+1}(x) = \sinh 2x$$

$$\lim_{n\to\infty} U_{n+1}(x) = \lim_{n\to\infty} \sinh 2x \Rightarrow \boxed{U(x) = \sinh 2x}$$

• References:-

W.V. Lovitt (1950) Linear Integral Equation Dover Represented
Abdul-Majid Wazwaz. A first course in Integral Equations solutions Manual, 2nd ed. World scientific Publishing
T.A Burton Eds. (2005) volterra Integral pre and differential equations, 2nd ed; Academic press, Elsevier.