VARIATIONAL ITERATION METHOD FOR SOLVING VOLTERA INTEGRAL EQUATION OF 2nd KIND-

Variational iteration method (VIM) is used to give the approximation solution of Voltera integral equations of second kind. The method constructs a convergent sequence of functions, which approximates the exact solution with few iterations. To illustrate the ability and reliabity of the method, some examples are given, revealing its effectiveness and simplicity.

Introduction: Let $u(x)$ is unknown function, $f(x)$ is given known function, and $k(x,t)$ a known integral kernel.

The Voltera integral equation of the second kind is an integral equation of the form

$$u(x) = f(x) + \int_a^x k(x,t)\,u(t)\,dt, \longrightarrow (1)$$

Variational iteration method is a method for solving linear and non-linear problems, introduced by Chinese mathematician, He modified the General Lagrange multiplier method and constructed an iterative sequence of functions which converges to the exact solution. In most linear problems, the Langrange multiplier, the approximate solution turns into the exact solution and is available with just one iteration

To illustrate the method, consider the following general functional equation

$$Lu(x) + Nu(x) = g(t) \longrightarrow (2)$$

where L is linear and N is non-linear operator and $g(t)$ is a known analytical function.

According to the variational iteration method, we can construct the following correction functional,

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\{Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\}\,d\xi \longrightarrow (3)$$

where λ is general Langrange multiplier which can be identified optimally via variationally theory, u_0 is an initiall approximation with possible unknown, and $\tilde{u}_n$ is considered as restricted variation

i-e $\delta\tilde{u}_n = 0$

SOLUTION OF V-I-Eq of 2nd KIND

Consider the V-I-Eq of 2nd kind given as eq (1) first we take the derivative of (1) w.r.t x.

For V-I-Eq of 2^{nd} kind, we have

$$u'(x) = f'(x) + \frac{d}{dx}\int_0^x k(x,t)\,u(t)\,dt \longrightarrow (4)$$

and

Now, consider, $\frac{d}{dx}\int_0^x k(x,t)\,u(t)\,dt$,

as a restricted value (that mean it behaves as constant)
we use the variational iteration method.
Then we have following iteration sequence

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left\{u_n'(\xi) - f'(\xi) - \frac{d}{d\xi}\int_0^{\xi} k(x,t)\,u_n(t)\,dt\right\} d\xi \quad (5)$$

Taking the derivative w.r.t independent variable u_n and noticing that $\delta u_n(0) = 0$,

$$\delta u_{n+1} = \delta u_n + \lambda\,\delta u_n\big|_{\xi=x} - \int_0^x \lambda'\,\delta u_n\,d\xi \longrightarrow (6)$$

The extremum conditions of u_{n+1} requires that $\delta u_{n+1} = 0$ and result of R-H-S of (6) is zero as well. This yields the stationary conditions

$$1 + \lambda\big|_{\xi=x} = 0, \quad \lambda'(\xi)\big|_{\xi=x} = 0 \longrightarrow (7)$$

Thus, from equation (7),

The General Langrangian multiplier

therefore can be readily identified:

$$\lambda = -1$$

And as result, we obtain the following iteration formula

$$u_{n+1}(x) = u_n(x) - \int_0^x \left\{ u_n'(\xi) - f'(\xi) - \frac{d}{d\xi} \int_a^\xi k(\xi,t)\, u_n(t)\,dt \right\} d\xi \longrightarrow (8)$$

NUMERICAL EXAMPLE:

To show the effeciency of the approch few examples are stated below:

Example NO 1: Solve V-I-Eq by using Variational iteration method

$$u(x) = 1 + \int_0^x u(t)\,dt \longrightarrow (1)$$

Solution:

Differenciating equation (1) by using Leibnitz rule

$$1) \Rightarrow \quad u'(x) = u(x) \longrightarrow (2)$$

using variational iteration method, the correction functional for eq (2) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(u_n'(\xi) - u_n(\xi)\right)d\xi \longrightarrow (3)$$

take

$$\lambda = -1$$

Putting the value of Langrange multiplier $\lambda = -1$ into the functional (3)

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$$u_{n+1}(x) = u_n(x) - \int_0^x (u_n'(\xi) - u_n(\xi))\,d\xi$$

Putting $x=0$ in (1)

$u(0) = 1 \implies u_0(x) = 1$

$u'(0) = 0$

$\implies u_0'(x) = 0$

for $u_1(x)$ put $n=0$ (3)

$$u_1(x) = u_0(x) - \int^x (u_0'(\xi) - u_0(\xi))\,d\xi$$

$$u_1(x) = 1 - \int_0^x (0-1)\,d\xi$$

$$\boxed{u_1(x) = 1 + x}$$

$$u_2(x) = u_1(x) - \int_0^x (u_1'(\xi) - u_1(\xi))\,d\xi$$

$$= 1 + x - \int_0^x (1 - (1+\xi))\,d\xi$$

$$= 1 + x + \int_0^x \xi\,d\xi$$

$$\boxed{u_2(x) = 1 + x + \frac{x^2}{2!}}$$

$$u_3(x) = u_2(x) - \int_0^x (u_2'(\xi) - u_2(\xi))\,d\xi$$

$$= 1 + x + \frac{x^2}{2!} - \int_0^x \left(\left(1 + \frac{2\xi}{2}\right) - \left(1 + \xi + \frac{\xi^2}{2}\right)\right)d\xi$$

$$= 1 + x + \frac{x^2}{2!} + \int_0^x \frac{\xi^2}{2}\,d\xi$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

and so on,

(VIM) admits the use of

$$u(x) = \lim_{n \to \infty} u_n(x)$$

and

$$u_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

So, $u(x) = \lim_{n \to \infty} u_n(x)$

$$= \lim_{n \to \infty} \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right)$$

$$u(x) = e^x$$

Example No 2:

$$u(x) = x + \int_0^x (x-t)u(t)\,dt \longrightarrow (1)$$

Differenciating (1) w.r.t x using Leibnitz rule

$$u'(x) = 1 + \int_0^x u(t)\,dt \longrightarrow (2)$$

using Variational iteration method

The correction Functional for (2) is

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(s) - 1 - \int_0^s u_n(\xi)\,d\xi \right) ds$$

$$\longrightarrow (3)$$

where the value of Langrangian multiplied is $\lambda = -1$

Now put $x = 0$ in (1)

gives the initial conditions;

$u(0) = (0)$; $u'(0) = 0$

$\Rightarrow$ $u_0(x) = 0$; $u_0'(x) = 0$

Put $n=0$ in 3:

$$u_1(x) = u_0(x) - \int_0^x u_0'(s) - 1 - \int_0^s u_0(\xi)d\xi)\,ds$$

$$= 0 - \int_0^x (0 - 1 - \int_0^s 0 \cdot d\xi)\,ds$$

$$= \int_0^x 1 \cdot ds = x$$

$$\boxed{u_1(x) = x}$$

Put $n=1$ in 3:

$$u_2(x) = u_1(x) - \int_0^x (u_1'(s) - 1 - \int_0^s u_1(\xi)d\xi)\,ds$$

$$= x - \int_0^x (1 - 1 - \int_0^s \xi d\xi)\,ds$$

$$= x - \int_0^x -\left(\frac{s^2}{2}\right) ds$$

$$\boxed{u_2(x) = x + \frac{x^3}{3!}}$$

$$u_3(x) = u_2(x) - \int_0^x (u_2'(s) - 1 - \int_0^s u_2(\xi)d\xi)\,ds$$

$$= x + \frac{x^3}{3!} - \int_0^x \left[\cancel{1} + \frac{s^2}{2} \cancel{-1} - \int_0^s \left(\xi + \frac{\xi^3}{3!}\right)d\xi\right] ds$$

$$= x + \frac{x^3}{3!} - \int_0^x \left(\cancel{\frac{s^2}{2}} - \cancel{\frac{s^2}{2}} - \frac{s^4}{4!}\right) ds$$

$$\boxed{u_3(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!}}$$

and so on,

$$u_n(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n-1}}{(2n-1)!}$$

So,
$$u(x) = \lim_{n\to\infty} u_n(x)$$

$$= \lim_{n\to\infty}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n-1}}{(2n-1)!}\right)$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \infty$$

$$u(x) = \sinh x$$

Example NO 3:

$$u(x) = 1 + x + \frac{x^3}{3!} - \int_0^x (x-t)u(t)\,dt \longrightarrow (1)$$

Differenciating (1) w.r.t x by Leibnitz rule

$$u'(x) = 1 + \frac{x^2}{2} - \int_0^x u(t)\,dt \longrightarrow (2)$$

using VIM, the correction function for (2) is

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(s) - 1 - \frac{s^2}{2} + \int_0^s u_n(r)\,dr\right)ds \longrightarrow (3)$$

where the Langrange multiplier is

$$\lambda = -1$$

Put $x = 0$ in (1)

$$u(0) = 1$$

$$u_0(x) = 1 \implies u_0'(x) = 0$$

using $n = 0$ in (3)

$$u_1(x) = u_0(x) - \int_0^x \left[u_0'(s) - 1 - \frac{s^2}{2} + \int_0^s u_0(r)\,dr\right]ds$$

$$= 1 - \int_0^x \left(0 - 1 - \frac{s^2}{2} + \int_0^s 1\,dr\right) ds$$

$$= 1 - \int_0^x \left(-1 - \frac{s^2}{2} + s\right) ds$$

$$\boxed{u_1(x) = 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!}}$$

$$u_1'(x) = 1 - x + x^2/2!$$

$$u_2(x) = u_1(x) - \int_0^x \left[u_1'(s) - 1 - \frac{s^2}{2!} + \int_0^s u_1(r)\,dr \right] ds$$

$$= 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left[\cancel{1} - s + \cancel{\frac{s^2}{2!}} - \cancel{1} - \cancel{\frac{s^2}{2!}} \right.$$

$$\left. + \int_0^s \left(1 + r - \frac{r^2}{2!} + \frac{r^3}{3!}\right) dr \right] ds$$

$$= 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left(-\cancel{s} + \cancel{s} + \frac{s^2}{2} - \frac{s^3}{3!} + \frac{s^4}{4!} \right) ds$$

$$= 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left(\frac{s^2}{2} - \frac{s^3}{3!} + \frac{s^4}{4!} \right) ds$$

$$= 1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} - \left(\frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} \right)$$

$$\boxed{u_2(x) = 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^5}{5!}}$$

$$u_2'(x) = 1 - x + \frac{x^3}{3!} - \frac{x^4}{4!}$$

$$u_3(x) = u_2(x) - \int_0^x \left[u_2'(s) - 1 - \frac{s^2}{2} + \int_0^s u_2(r)\,dr \right] ds$$

$$= 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \int_0^x \left[\cancel{1} - s + \frac{s^3}{3!} - \frac{s^4}{4!} - \cancel{1} - \frac{s^2}{2} \right.$$

$$\left. + \int_0^s \left(1 + r - \frac{r^2}{2!} + \frac{r^4}{4!} - \frac{r^5}{5!} \right) dr \right] ds$$

$\therefore -s^4/4!$

$$= 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \int_0^x \Big[\cancel{-s} + \cancel{\frac{s^3}{3!}} - \cancel{\frac{s^2}{2!}} + \cancel{s} + \cancel{\frac{s^2}{2!}} - \cancel{\frac{s^3}{3!}}$$

$$+ \frac{s^5}{5!} - \frac{s^6}{6!} \Big] ds$$

$$= 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \int_0^x \left(-\frac{s^4}{4!} + \frac{s^5}{5!} - \frac{s^6}{6!} \right) ds$$

$$= 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^7}{7!}$$

$$\boxed{u_3(x) = 1 + x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^7}{7!}}$$

and so on,

$$u_n(x) = x + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!}\right)$$

$$u(x) = \lim_{n \to \infty} u_n(x)$$

$$= \lim_{n \to \infty} \left[x + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n}{(2n)!} x^{2n}\right)\right]$$

$$= x + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \infty\right)$$

$$u(x) = x + \cos x$$

Example NO 4:

$$u(x) = 1 + x + \frac{x^2}{2} + \frac{1}{2} \int_0^x (x-t)^2 u(t)\, dt \rightarrow (1)$$

differenciating (1) w.r.t x using Leibnitz rule

$$u'(x) = 1 + x + \int_0^x (x-t) u(t)\, dt \longrightarrow (2)$$

using VIM, the correction functional for (2)

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[u_n'(s) - 1 - s - \int_0^s (s-r)\,u_n(r)\,dr\right] ds \longrightarrow (3)$$

where the Langrangian multiplier is

$$\lambda = -1.$$

Put $x = 0$ in (1)

$$u_0(x) = 1 \qquad u_0'(x) = 0$$

for $n = 0$

$$u_1(x) = u_0(x) - \int_0^x \left[u_0'(s) - 1 - s - \int_0^s (s-r)\,u_0(r)\,dr\right] ds$$

$$= 1 - \int_0^x \left[0 - 1 - s - \int_0^s (s-r)\,dr\right] ds$$

$$= 1 - \int_0^x \left[-1 - s - sr\Big|_0^s + r^2/2\Big|_0^s\right] ds$$

$$= 1 - \int_0^x \left[-1 - s - s^2 + \frac{s^2}{2}\right] ds$$

$$= 1 - \int_0^x \left(-1 - s - \frac{s^2}{2}\right) ds$$

$$\boxed{u_1(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}}$$

$$u_1'(x) = 1 + x + \frac{x^2}{2!}$$

for $n = 1$

$$u_2(x) = u_1(x) - \int_0^x \left[u_1'(s) - 1 - s - \int_0^s (s-r)\,u_1(r)\,dr\right] ds$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \Big(\cancel{1} + \cancel{s} + \frac{s^2}{2!} - \cancel{1} - \cancel{s} -$$

$$\int_0^s (s-r)\left(1 + r + \frac{r^2}{2!} + r^3/3!\right) dr\Big)\, ds$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left[\frac{s^2}{2!} - \int_0^s s\left(1 + r + \frac{r^2}{2!} + \frac{r^3}{3!}\right) dr - \int_0^s \left(r + r^2 + \frac{r^3}{2!} + \frac{r^4}{3!}\right) dr\right] ds$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left[\frac{s^2}{2!} - \left\{ s\left(s + \frac{s^2}{2} + \frac{s^3}{3!} + \frac{s^4}{4!} \right) \right\} \right.$$

$$\left. + \left\{ \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{2!\cdot 4} + \frac{s^5}{3!\cdot 5} \right\} \right] ds$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left[\cancel{\frac{s^2}{2!}} - \cancel{s^2} - \frac{s^3}{2} - \frac{s^4}{3!} - \frac{s^5}{4!} \right.$$

$$\left. + \cancel{\frac{s^2}{2}} + \frac{s^3}{3} + \frac{s^4}{2\cdot 4} + \frac{s^5}{5\cdot 3\cdot 2} \right] ds$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left[\left(\frac{-s^3}{2} + \frac{s^3}{3} \right) + \left(\frac{-s^4}{6} + \frac{s^4}{8} \right) + \right.$$

$$\left. \left(\frac{-s^5}{24} + \frac{s^5}{30} \right) \right] ds$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left[\left(-\frac{s^3}{6} \right) + \left(-\frac{s^4}{24} \right) + \left(\frac{-s^5}{120} \right) \right] ds$$

$$\boxed{u_2(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}}$$

$$u(x) = \lim_{n \to \infty} u_n(x)$$

$$= \lim_{n \to \infty} \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} \right]$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \infty$$

$$u(x) = e^x$$

EXERCISE 3.2.4

Use the variational iteration method to solve the following Voltera Integral Equations by converting the equation into the initial Value problem or to an equivelent to integro differential equation:

Question No 1: $u(x) = 1 - \int_0^x u(t)dt \rightarrow (1)$

differenciating (1) w.r.t x

using Leibnitz rule

$u'(x) = -u(x)$

$u'(x) + u(x) = 0 \longrightarrow (2)$

Put $x = 0$ in (1) to find $u_0(x)$

$u_0(x) = 1, \quad u_0'(x) = 0$

The correction (~~factor~~) function for (2) is

$u_{n+1}(x) = u_n(x) + \lambda \int_0^1 (u_n'(s) + u_n(s))\, ds$

$u_{n+1}(x) = u_n(x) - \int_0^1 (u_n'(s) + u_n(s))\, ds \longrightarrow (3)$

where $\lambda = -1$

Put $n=0$ in 3

$u_1(x) = u_0(x) - \int_0^1 (u_0'(s) + u_0(s))\, ds$

$= 1 - \int_0^x (0+1) ds$

$= 1 - \int_0^x ds = 1 - x$

$\boxed{u_1(x) = 1 - x}$

$u_1'(x) = -1$

for $n=1$; $u_2(x) = u_1(x) - \int_0^x (u_1'(s) + u_1(s))ds$

$$u_2(x) = (1-x) - \int_0^x [\cancel{-1} + 1 + s]ds$$

$$\boxed{u_2(x) = 1 - x + \frac{x^2}{2}} \qquad u_2'(x) = -1 + x$$

for $n=2$; $u_3(x) = u_2(x) - \int_0^x (u_2'(s) + u_2(s))ds$

$$u_3(x) = 1 - x + \frac{x^2}{2} - \int_0^x \left((-1+s) + \left(1 - s + \frac{s^2}{2}\right)\right)ds$$

$$= 1 - x + \frac{x^2}{2} - \int_0^x \frac{s^2}{2}ds$$

$$\boxed{u_3(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}}$$

$$u_{n+1}(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!}$$

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \lim_{n\to\infty}\left(1 - x + \frac{x^2}{2!} + \cdots + \infty\right)$$

$$= e^{-x}$$

Question NO 2:

$$u(x) = x + x^4 + \frac{1}{2}x^2 + \frac{1}{5}x^5 - \int_0^x u(t)dt \longrightarrow (1)$$

differenciating (1) w.r.t x

$$u'(x) = 1 + 4x^3 + x + x^4 - u(x) \longrightarrow (2)$$

the correction factor for variational iteration method for (2) is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x [u_n'(s) - 1 - 4s^3 - s - s^4 + u_n(s)]ds \longrightarrow (3)$$

for zeroth aproximation, put $x=0$ in (1)

$u(0)=0$

$u_0(x)=0$

$u_0'(x)=0$

for $n=0$ in (3)

$$u_1(x) = \cancel{u_0(x)}^{0} - \int_0^x [\cancel{u_0'(s)}^{0} - 1 - 4s^3 - s - s^4 + \cancel{u_0(s)}^{0}]\,ds$$

$$u_1(x) = -\int_0^x [-1-4s^3-s-s^4]\,ds$$

$$= -\left[-x - \frac{4x^4}{4} - \frac{x^2}{2} - \frac{x^5}{5}\right]$$

$$\boxed{u_1(x) = x + x^4 + \frac{x^2}{2} + \frac{x^5}{5}}$$

$$u_1'(x) = 1 + 4x^3 + x + x^4$$

for $n=1$ in (3)

$$u_2(x) = u_1(x) - \int_0^x [u_1'(s) - 1 - 4s^3 - s - s^4 + u_1(s)]\,ds$$

$$= x + x^4 + \frac{x^2}{2} + \frac{x^5}{5} - \left[\int_0^x \left(\cancel{1} + \cancel{4s^3} + \cancel{s} + \cancel{s^4} - \cancel{1} - \cancel{4s^3} - \cancel{s} - \cancel{s^4} + s + s^4 + \frac{s^2}{2} + \frac{s^5}{5}\right)ds\right]$$

$$= x + x^4 + \frac{x^2}{2} + \frac{x^5}{5} - \int_0^x \left(s + s^4 + \frac{s^2}{2} + \frac{s^5}{5}\right)ds$$

$$= x + x^4 + \frac{x^2}{2} + \frac{x^5}{5} - \left(\frac{x^2}{2} - \frac{x^5}{5} - \frac{x^3}{6} + \frac{x^6}{30}\right)$$

$$\boxed{u_2(x) = x + x^4 - \frac{x^3}{6} - \frac{x^6}{30}}$$

$$u_2'(x) = 1 + 4x^3 - \frac{x^2}{2} - \frac{x^5}{5}$$

for $n=2$ in (3)

$$u_3(x) = u_2(x) - \int_0^x [u_2'(s) - 1 - 4s^3 - s - s^4 + u_2(s)]\,ds$$

$$= x + x^4 + \frac{x^3}{6} + \frac{x^6}{30} - \left[\int_0^x \left(\cancel{1} + \cancel{4s^3} - \frac{s^2}{2} - \frac{s^5}{5} - \cancel{1} - \cancel{4s^3} - \cancel{s} - \cancel{s^4} + \cancel{s} + \cancel{s^4} - \frac{s^3}{6} - \frac{s^6}{30}\right) ds\right]$$

$$= x + x^4 + \frac{x^3}{6} + \frac{x^6}{30} - \int_0^x \left(-\frac{s^2}{2} - \frac{s^5}{5} - \frac{s^3}{6} - \frac{s^6}{30}\right) ds$$

$$= x + x^4 + \cancel{\frac{x^3}{6}} - \cancel{\frac{x^6}{30}} + \cancel{\frac{x^3}{6}} + \cancel{\frac{x^6}{30}} + \frac{x^4}{24} + \frac{x^7}{210}$$

$$\boxed{u_3(x) = x + x^4 + \frac{x^4}{24} + \frac{x^7}{210}}$$

$\frac{x^4}{24}$, $\frac{x^7}{210}$ are noise terms so cancelled

with next approximation of $u_{n+1}(x)$

So, in the same maner continuing, we get

$$u_{n+1}(x) = x + x^4$$

$$\lim_{n \to \infty} = \lim_{n \to \infty} (x + x^4) = x + x^4 \text{ Answer-}$$

Question NO 3:

$$u(x) = 1 - \frac{1}{2}x^2 + \int_0^x u(t)\,dt \rightarrow (1)$$

differiating w.r.t x

$$u'(x) = -x + u(x) \rightarrow (2)$$

For VIM the correction Function is

$$u_{n+1}(x) = u_n(x) - \int^x [u_n'(s) + s - u_n(s)]\,ds \rightarrow (3)$$

For zeroth aproximation at $x=0$

$1) \Rightarrow \quad u_0(x) = 1 \quad , \quad u_0'(x) = 0$

for $n=0$ in (3)

$$u_1(x) = u_0(x) - \int_0^x [u_0'(s) + s - u_0(s)]\,ds$$

$$= 1 - \int_0^x [0 + s - 1]\,ds$$

$$= 1 - \frac{x^2}{2} + x$$

$$\boxed{u_1(x) = 1 + x - \frac{x^2}{2!}} \quad ; \quad u_1'(x) = 1 - x$$

$$u_2(x) = u_1(x) - \int_0^x [u_1'(s) + s - u_1(s)]\,ds$$

$$= 1 + x - \frac{x^2}{2} - \int_0^x [\cancel{1} - \cancel{s} + \cancel{s} - \cancel{1} - s + \frac{s^2}{2}]\,ds$$

$$= 1 + x - \frac{x^2}{2} - \int_0^x \left(-s + \frac{s^2}{2}\right)ds$$

$$= 1 + x - \frac{x^2}{2} - \left(-\frac{x^2}{2!} + \frac{x^3}{3!}\right)$$

$$\boxed{u_2(x) = 1 + x - \frac{x^3}{3!}} \quad ; \quad u_2'(x) = 1 - \frac{x^2}{2}$$

$$u_3(x) = u_2(x) - \int_0^x [u_2'(s) + s - u_1(s)]\,ds$$

$$u_3(x) = 1 + x - \frac{x^3}{3!} - \int_0^x \left[1 - \frac{s^2}{2} + s - \left(1 + s - \frac{s^3}{3!}\right)\right] ds$$

$$= 1 + x - \frac{x^3}{3!} - \int_0^x \left(-\frac{s^2}{2} + \frac{s^3}{3!}\right) ds$$

$$= 1 + x - \frac{x^3}{3!} - \left(-\frac{x^3}{3!} + \frac{x^4}{4!}\right)$$

$$\boxed{u_3(x) = 1 + x - \frac{x^4}{4!}}$$

and so, on

$$u_{n+1}(x) = 1 + x - \frac{x^{n+2}}{(n+2)!}$$

$$u(x) = \lim_{n \to \infty} u_{n+1}(x) = \lim_{n \to \infty} \left(1 + x - \frac{x^{n+2}}{(n+2)!}\right)$$

$$u(x) = 1 + x$$

Question NO 4:

$$u(x) = 1 - x - \frac{x^2}{2} + \int_0^x (x - t) u(t) dt \longrightarrow (1)$$

differenciating w.r.t x

$$u'(x) = -1 - x + \int_0^x u(t) dt \longrightarrow (2)$$

using VIM the correction function for (2) is

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[u_n'(s) + 1 + s - \int_0^s u_n(r) dr\right] ds$$

for $u_0(x)$ Put $x = 0$ in (1)

$$u_0(x) = 1 \quad ; \quad u_0'(x) = 0$$

$$u_1(x) = u_0(x) - \int_0^x \left[u_0'(s) + 1 + s - \int_0^s u_0(r) dr\right] ds$$

$$u_1(x) = 1 - \int_0^x \left[0 + 1 + s - \int_0^s 1 \cdot dr \right] ds$$

$$= 1 - \int_0^x (\cancel{1} + s - \cancel{s}) ds$$

$$u_1(x) = 1 - \int_0^x s\,ds = 1 + x$$

$$\boxed{u_1(x) = 1 - x} \qquad u_1'(x) = -1$$

$$u_2(x) = u_1(x) - \int_0^x \left[u_1'(s) + 1 + s - \int_0^s u_1(r)\,dr \right] ds$$

$$= 1 - x - \int_0^x \left[-1 + 1 + s - \int_0^s (1 - r)\,dr \right] ds$$

$$= 1 - x - \int_0^x \left[s - s + \frac{s^2}{2} \right] ds$$

$$= 1 - x - \int_0^x \frac{s^2}{2} ds = 1 - x - \frac{x^3}{3!}$$

$$\boxed{u_2(x) = 1 - x - \frac{x^3}{3!}} \; ; \qquad u_2'(x) = -1 - \frac{x^2}{2}$$

$$u_3(x) = u_2(x) - \int_0^x \left[u_2'(s) + 1 + s - \int_0^s u_2(r)\,dr \right] ds$$

$$= 1 - x - \frac{x^3}{3!} - \int_0^x \left[\cancel{-1} - \frac{s^2}{2} \cancel{+1} + s - \int_0^s \left(1 - r - \frac{r^3}{3!}\right) dr \right] ds$$

$$= 1 - x - \frac{x^3}{3!} - \int_0^x \left[\left(s - \frac{s^2}{2} \right) - \left(s - \frac{s^2}{2} - \frac{s^4}{4!} \right) \right] ds$$

$$= 1 - x - \frac{x^3}{3!} - \int_0^x \frac{s^4}{4!} ds = 1 - x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\boxed{u_3(x) = 1 - x - \frac{x^3}{3!} - \frac{x^5}{5!}}$$

and so on,

$$u_{n+1}(x) = 1 - \left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right]$$

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \lim_{n\to\infty}\left[1 - \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\right]$$

$$u(x) = 1 - \sinh x$$

Question NO 5: $u(x) = 1 - \int_0^x (x-t)u(t)\,dt \rightarrow (1)$

differenciating w.r.t x

$$u'(x) = 0 - \int_0^x u(t)\,dt \longrightarrow (2)$$

Using VIM, the correction function for (2) is

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[u_n'(s) + \int_0^s u_n(r)\,dr\right] ds \longrightarrow (3)$$

For $u_0(x)$ Put $x=0$ in (1)

$$u_0(x) = 1 \quad ; \quad u_0'(x) = 0$$

For $n=0$, in (3)

$$u_1(x) = u_0(x) - \int_0^x \left[u_0'(s) + \int_0^s u_0(r)\,dr\right] ds$$

$$= 1 - \int_0^x \left[0 + \int_0^s 1\,dr\right] ds$$

$$= 1 - \int_0^x s\,ds = 1 - \frac{x^2}{2!}$$

$$\boxed{u_1(x) = 1 - \frac{x^2}{2!}} \quad ; \quad u_1'(x) = -x$$

$$u_2(x) = u_1(x) - \int_0^x \left[u_1'(s) + \int_0^s u_1(r)\,dr\right] ds$$

$$= 1 - \frac{x^2}{2!} - \int_0^x \left[-s + \int_0^s \left(1 - \frac{r^2}{2!}\right) dr\right] ds$$

$$u_2(x) = 1 - \frac{x^2}{2!} - \int_0^x \left[-s + s - \frac{s^3}{3!}\right] ds$$

$$= 1 - \frac{x^2}{2!} - \int_0^x -\frac{s^3}{3!}\, ds = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\boxed{u_2(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}} \; ; \; u_2'(x) = -x + \frac{x^3}{3!}$$

$$u_3(x) = u_2(x) - \int_0^x \left[u_2'(s) + \int_0^s u_2(r)\, dr\right] ds$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \int_0^x \left[-s + \frac{s^3}{3!} + \int_0^s \left(1 - \frac{r^2}{2!} + \frac{r^4}{4!}\right) dr\right] ds$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \int_0^x \left[\cancel{-s} + \cancel{\frac{s^3}{3!}} + \cancel{s} - \cancel{\frac{s^3}{3!}} + \frac{s^5}{5!}\right] ds$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \int_0^x \frac{s^5}{5!}\, ds = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\boxed{u_3(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}}$$

and So on;

$$u_{n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$u(x) = \lim_{n \to \infty} u_{n+1}(x) = \lim_{n \to \infty} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

$$u(x) = \cos x$$

Question NO 6: $u(x) = x + \int_0^x (x-t)\, u(t)\, dt \longrightarrow (1)$

differenciating w.r.t x

$$u'(x) = 1 + \int_0^x u(t)\, dt \longrightarrow (2)$$

Using VIM the correction formular for (2) is

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[u_n'(s) - 1 - \int_0^s u_n(r)\,dr \right] ds \longrightarrow (3)$$

Put $x=0$ in (1) then,

$u_0(x) = 0$; $u_0'(x) = 0$

Put $n=0$ in 3

$$u_1(x) = u_0(x) - \int_0^x \left[u_0'(s) - 1 - \int_0^s u_0(r)\,dr \right] ds$$

$$= 0 - \int_0^x \left[0 - 1 - \int_0^s 0 \cdot dr \right] ds$$

$$= - \int_0^x (-1)\,ds = x$$

$\boxed{u_1(x) = x}$ $u_1'(x) = 1$

Put $n=1$, in 3

$$u_2(x) = u_1(x) - \int_0^x \left[u_1'(s) - 1 - \int_0^s u_1(r)\,dr \right] ds$$

$$= x - \int_0^x \left[\cancel{1} - \cancel{1} - \int_0^s r\,dr \right] ds$$

$$= x - \int_0^x \left(- \frac{s^2}{2!} \right) ds$$

$\boxed{u_2(x) = x + \frac{x^3}{3!}}$; $u_2'(x) = 1 + \frac{x^2}{2!}$

Put $n=2$ in 3

$$u_3(x) = u_2(x) - \int_0^x \left[u_2'(s) - 1 - \int_0^s u_2(r)\,dr \right] ds$$

$$= x + \frac{x^3}{3!} - \int_0^x \left(1 + \frac{s^2}{2!}\right) - 1 - \int_0^s \left(r + \frac{r^3}{3!}\right) dr \Big] ds$$

$$= x + \frac{x^3}{3!} - \int_0^x \left(\frac{\cancel{s^2}}{\cancel{2!}} - \frac{\cancel{s^2}}{\cancel{2}} - \frac{s^4}{4!} \right) ds$$

$\boxed{u_3(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!}}$ and So on,

$u_{n+1}(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$

$u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty}\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots\right]$

$u(x) = \sinh x$

Question NO 7: $u(x) = 1 + 2x + 4\int_0^x (x-t)u(t)\,dt \longrightarrow (1)$

differenciating (1) w.r.t x

$u'(x) = 2 + 4\int_0^x u(t)\,dt \longrightarrow (2)$

using $x = 0$ in (1)

$\Rightarrow \quad u_0(x) = 1 \;; \quad u_0'(x) = 0$

The correction formula for (2) is

$u_{n+1}(x) = u_n(x) - \int_0^x \left[u_n'(s) - 2 - 4\int_0^s u_n(r)\,dr\right]ds \longrightarrow (3)$

for $n=0$ in 3

$u_1(x) = u_0(x) - \int_0^x \left[u_0'(s) - 2 - 4\int_0^s u_0(r)\,dr\right]ds$

$= 1 - \int_0^x \left[0 - 2 - 4\int_0^s 1\cdot dr\right]ds$

$= 1 - \int_0^x (-2 - 4s)\,ds$

$= 1 - (-2x - 2x^2)$

$\boxed{u_1(x) = 1 + 2x + 2x^2} \;; \quad u_1'(x) = 2 + 4x$

for $n=1$;

3) $\Rightarrow$ $u_2(x) = u_1(x) - \int_0^x \left[u_1'(s) - 2 - 4\int_0^s u_1(r)dr\right]ds$

$$= 1+2x^2+2x - \int_0^x \left[\cancel{2}+4s-\cancel{2} - 4\int_0^s (1+2r+2r^2)dr\right]ds$$

$$= 1+2x^2+2x - \int_0^x \left[4s - 4\left(s+s^2+\frac{2s^3}{3}\right)\right]ds$$

$$= 1+2x^2+2x - \int_0^x \left(\cancel{4s} - \cancel{4s} - 4s^2 - \frac{8s^3}{3}\right)ds$$

$$= 1+2x^2+2x - \int_0^x \left(-4s^2 - \frac{8s^3}{3}\right)ds$$

$$\boxed{u_2(x) = 1+2x^2+2x+\frac{4x^3}{3}+\frac{2x^4}{3}}$$

$$u_2'(x) = 2+4x+4x^2+8x^3/3$$

for $n=2$;

3) $\Rightarrow$ $u_3(x) = u_2(x) - \int_0^x \left[u_2'(s) - 2 - 4\int_0^s u_2(r)dr\right]ds$

$$= 1+2x+2x^2+\frac{4x^3}{3}+\frac{2x^4}{3} - \int_0^x \left[\cancel{2}+4s+4s^2+\frac{8s^3}{3}-\cancel{2}\right.$$

$$\left. -4\int_0^s \left(1+2r+2r^2+\frac{4r^3}{3}+\frac{2r^4}{3}\right)dr\right]ds$$

$$= 1+2x+2x^2+\frac{4x^3}{3}+\frac{2x^4}{3} - \int_0^x \left[4s^2+\frac{8s^3}{3}+4s - \right.$$

$$\left. 4\left(s+s^2+\frac{2s^3}{3}+\frac{s^4}{3}+\frac{2s^5}{3\cdot 5}\right)\right]ds$$

$$= 1+2x+2x^2+\frac{4x^3}{3}+\frac{2x^4}{3} - \int_0^x \left[\frac{-4s^4}{3}+\frac{8s^5}{15}\right]ds$$

$$\boxed{u_3(x) = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \frac{4x^5}{15} + \frac{8x^6}{90}}$$

and so on,

$$u_{n+1}(x) = 1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + \frac{(2x)^4}{24} + \cdots$$

$$u_{n+1}(x) = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots$$

$$u(x) = \lim_{n \to \infty} (u_{n+1}(x))$$
$$= e^{2x}$$

Question NO 8: $u(x) = 5 + 2x^2 - \int_0^x (x-t)u(t)\,dt \longrightarrow (1)$

differenciate w.r.t x

$$u'(x) = 4x - \int_0^x u(t)\,dt \longrightarrow (2)$$

Put $x = 0$ in (1)

$u_0(x) = 5$; $u_0'(x) = 0$

The correction function for (2) is

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[u_n'(s) - 4s + \int_0^s u_n(r)\,dr\right] ds \longrightarrow (3)$$

for $n = 0$;

$$3) \Rightarrow \quad u_1(x) = u_0(x) - \int_0^x \left[u_0'(s) - 4s + \int_0^s u_0(r)\,dr\right] ds$$

$$= 5 - \int_0^x \left[0 - 4s + \int_0^s 5\,dr\right] ds$$

$$= 5 - \int_0^x (-4s + 5s)\,ds = 5 - \int_0^x s\,ds$$

$$\boxed{u_1(x) = 5 - \frac{x^2}{2}} \qquad \boxed{u_1'(x) = -x}$$

for $n=1$;

$$3) \Rightarrow \quad u_2(x) = u_1(x) - \int_0^x \left[u_1'(s) - 4s + \int_0^s u_1(r)\,dr\right] ds$$

$$= 5 - \frac{x^2}{2} - \int_0^x \left[-s - 4s + \int_0^s \left(5 - \frac{r^2}{2}\right) dr\right] ds$$

$$= 5 - \frac{x^2}{2} - \int_0^x \left(-\cancel{5s} + \cancel{5s} - \frac{s^3}{3!}\right) ds$$

$$\boxed{u_2(x) = 5 - \frac{x^2}{2!} + \frac{x^4}{4!}} \quad ; \; u_2'(x) = -x + \frac{x^3}{3!}$$

for $n=2$;

$$3) \Rightarrow \quad u_3(x) = u_2(x) - \int_0^x \left[u_2'(s) - 4s + \int_0^s u_2(r)\,dr\right] ds$$

$$= 5 - \frac{x^2}{2!} + \frac{x^4}{4!} - \int_0^x \left[-s + \frac{s^3}{3!} - 4s + \int_0^s \left(5 - \frac{r^2}{2!} + \frac{r^4}{4!}\right) dr\right] ds$$

$$= 5 - \frac{x^2}{2!} + \frac{x^4}{4!} - \int_0^x \left(-\cancel{5s} + \cancel{\frac{s^3}{3!}} + \cancel{5s} - \cancel{\frac{s^3}{3!}} + \frac{s^5}{5!}\right) ds$$

$$\boxed{u_3(x) = 5 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}} \quad \text{and so on}$$

$$u_{n+1}(x) = 5 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$u(x) = \lim_{n \to \infty} u_{n+1}(x)$$

$$= \lim_{n \to \infty} \left(4 + 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$u(x) = 4 + \cos x$$

Question No 9:

$$u(x) = 1 + x + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt \longrightarrow (1)$$

differenciate w.r.t x

$$u'(x) = 1 + x + \int_0^x (x-t)\,u(t)\,dt \longrightarrow (2)$$

Put $x = 0$ in (1)

$u_0(x) = 1$ $\quad$ $u_0'(x) = 0$

The correction function for (2) is

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[u_n'(s) - 1 - s - \int_0^s (s-r)u_n(r)\,dr\right]ds \longrightarrow (3)$$

for $n = 0$ in (3)

$$u_1(x) = u_0(x) - \int_0^x \left[u_0'(x) - 1 - s - \int_0^s (s-r)\,u_0(r)\,dr\right]ds$$

$$= 1 - \int_0^x \left[0 - 1 - s - \int_0^s (s-r)\,dr\right]ds$$

$$= 1 - \int_0^x \left[0 - 1 - s - s^2 + \frac{s^2}{2}\right]ds$$

$$= 1 - \int_0^x (-1 - s - s^2/2)\,ds$$

$$\boxed{u_1(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}}$$

$$u_1'(x) = 1 + x + x^2/2!$$

for $n = 1$;

$$3) \rightarrow \quad u_2(x) = u_1(x) - \int_0^x \left[u_1'(s) - 1 - s - \int_0^s (s-r)\underset{u_1(r)}{}\,dr\right]ds$$

$$u_2(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left(\cancel{1} + \cancel{s} + \frac{s^2}{2!} - \cancel{1} - \cancel{s}\right.$$

$$\left. - \int_0^s (s-r)\left(1 + r + \frac{r^2}{2!} + \frac{r^3}{3!}\right) dr\right] ds$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left[\frac{s^2}{2} - \int_0^s s\left(1 + r + \frac{r^2}{2!} + \frac{r^3}{3!}\right) dr\right.$$

$$\left. + \int_0^s \left(r + r^2 + \frac{r^3}{2!} + \frac{r^4}{3!}\right) dr\right] ds$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left[\frac{s^2}{2!} - s\left(s + \frac{s^2}{2} + \frac{s^3}{3!} + \frac{s^4}{4!}\right)\right.$$

$$\left. + \left(\frac{s^2}{2!} + \frac{s^3}{3} + \frac{s^4}{2 \cdot 4} + \frac{s^5}{3! \cdot 5}\right)\right] ds$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left[\cancel{\frac{s^2}{2!}} - \cancel{s^2} - \frac{s^3}{2} - \frac{s^4}{6} - \frac{s^5}{24}\right.$$

$$\left. + \cancel{\frac{s^2}{2}} + \frac{s^3}{3} + \frac{s^4}{8} + \frac{s^5}{30}\right] ds$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left[\left(-\frac{s^3}{2} + \frac{s^3}{2}\right) + \left(\frac{-s^4}{6} + \frac{s^4}{8}\right)\right.$$

$$\left. + \left(-\frac{s^5}{8} + \frac{s^5}{30}\right)\right] ds$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \int_0^x \left(-\frac{s^3}{6} - \frac{s^4}{24} - \frac{s^5}{120}\right) ds$$

$$\boxed{u_2(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}}$$

$$u(x) = \lim_{n \to \infty} (u_{n+1}(x))$$

$$= \lim_{n \to \infty} \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \right)$$

$u(x) = e^x$

Question NO 10:

$$u(x) = 1 + \frac{x}{2} + \frac{1}{2} \int_0^x (x - t + 1)\, u(t)\, dt \longrightarrow (1)$$

differenciating w.r.t x

$$u'(x) = \frac{1}{2} + \frac{1}{2} \int_0^x u(t)\, dt + \frac{1}{2} u(t) \longrightarrow (2)$$

Put $x = 0$ in (1)

$u_0(x) = 1$, $u_0'(x) = 0$

The correction formula for (2) is

$$u_{n+1}(x) = u_n(x) - \int_0^x \left[u_n'(s) - \frac{1}{2} - \frac{1}{2} u_n(s) - \frac{1}{2} \int_0^s u_n(r)\, dr \right] ds \longrightarrow (3)$$

Put $n = 0$ in (3)

$$u_1(x) = u_0(x) - \int_0^x \left[u_0'(s) - \frac{1}{2} - \frac{1}{2} u_0(s) - \frac{1}{2} \int_0^s u_0(r)\, dr \right] ds$$

$$= 1 - \int_0^x \left[0 - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} \int_0^s dr \right] ds$$

$$= 1 - \int_0^x \left(-1 - \frac{1}{2} s \right) ds$$

$$= 1 - \left(-x - x^2/4 \right)$$

$$\boxed{u_1(x) = 1 + x + x^2/4}$$

$$u_1'(x) = 1 + \frac{x}{2}$$

for $n=1$ in 3

$$u_2(x) = u_1(x) - \int_0^x \left[u_1'(s) - \frac{1}{2} - \frac{1}{2} u_1(s) - \frac{1}{2}\int_0^s u_1(r)\,dr \right] ds$$

$$= 1 + x + \frac{x^2}{4} - \int_0^x \left[1 + \frac{s}{2} - \frac{1}{2} - \frac{1}{2}\left(1 + s + \frac{s^2}{4}\right) - \frac{1}{2}\int_0^s (1 + r + r^2/4)\,dr \right] ds$$

$$= 1 + x + \frac{x^2}{4} - \int_0^x \left[\cancel{1} + \cancel{\frac{s}{2}} - \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} - \cancel{\frac{s}{2}} - \frac{s^2}{8} - \frac{1}{2}\left(s + \frac{s^2}{2} + \frac{s^3}{12}\right) \right] ds$$

$$= 1 + x + \frac{x^2}{4} - \int_0^x \left(-\frac{s^2}{8} - \frac{s}{2} - \frac{s^2}{4} - \frac{s^3}{24} \right) ds$$

$$= 1 + x + \frac{x^2}{4} - \int_0^x \left(-\frac{s}{2} - \frac{3s^2}{8} - \frac{s^3}{24} \right) ds$$

$$= 1 + x + \frac{x^2}{4} + \frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{96}$$

$$\boxed{u_2(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{8} + \frac{x^4}{96}}$$

$$u_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{8} + \frac{x^4}{96} - \int_0^x \left[1 + s + \frac{3s^2}{8} + \frac{s^3}{24} - \frac{1}{2} - \frac{1}{2}\left(1 + s + \frac{s^2}{2} + \frac{s^3}{8} + \frac{s^4}{96}\right) - \frac{1}{2}\int_0^s \left(1 + r + \frac{r^2}{2} + \frac{r^3}{8} + \frac{r^4}{96}\right) dr \right] ds$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{8} + \frac{x^4}{96} - \int_0^x \Big[1 + s + \frac{3s^2}{8} + \frac{s^3}{24} - \frac{1}{2}$$

$$- \frac{1}{2} - \frac{s}{2} - \frac{s^2}{4} - \frac{s^3}{16} - \frac{s^4}{192} - \frac{s}{s} - \frac{s^2}{4} - \frac{s^3}{12}$$

$$- \frac{s^4}{64} - \frac{s^5}{960} \Big] ds$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{8} + \frac{x^4}{96} - \int_0^x \Big[- \frac{s^2}{8} - \frac{s^3}{24} - \frac{s^4}{64} - \frac{s^5}{960} \Big] ds$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{8} + \frac{x^4}{96} + \frac{x^3}{24} + \frac{x^4}{96} + \frac{x^5}{320} + \frac{x^6}{5760}$$

$$u_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{48} + \frac{x^5}{320} + \frac{x^6}{5760}$$

$$u_{n+1}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$u(x) = \lim_{n \to \infty} (u_{n+1}(x)) = e^x$$