The Noise Terms Phenomenon For Voltera Integral Equation Of Second Kind

Earlier, the modified decomposition method use for accelerating the Computational work. But the proper selection of $f_1(x)$ and $f_2(x)$ was necessary for this method.

Then the new technique that accelerate the Convergence of Adomian decomposition method was invented termed as "Noise term Phenomenon". It provides the fastest Convergence of solution. It can be used for all types of differential and integral equation.

The noise term method provides the exact solution by using only first two iteration $U_0(x)$ and $U_1(x)$

The main concept of noise term phenomenon for solving the Volterra integral equations are as follows:-

(1) The noise terms are the identical terms having opposite signs that appear in the $U_0(x)$ and $U_1(x)$. The noise terms may exist for some equations and may not appear for other equations.

(2) The non-cancelled terms of $U_0(x)$ can give the exact solution of the Volterra integral equation. When noise terms between $U_0(x)$ and $U_1(x)$ are cancelled out, even $U_1(x)$ may contains further terms. The existence of noise terms between $U_0(x)$ and $U_1(x)$ is not sufficient to get the exact solution by the cancelation of noise terms. Therefore, the non-cancelled terms of $U_0(x)$ must satisfy the given integral equation. On the other hand if non-cancelled terms of $U_0(x)$ does not satisfy the given Volterra integral equation or noise terms between $U_0(x)$ and $U_1(x)$ did not appear, then the determination of more components of $U(x)$ is necessary to get the series form of solution.

(3) The noise terms appear for the special cases of inhomogenous differential and integral equations, On the other hand the homogenous do not give noise terms.

(4) The appearance of noise terms depends upon necessary conditions. The zeroth component $U_0(x)$ must contain the exact solution $U(x)$ among other terms. It was also proved that inhomogenety condition of equations does not always provide the noise terms.

From the above discussion, we can conclude the following summary:-

The noise terms are identical opposite signs terms that may appear between $U_0(x)$ and $U_1(x)$ or not. They appear only for special cases of inhomogeneous equations.

The phenomenon of Noise terms method can be explained with the help of following examples.

Example #01

$$U(x) = 8x + x^3 - \frac{3}{8}\int_0^x t\,u(t)\,dt$$

Solution:

The standard Adomian method gives the recurrence relation

$$U_0(x) = 8x + x^3$$

$$u_{k+1}(x) = \frac{-3}{8}\int_0^x t\,u_k(t)\,dt\ ,\ k \geq 0$$

This gives

$$u_0(x) = 8x + x^3$$

$$u_1(x) = \frac{-3}{8}\int_0^x t\,u_0(t)\,dt$$

$$u_1(x) = \frac{-3}{8}\int_0^x t\,(8t + t^3)\,dt$$

$$u_1(x) = \frac{-3}{8}\int_0^x (8t^2 + t^4)\,dt$$

$$u_1(x) = \frac{-3}{8}\left[8\frac{t^3}{3}\Big|_0^x + \frac{t^5}{5}\Big|_0^x\right]$$

$$= -\frac{3}{8}\left[\frac{8}{3}x^3 + \frac{x^5}{5}\right]$$

$$u_1(x) = -x^3 - \frac{3}{40}x^5$$

The noise terms $\pm x^3$ appear in $u_0(x)$ and $u_1(x)$. Cancelling this term from zeroth Component $u_0(x)$ gives the exact solution

$$u(x) = 8x$$

Verification

R.H.S of given Volterra integral equation is

$$= 8x + x^3 - \frac{3}{8}\int_0^x t\,u(t)\,dt$$

$$= 8x + x^3 - \frac{3}{8}\int_0^x t\,(8t)\,dt$$

$= 8x + x^3 - \frac{3}{8}\left[8\frac{t^3}{3}\Big|_0^x\right]$

$= 8x + x^3 - x^3$

$= 8x$

$= U(x) = L.H.S$

So exact solution satisfy the given integral equation.

Example #02

$$U(x) = -2 + x + x^2 + \frac{1}{12}x^4 + \sin x + 2\cos x - \int_0^x (x-t)^2 u(t)\,dt$$

Solution:

The standard Adomian method gives the recurrence relation

$$U_0(x) = -2 + x + x^2 + \frac{1}{12}x^4 + \sin x + 2\cos x$$

$$U_{k+1}(x) = \int_0^x (x-t)^2 U_k(t)\,dt \quad , \quad k \geq 0$$

This gives

$$U_0(x) = -2 + x + x^2 + \frac{1}{12}x^4 + \sin x + 2\cos x$$

$$U_1(x) = -\int_0^x (x-t)^2 U_0(t)\,dt$$

$$U_1(x) = -\int_0^x (x-t)^2 \left(-2 + t + t^2 + \frac{1}{12}t^4 + \sin t + 2\cos t\right)dt$$

$$= -\int_0^x (x^2 + t^2 - 2xt)\left(-2 + t + t^2 + \frac{1}{12}t^4 + \sin t + 2\cos t\right)dt$$

$$= -\int_0^x \Big(-2x^2 + x^2 t + x^2 t^2 + \frac{1}{12}x^2 t^4 + x^2 \sin t + 2x^2 \cos t$$
$$- 2t^2 + t^3 + t^4 + \frac{1}{12}t^6 + t^2 \sin t + 2t^2 \cos t$$

$+2x\sin x + 2\cos x - 2 + 2x^2\sin x - 4\left(-t\cos t\Big|_0^x\right.$
$\left.+\int_0^x \cos t\, dt\right) + \cancel{2x^3} - \frac{1}{36}x^7 + 2\cancel{x^2}\cos x - 2\cancel{x}\sin x\Big)$
$-4x^2\sin x - 4x\cos x + 4x\Big]$

$$= -\Big[\frac{1}{12}x^4 + \frac{1}{30}x^5 - \frac{2}{3}x^3 + \frac{1}{1260}x^7 + x^2 + 2\cancel{x^2\sin x}$$

$$+2\cos x - 2 + 2\cancel{x^2\sin x} + \cancel{4x\cos x} + (-4\sin x)$$

$$-4x^2\cancel{\sin x} - 4x\cancel{\cos x} + 4x\Big]$$

$$u_1(x) = 2 - \frac{1}{12}x^4 - \frac{1}{30}x^5 + \frac{2}{3}x^3 - \frac{1}{1260}x^7 - x^2 - 2\cos x$$

$$+4\sin x - 4x$$

The noise terms $\pm 2, \pm x^2, \pm \frac{1}{12}x^4 \pm 2\cos x$ appear in $u_0(x)$ and $u_1(x)$. Cancelling these terms from zeroth component $u_0(x)$ give the exact solution

$$u(x) = x + \sin x$$

Verification:

R.H.S of given Voltera integral equation is

$$= -2 + x + x^2 + \frac{1}{12}x^4 + \sin x + 2\cos x - \int_0^x (x-t)^2 u(t)\, dt$$

$$= -2 + x + x^2 + \frac{1}{12}x^4 + \sin x + 2\cos x - \int_0^x (x^2 + t^2 - 2xt)\, u(t)\, dt$$

$$= -2 + x + x^2 + \frac{1}{12}x^4 + \sin x + 2\cos x - \int_0^x (x^2 + t^2 - 2xt)(t + \sin t)\,dt$$

$$= -2 + x + x^2 + \frac{1}{12}x^4 + \sin x + 2\cos x - \int_0^x [x^2 t + x^2 \sin t + t^3 + t^2 \sin t - 2xt^2 - 2xt\sin t]\,dt$$

$$= -2 + x + x^2 + \frac{1}{12}x^4 + \sin x + 2\cos x - \Big[x^2 \frac{t^2}{2}\Big|_0^x + x^2 \Big| -\cos t \Big|_0^x$$

$$+ \frac{t^4}{4}\Big|_0^x - t^2 \cos t \Big|_0^x + 2\int_0^x t\cos t\,dt - 2x\frac{t^3}{3}\Big|_0^x$$

$$+ 2x\, t\cos t\Big|_0^x - 2x\int_0^x \cos t\,dt \Big]$$

$$= -2 + x + x^2 + \frac{1}{12}x^4 + \sin x + 2\cos x - \Big[\frac{x^4}{2} - \cancel{x^2 \cos x}$$

$$+ x^2 + \frac{x^4}{4} - \cancel{x^2 \cos x} + 2(t\sin t\big|_0^x + \cos t\big|_0^x)$$

$$- \frac{2x^4}{3} + \cancel{2x^2 \cos x} - 2x\sin t\Big|_0^x \Big]$$

$$= -2 + x + x^2 + \frac{1}{12}x^4 + \sin x + 2\cos x - \Big[\frac{1}{12}x^4 + x^2$$

$$\cancel{2x\sin x} + 2\cos x - 2 - \cancel{2x\sin x} \Big]$$

$$= \cancel{-2} + x + \cancel{x^2} + \cancel{\frac{1}{12}x^4} + \sin x + \cancel{2\cos x} - \cancel{\frac{1}{12}x^4} - \cancel{x^2}$$

$$- \cancel{2\cos x} + \cancel{2}$$

$$= x + \sin x = u(x) = \text{L.H.S}$$

So the exact Solution Satisfy the given Volterra integral equation.

Example #03

$$U(x)=\frac{1}{2}x-\frac{1}{4}\sinh(2x)+\sinh^2(x)+\int_0^x u(t)dt$$

Solution:-

The standard Adomian method gives the recurrence relation

$$U_0(x)=\frac{1}{2}x-\frac{1}{4}\sinh(2x)+\sinh^2(x)$$

$$U_{k+1}(x)=\int_0^x U_k(t)dt, \quad k\geq 0$$

This gives

$$U_0(x)=\frac{1}{2}x-\frac{1}{4}\sinh(2x)+\sinh^2(x)$$

$$U_1(x)=\int_0^x U_0(t)dt$$

$$=\int_0^x\left[\frac{1}{2}t-\frac{1}{4}\sinh(2t)+\sinh^2(t)\right]dt$$

$$=\frac{1}{2}\frac{t^2}{2}\Big|_0^x-\frac{1}{4}\frac{\cosh(2t)}{2}\Big|_0^x+\int_0^x\frac{-1+\cosh(2t)}{2}dt$$

$$=\frac{1}{4}x^2-\frac{1}{8}\cosh(2x)+\frac{1}{8}-\frac{1}{2}x+\frac{\sinh(2t)}{4}\Big|_0^x$$

$$U_1(x)=\frac{1}{4}x^2-\frac{1}{8}\cosh(2x)+\frac{1}{8}-\frac{1}{2}x+\frac{\sinh(2x)}{4}$$

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The noise terms $\pm\frac{1}{2}x$ and $\mp\frac{1}{4}\sinh(2x)$ appear in $u_0(x)$ and $u_1(x)$. Canceling these terms from the zeroth component $u_0(x)$ gives the exact solution.

$$u(x) = \sinh^2 x$$

Verification:

R.H.S of given Voltera integral equation is

$$= \frac{1}{2}x - \frac{1}{4}\sinh(2x) + \sinh^2(x) + \int_0^x u(t)\,dt$$

$$= \frac{1}{2}x - \frac{1}{4}\sinh(2x) + \sinh^2(x) + \int_0^x \sinh^2(t)\,dt$$

$$= \frac{1}{2}x - \frac{1}{4}\sinh(2x) + \sinh^2 x + \int_0^x \frac{\cosh(2t) - 1}{2}\,dt$$

$$= \cancel{\frac{1}{2}x} - \frac{1}{4}\sinh(2x) + \sinh^2 x + \frac{\sinh(2t)}{4}\Big|_0^x - \cancel{\frac{1}{2}x}$$

$$= \cancel{-\frac{1}{4}\sinh(2x)} + \sinh^2 x + \cancel{\frac{\sinh(2x)}{4}}$$

$$= \sinh^2 x$$

$$= u(x)$$

$$= L.H.S$$

So the exact solution satisfy the given Volterra integral equation.

Example #04

$$u(x) = -1 + x + \frac{1}{2}x^2 + 2e^x - \int_0^x u(t)\,dt$$

Solution:-

The standard Adomian method gives the recurrence relation

$$u_0(x) = -1 + x + \frac{1}{2}x^2 + 2e^x$$

$$u_{k+1}(x) = -\int_0^x u_k(t)\,dt, \quad k \geq 0$$

This gives

$$u_0(x) = -1 + x + \frac{1}{2}x^2 + 2e^x$$

$$u_1(x) = -\int_0^x u_0(t)\,dt$$

$$= -\int_0^x \left(-1 + t + \frac{1}{2}t^2 + 2e^t\right) dt$$

$$= -\left[-t\Big|_0^x + \frac{t^2}{2}\Big|_0^x + \frac{1}{2}\frac{t^3}{3}\Big|_0^x + 2e^t\Big|_0^x\right]$$

$$= -\left[-x + \frac{x^2}{2} + \frac{x^3}{6} + 2e^x - 2\right]$$

$$u_1(x) = x - \frac{x^2}{2} - \frac{x^3}{6} - 2e^x + 2$$

The noise term $\pm\frac{1}{2}x^2, \pm 2e^x$ appear in $u_0(x)$ and $u_1(x)$. Canceling these terms from zeroth component $u_0(x)$ gives

$$u(x) = x - 1$$

Verification:

R.H.S of given integral equation is

$$= -1 + x + \frac{1}{2}x^2 + 2e^x - \int_0^x (t-1)\,dt$$

$$= -1 + x + \frac{x^2}{2} + 2e^x - \frac{t^2}{2}\Big|_0^x + t\Big|_0^x$$

$$= -1 + x + \cancel{\frac{x^2}{2}} + 2e^x - \cancel{\frac{x^2}{2}} + x$$

$$= -1 + 2x + 2e^x$$

$$\neq U(x)$$

This shows that $x-1$ is not exact. Solution of the given equation. This Confirm our belief that non-Cancelled terms in $U_0(x)$ do not always give exact Solution. and therefore justification is necessary.

The exact Solution of given integral equation Can be obtained by using modified decomposition method.

Let $f(x) = -1 + x + \frac{1}{2}x^2 + 2e^x$

$f_1(x) = x + e^x$

$f_2(x) = -1 + \frac{1}{2}x^2 + e^x$

$U_0(x) = f_1(x) = x + e^x$

and

$$U_1(x) = f_2(x) - \int_0^x U_0(t)\,dt$$

$$= -1 + \frac{1}{2}x^2 + e^x - \int_0^x (t + e^t)\,dt$$

$$= -1 + \frac{1}{2}x^2 + e^x - \left[\frac{t^2}{2}\Big|_0^x + e^t\Big|_0^x\right]$$

$$= -1 + \frac{1}{2}x^2 + e^x - \frac{x^2}{2} - e^x + 1$$

$$u_1(x) = 0$$

$$u_{k+1}(x) = +\int_0^x u_k(t)\,dt \;;\; k \geq 1$$

$$= 0$$

Hence the required Solution is

$$u(x) = x + e^x$$

$\Rightarrow$ Here we have some other examples to understand the noise term phenomenon for Volterra integral equation of second kind.

Question # 01

$$U(x) = 6x + 3x^2 - \int_0^x u(t)\,dt$$

Solution:-

The standard Adomian method gives the recurrence relation

$$U_0(x) = 6x + 3x^2$$

$$U_{k+1}(x) = -\int_0^x u_k(t)\,dt \;,\; k \geq 0$$

This gives

$$U_0(x) = 6x + 3x^2$$

$$U_1(x) = -\int_0^x U_0(t)dt$$

$$= -\int_0^x (6t+3t^2)dt$$

$$= -\left[6\int_0^x t\,dt + 3\int_0^x t^2 dt\right]$$

$$= -\left[6\frac{t^2}{2}\Big|_0^x + 3\frac{t^3}{3}\Big|_0^x\right]$$

$$U_1(x) = -3x^2 - x^3$$

The noise terms $\pm 3x^2$ appear in $U_0(x)$ and $U_1(x)$. Canceling this term from zeroth component $U_0(x)$ gives

$$U(x) = 6x$$

Verification:

R.H.S of given Volterra integral equation is

$$= 6x + 3x^2 - \int_0^x u(t)dt$$

$$= 6x + 3x^2 - \int_0^x 6t\,dt$$

$$= 6x + 3x^2 - 6\frac{t^2}{2}\Big|_0^x$$

$$= 6x + \cancel{3x^2} - \cancel{3x^2}$$

$$= 6x$$

$$= U(x)$$

$$= L.H.S$$

This satisfy the given integral equation so $U(x) = 6x$ is the exact solution.

Question #02

$$U(x) = 6x + 3x^3 - \int_0^x x\, u(t)\,dt$$

Solution:-

The standard Adomian method gives the recurrence relation

$$U_0(x) = 6x + 3x^3$$

$$U_{k+1}(x) = -\int_0^x x\, U_k(t)\,dt \;;\; k \geq 0$$

This gives

$$U_0(x) = 6x + 3x^3$$

$$U_1(x) = -\int_0^x x\, U_0(t)\,dt$$

$$U_1(x) = -\int_0^x x\,(6t + 3t^3)\,dt$$

$$U_1(x) = -x\int_0^x (6t + 3t^3)\,dt$$

$$= -x\left[6\frac{t^2}{2}\Big|_0^x + 3\frac{t^4}{4}\Big|_0^x\right]$$

$$= -x\left[3x^2 + \frac{3x^4}{4}\right]$$

$$U_1(x) = -3x^3 - \frac{3}{4}x^4$$

The noise term $\pm 3x^3$ appear in $U_0(x)$ and $U_1(x)$. Canceling this term from $U_0(x)$. $U_0(x)$ gives

$U(x) = 6x$

Verification:

R.H.S of given integral equation is

$$= 6x + 3x^3 - \int_0^x x\,U(t)\,dt$$

$$= 6x + 3x^3 - \int_0^x x(6t)\,dt$$

$$= 6x + 3x^3 - x\left|6\frac{t^2}{2}\right|_0^x$$

$$= 6x + \cancel{3x^3} - \cancel{3x^3}$$

$$= 6x$$

$$= U(x) = \text{L.H.S}$$

So this satisfy the given integral equation

So. $U(x) = 6x$ is the exact solution.

Question #03

$$U(x) = 6x + 2x^3 - \int_0^x t\,u(t)\,dt$$

Solution:-

The standard Adomian method gives the recurrence relation

$$U_0(x) = 6x + 2x^3$$

$$U_{k+1}(x) = -\int_0^x t\,U_k(t)\,dt \quad ; \quad k \geq 0$$

This gives

$$U_0(x) = 6x + 2x^3$$

$$u_1(x) = -\int_0^x t\, u_0(t)\,dt$$

$$= -\int_0^x t\,(6t+2t^3)\,dt$$

$$= -\int_0^x (6t^2+2t^4)\,dt$$

$$= -\left[\int_0^x 6t^2\,dt + 2\int_0^x t^4\,dt\right]$$

$$= -\left[6\,\frac{t^3}{3}\Big|_0^x + 2\,\frac{t^5}{5}\Big|_0^x\right]$$

$$u_1(x) = -2x^3 - \frac{2}{5}x^5$$

The noise terms $\pm 2x^3$ appear in $u_0(x)$ and $u_1(x)$. Canceling this term from zeroth component $u_0(x)$ gives

$$u(x) = 6x$$

Verification:-

R.H.S of given voltera integral equation is

$$= 6x + 2x^3 - \int_0^x t\, u(t)\,dt$$

$$= 6x + 2x^3 - \int_0^x t\,(6t)\,dt$$

$$= 6x + 2x^3 - 6\int_0^x t^2\,dt$$

$$= 6x + 2x^3 - 6\,\frac{t^3}{3}\Big|_0^x$$

$$= 6x + \cancel{2x^3} - \cancel{2x^3}$$

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$= 6x$

$= U(x) = L.H.S$

This satisfy the given Volterra integral equation

So $U(x) = 6x$ is the exact Solution.

Question #04

$$U(x) = x + x^2 - 2x^3 - x^4 + 12\int_0^x (x-t)u(t)dt$$

Solution:-

The standard Adomian method gives the recurrence relation

$U_0(x) = x + x^2 - 2x^3 - x^4$

$U_{k+1}(x) = 12\int_0^x (x-t)U_k(t)dt \; ; k \geq 0$

This gives

$U_0(x) = x + x^2 - 2x^3 - x^4$

$U_1(x) = 12\int_0^x (x-t)U_0(t)dt$

$U_1(x) = 12\int_0^x (x-t)(t+t^2-2t^3-t^4)dt$

$= 12\left[\int_0^x (xt + xt^2 - 2xt^3 - xt^4 - t^2 - t^3 + 2t^4 + t^5)dt\right]$

$= 12\left[x\frac{t^2}{2}\Big|_0^x + x\frac{t^3}{3}\Big|_0^x - 2x\frac{t^4}{4}\Big|_0^x - x\frac{t^5}{5}\Big|_0^x - \frac{t^3}{3}\Big|_0^x \right.$

$\left. - \frac{t^4}{4}\Big|_0^x + \frac{2}{5}t^5\Big|_0^x + \frac{t^6}{6}\Big|_0^x \right]$

$$= 12\left[\frac{x^3}{3} + \frac{x^4}{3} - \frac{1}{2}x^5 - \frac{x^6}{5} - \frac{x^3}{3} - \frac{x^4}{4} + \frac{2}{5}x^5 + \frac{x^6}{6}\right]$$

$$= 12\left[\frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{10} - \frac{x^6}{30}\right]$$

$$u_1(x) = 2x^3 + x^4 - \frac{6}{5}x^5 - \frac{2}{5}x^6$$

The noise terms $\mp 2x^3, \mp x^4$ appear in $u_0(x)$ and $u_1(x)$. Canceling this term from Zeroth Component. $U_0(x)$ gives

$$u(x) = x + x^2$$

Verification:

R.H.S of given integral equation is

$$= x + x^2 - 2x^3 - x^4 + 12\int_0^x (x-t)u(t)\,dt$$

$$= x + x^2 - 2x^3 - x^4 + 12\int_0^x (x-t)(t+t^2)\,dt$$

$$= x + x^2 - 2x^3 - x^4 + 12\left[\int_0^x (xt + xt^2 - t^2 - t^3)\,dt\right]$$

$$= x + x^2 - 2x^3 - x^4 + 12\left[x\frac{t^2}{2}\Big|_0^x + x\frac{t^3}{3}\Big|_0^x - \frac{t^3}{3}\Big|_0^x - \frac{t^4}{4}\Big|_0^x\right]$$

$$= x + x^2 - 2x^3 - x^4 + 12\left[\frac{x^3}{2} + \frac{x^4}{3} - \frac{x^3}{3} - \frac{x^4}{4}\right]$$

$$= x + x^2 - 2x^3 - x^4 + 12\left[\frac{x^3}{6} + \frac{x^4}{12}\right]$$

$$= x + x^2 - \cancel{2x^3} - \cancel{x^4} + \cancel{2x^3} + \cancel{x^4}$$

$= x + x^2$

$= U(x) = L.H.S$

This Satisfy the given Volterra integral equation so the exact Solution is

$$U(x) = x + x^2$$

Question #05

$$U(x) = -2 + x^2 + \sin x + 2\cos x - \int_0^x (x-t)^2 u(t)\,dt$$

Solution:

The standard Adomian method gives the recurrence relation

$$U_0(x) = -2 + x^2 + \sin x + 2\cos x$$

$$U_{k+1}(x) = -\int_0^x (x-t)^2 U_k(t)\,dt, \quad k \geq 0$$

This gives

$$U_0(x) = -2 + x^2 + \sin x + 2\cos x$$

$$U_1(x) = -\int_0^x (x-t)^2 U_0(t)\,dt$$

$$U_1(x) = -\int_0^x (x^2 + t^2 - 2xt)(-2 + t^2 + \sin t + 2\cos t)\,dt$$

$$= -\int_0^x \Big[-2x^2 + x^2t^2 + x^2\sin t + 2x^2\cos t - 2t^2 + t^4 + t^2\sin t$$

$$+ 2t^2\cos t + 4xt - 2xt^3 - 2xt\sin t - 4xt\cos t\Big]\,dt$$

$$= -\Big[-2x^2\int_0^x dt + x^2\int_0^x t^2 dt + x^2\int_0^x \sin t\, dt + 2x^2\int_0^x \cos t\, dt$$

$$-2\int_0^x t^2 dt + \int_0^x t^2\sin t\, dt + \int_0^x t^4 dt + 2\int_0^x t^2\cos t\, dt$$

$$+4x\int_0^x t\,dt - 2x\int_0^x t^3 dt - 2x\int_0^x t\sin t\, dt - 4x\int_0^x t\cos t\, dt\Big]$$

$$= -\Big[-2x^2 t\Big|_0^x + x^2\frac{t^3}{3}\Big|_0^x + x^2\Big|-\cos t\Big|_0^x + 2x^2\sin t\Big|_0^x$$

$$-\frac{2}{3}t^3\Big|_0^x + \left(-t^2\cos t\Big|_0^x + 2\int_0^x t\cos t\, dt\right) + \frac{t^5}{5}\Big|_0^x$$

$$+2\left(t^2\sin t\Big|_0^x - 2\int_0^x t\sin t\,dt\right) + 4x\frac{t^2}{2}\Big|_0^x - 2x\frac{t^4}{4}\Big|_0^x$$

$$-2x\left(-t\cos t\Big|_0^x + \int_0^x \cos t\, dt\right) - 4x\left(t\sin t\Big|_0^x - \int_0^x \sin t\, dt\right)\Big]$$

$$= -\Big[-2x^3 + \frac{x^5}{3} - x^2\cos x + x^2 + 2x^2\sin x - \frac{2}{3}x^3 - x^2\cos x$$

$$+2\left(t\sin t\Big|_0^x - \int_0^x \sin t\, dt\right) + \frac{x^5}{5} + 2\left(x^2\sin x - 2\left(-t\cos t\Big|_0^x\right.\right.$$

$$\left.\left.+\int_0^x \cos t\, dt\right)\right) + 4\frac{x^3}{2} - \frac{x^5}{2} - 2x(-x\cos x + \sin x)$$

$$-4x(x\sin x + \cos x - 1)\Big]$$

$$= -\Big[-\cancel{2x^3} + \frac{x^5}{3} - \frac{x^5}{2} + \frac{x^5}{5} + x^2 - \cancel{x^2\cos x} + \cancel{2x^2\sin x} - \frac{2}{3}x^3$$

$$+ 2x\cancel{\sin x} + 2\cos x - 2 + \cancel{2x^2\sin x} + 4x\cancel{\cos x} - 4\sin x - \cancel{x^2\cos x}$$

$$+\cancel{2x^3} + 2x^2\cancel{\cos x} - 2\cancel{x}\sin x - 4\cancel{x^2}\sin x - 4\cancel{x}\cos x$$

$$+4x\Big]$$

$$= -\left[\frac{-2}{3}x^3 + \frac{x^5}{30} + x^2 + 2\cos x - 2 - 4\sin x + 4x\right]$$

$$= \frac{2}{3}x^3 - \frac{x^5}{30} - x^2 - 2\cos x + 2 + 4\sin x - 4x$$

The noise terms $\pm 2, \pm x^2, \pm 2\cos x$ appear in $U_0(x)$ and $U_1(x)$. Canceling these terms from zeroth Component. $U_0(x)$ gives

$U(x) = \sin x$

Verification:-

R.H.S of given integral equation is

$$= -2 + x^2 + \sin x + 2\cos x - \int_0^x (x-t)^2 u(t)\,dt$$

$$= -2 + x^2 + \sin x + 2\cos x - \int_0^x (x-t)^2 \sin t\,dt$$

$$= -2 + x^2 + \sin x + 2\cos x - \int_0^x (x^2 + t^2 - 2xt)\sin t\,dt$$

$$= -2 + x^2 + \sin x + 2\cos x - \left[\int_0^x (x^2\sin t + t^2\sin t - 2xt\sin t)\,dt\right]$$

$$= -2 + x^2 + \sin x + 2\cos x - \int_0^x x^2\sin t\,dt - \int_0^x t^2\sin t\,dt$$

$$+ 2x\int_0^x t\sin t\,dt$$

$$= -2 + x^2 + \sin x + 2\cos x + x^2\cos t\Big|_0^x - \left(-t^2\cos t\Big|_0^x\right.$$

$$\left. + \int_0^x 2t\cos t\,dt\right) + 2x\left(-t\cos t\Big|_0^x + \int_0^x \cos t\,dt\right.$$

$$= -2 + \cancel{x^2} + \sin x + 2\cos x + x^2\cos x - \cancel{x^2} - \left(-x^2\cos x\right.$$

$$\left. + 2\left(t\sin t\Big|_0^x + \cos t\Big|_0^x\right)\right) + 2x\left(-x\cos x + \sin x\right)$$

$= \cancel{2} + \sin x + \cancel{2\cos x} + x^2\cancel{\cos x} + \cancel{x^2\cos x} - \cancel{2x\sin x}$
$- \cancel{2\cos x} + \cancel{2} - \cancel{2x^2\cos x} + \cancel{2x\sin x}$

$= \sin x$
$= U(x)$
$= L.H.S$

This satisfy the given Volterra integral equation so
$U(x) = \sin x$ is the exact Solution.

Question #06

$$U(x) = 2x - 2\sin x + \cos x - \int_0^x (x-t)^2 u(t)\,dt$$

Solution:-

The standard Adomian method gives the recurrence relation

$$U_0(x) = 2x - 2\sin x + \cos x$$
$$U_{k+1}(x) = -\int_0^x (x-t)^2 U_k(t)\,dt, \quad k \geq 0$$

This gives

$$U_0(x) = 2x - 2\sin x + \cos x$$
$$U_1(x) = -\int_0^x (x-t)^2 U_0(t)\,dt$$
$$U_1(x) = -\int_0^x (x^2+t^2-2xt)(2t - 2\sin t + \cos t)\,dt$$

$$= -\left[\int_0^x \left(2x^2t - \cancel{2x^2t} - 2x^2\sin t + x^2\cos t + 2t^3 - 2t^2\sin t + t^2\cos t - 4xt^2 + 4xt\sin t - 2xt\cos t\right)dt\right]$$

$$= -\left[2x^2\int_0^x t\,dt - \cancel{2x^2\int t} - 2x^2\int_0^x \sin t\,dt + x^2\int_0^x \cos t\,dt\right.$$

$$+ 2\int_0^x t^3dt - 2\int_0^x t^2\sin t\,dt + \int_0^x t^2\cos t\,dt - 4x\int_0^x t^2dt$$

$$\left. + 4x\int_0^x t\sin t\,dt - 2x\int_0^x t\cos t\,dt\right]$$

$$= -\left[2x^2\frac{t^2}{2} + 2x^2\cos t\Big|_0^x + x^2\sin t\Big|_0^x + 2\frac{t^4}{4}\Big|_0^x\right.$$

$$+ 2t^2\cos t\Big|_0^x - 4\int_0^x t\cos t\,dt + t^2\sin t\Big|_0^x - 2\int_0^x t\sin t\,dt.$$

$$-4x\frac{t^3}{3}\Big|_0^x + 4x\left(-t\cos t\Big|_0^x + \int_0^x \cos t\,dt\right)$$

$$\left.-2x\left(t\sin t\Big|_0^x - \int_0^x \sin t\,dt\right)\right]$$

$$= -\left[x^4 + 2x^2\cos x - 2x^2 + x^2\sin x + \frac{1}{2}x^4 + 2x^2\cos x\right.$$

$$-4\left(t\sin t\Big|_0^x - \int_0^x \sin t\,dt\right) + x^2\sin x - 2\left(-t\cos t\Big|_0^x + \sin x\right)$$

$$\left.-4\frac{x^4}{3} + 4x(-x\cos x + \sin x) - 2x^2\sin x - 2x\cos x + 2x\right]$$

$$= -\left[x^4 + 2x^2\cancel{\cos x} - 2x^2 + x^2\cancel{\sin x} + \frac{1}{2}x^4 + 2x^2\cancel{\cos x}\right.$$

$$-4x\cancel{\sin x} - 4\cos x + 4 + x^2\cancel{\sin x} + 2x\cancel{\cos x} - 2\sin x$$

$$-\frac{4}{3}x^4 - 4x^2\cancel{\cos x} + 4x\cancel{\sin x} - 2x^2\sin x - 2x\cancel{\cos x}$$

$$\left.+ 2x\right]$$

$$= -\left[x^4 + \frac{x^4}{2} - \frac{4}{3}x^4 - 2x^2 - 4\cos x + 4 - 2\sin x + 2x\right]$$

$$= -\left(\frac{6x^4+3x^4-8x^4}{6} - 2x^2 - 4\cos x + 4 - 2\sin x + 2x\right)$$

$$= -\frac{x^4}{6} + 2x^2 + 4\cos x - 4 + 2\sin x - 2x$$

The noise term $\pm 2x$, $\pm 2\sin x$ appear in $u_0(x)$ and $u_1(x)$. Canceling these terms from Zeroth Component. $u_0(x)$ gives

$$U(x) = \cos x$$

Verification:-

R.H.S of given Volterra integral equation is

$$= 2x - 2\sin x + \cos x - \int_0^x (x-t)^2 u(t)\,dt$$

$$= 2x - 2\sin x + \cos x - \int_0^x (x^2+t^2-2xt)\cos t\,dt$$

$$= 2x - 2\sin x + \cos x - \int_0^x x^2\cos t\,dt - \int_0^x t^2\cos t\,dt$$

$$+ 2x\int_0^x t\cos t\,dt$$

$$= 2x - 2\sin x + \cos x - x^2 \sin t\Big|_0^x - \left(t^2\sin t\Big|_0^x - 2\int_0^x t\sin t\,dt\right)$$

$$+ 2x\left(t\sin t\Big|_0^x - \int_0^x \sin t\,dt\right)$$

$$= \cancel{2x} - 2\sin x + \cos x - x^2\sin x - \left(x^2\sin x - 2\left(-t\cos t\right|_0^x + \int_0^x \cos t\,dt\right)\right) + 2x^2\sin x + 2x\cos x - \cancel{2x}$$

$$= -2\cancel{\sin x} + \cos x - x^2\cancel{\sin x} - \cancel{x^2}\sin x - 2\cancel{x}\cos x + 2\cancel{\sin x} + 2\cancel{x^2}\sin x + 2\cancel{x}\cos x$$

$$= \cos x$$
$$= U(x)$$
$$= L.H.S$$

This satisfy the given integral equation
So $U(x) = \cos x$ is the exact solution.

Question #07

$$U(x) = \sinh x + x\sinh x - x^2\cosh x + \int_0^x xt\,u(t)\,dt$$

Solution:

The standard Adomian method gives the recurrence relation

$$U_0(x) = \sinh x + x\sinh x - x^2\cosh x$$
$$U_{k+1}(x) = +\int_0^x xt\,U_k(t)\,dt \; ; \; k \geq 0$$

This gives

$$U_0(x) = \sinh x + x\sinh x - x^2\cosh x$$
$$U_1(x) = \int_0^x xt\,U_0(t)\,dt$$

$$U_1(x) = \int_0^x xt\left(\sinh t + t\sinh t - t^2\cosh t\right)dt$$

$$= x\left[\int_0^x t\sinh t\,dt + \int_0^x t^2\sinh t\,dt - \int_0^x t^3\cosh t\,dt\right]$$

$$= x\left[t\cosh t\Big|_0^x - \int_0^x \cosh t\,dt + t^2\cosh t\Big|_0^x - 2\int_0^x t\cosh t\,dt - \int_0^x t^3\cosh t\,dt\right]$$

$$= x\left[x\cosh x - \sinh x + x^2\cosh x - 2\left(t\sinh t\Big|_0^x - \int_0^x \sinh t\,dt\right) - \left(t^3\sinh t\Big|_0^x - 3\int_0^x t^2\sinh t\,dt\right)\right]$$

$$= x\left[x\cosh x - \sinh x + x^2\cosh x - 2x\sinh x + 2\cosh x - 2 - x^3\sinh x + 3\left(t^2\cosh t\Big|_0^x - 2\int_0^x t\cosh t\,dt\right)\right]$$

$$= x\left[x\cosh x - \sinh x + x^2\cosh x - 2x\sinh x + 2\cosh x - 2 - x^3\sinh x + 3x^2\cosh x - 6\left(t\sinh t\Big|_0^x - \int_0^x \sinh t\,dt\right)\right]$$

$$= x\left[x\cosh x - \sinh x + x^2\cosh x - 2x\sinh x + 2\cosh x - 2 - x^3\sinh x + 3x^2\cosh x - 6x\sinh x + 6\cosh x - 6\right]$$

$$= x^2\cosh x - x\sinh x + x^3\cosh x - 2x^2\sinh x + 2x\cosh x - x^4\sinh x + 3x^3\cosh x - 6x^2\sinh x + 6x\cosh x - 6x$$

The noise term $\pm x^2\cosh x$, $\pm x\sinh x$ appear in $U_0(x)$ and $U_1(x)$. Canceling these terms from Zeroth Component. $U_0(x)$ give

$$U(x) = \sinh x$$

Verification:

R.H.S of given integral equation is

$$= \sinh x + x\sinh x - x^2\cosh x + \int_0^x xt\,u(t)\,dt$$

$$= \sinh x + x\sinh x - x^2\cosh x + x\int_0^x t\sinh t\,dt$$

$$= \sinh x + x\sinh x - x^2\cosh x + x\left(t\cosh t\Big|_0^x - \int_0^x \cosh t\,dt\right)$$

$$= \sinh x + x\sinh x - x^2\cosh x + x(x\cosh x - \sinh x)$$

$$= \sinh x + \cancel{x\sinh x} - \cancel{x^2\cosh x} + \cancel{x^2\cosh x} - \cancel{x\sinh x}$$

$$= \sinh x$$

$$= U(x) = L.H.S$$

This is the exact solution and satisfies the given Volterra integral equation.

Question # 08

$$U(x) = x + \cosh x + x^2\sinh x - x\cosh x - \int_0^x xt\,u(t)\,dt$$

Solution:-

The standard Adomian method gives the recurrence relation

$$U_0(x) = x + \cosh x + x^2\sinh x - x\cosh x$$

$$U_{k+1}(x) = -\int_0^x xt\,U_k(t)\,dt \; ; \; k \geq 0$$

This gives

$$U_0(x) = x + \cosh x + x^2\sinh x - x\cosh x$$

$$U_1(x) = -\int_0^x xt\, U_0(t)\,dt$$

$$U_1(x) = -\int_0^x xt\,(t + \cosh t + t^2\sinh t - t\cosh t)\,dt$$

$$= -x\left[\int_0^x t^2\,dt + \int_0^x t\cosh t\,dt + \int_0^x t^3\sinh t\,dt - \int_0^x t^2\cosh t\,dt\right]$$

$$= -x\left[\frac{t^3}{3}\Big|_0^x + t\sinh t\Big|_0^x - \int_0^x \sinh t\,dt + t^3\cosh t\Big|_0^x - 3\int_0^x t^2\cosh t\,dt\right.$$

$$\left. - t^2\sinh t\Big|_0^x + 2\int_0^x t\sinh t\,dt\right]$$

$$= -x\left[\frac{x^3}{3} + x\sinh x - \cosh t\Big|_0^x + x^3\cosh x - 3\left(t^2\sinh t\Big|_0^x\right.\right.$$

$$\left.\left. - 2\int_0^x t\sinh t\,dt\right) - x^2\sinh x + 2\left(t\cosh t\Big|_0^x - \int_0^x \cosh t\,dt\right)\right]$$

$$= -x\left[\frac{x^3}{3} + x\sinh x - \cosh x + 1 + x^3\cosh x - 3x^2\sinh x\right.$$

$$+ 6\left(t\cosh t\Big|_0^x - \int_0^x \cosh t\,dt\right) - x^2\sinh x + 2x\cosh x$$

$$\left. - 2\sinh x\right]$$

$$= -x\left[\frac{x^3}{3} + x\sinh x - 3x^2\sinh x - \cosh x + 1 + x^3\cosh x\right.$$

$$+ 6x\cosh x - 6\sinh x - x^2\sinh x + 2x\cosh x$$

$$\left. - 2\sinh x\right]$$

$$= -x\left[\frac{x^3}{3} + x\sinh x - \cosh x + 1 + x^3\cosh x - 3x^2\sinh x\right.$$

$$\left. + 8x\cosh x - 8\sinh x - x^2\sinh x\right]$$

$$= -\frac{x^4}{3} - x^2\sinh x + x\cosh x - x - x^4\cosh x + 4x^3\sinh x$$

$$- 8x^2\cosh x + 8x\sinh x$$

The noise term $\pm x^2\sinh x$, $\pm x\cosh x$, $\pm x$ appear in $U_0(x)$ and $U_1(x)$. Canceling these terms from Zeroth Component $U_0(x)$ gives

$$U(x) = \cosh x$$

Verification:-

R.H.S of given integral equation is

$$= x + \cosh x + x^2\sinh x - x\cosh x - \int_0^x xt\,u(t)\,dt$$

$$= x + \cosh x + x^2\sinh x - x\cosh x - x\int_0^x t\cosh t\,dt$$

$$= x + \cosh x + x^2\sinh x - x\cosh x - x\left(t\sinh t\Big|_0^x\right.$$

$$\left. - \int_0^x \sinh t\,dt\right)$$

$$= x + \cosh x + x^2\sinh x - x\cosh x - x(x\sinh x$$

$$- \cosh x + 1)$$

$= \cancel{x} + \cosh x + x^2 \sinh x - x\cosh x - x^2 \sinh x + x\cosh x$
$\cancel{-x}$
$= \cosh x$
$= U(x)$
$= L.H.S$

This is exact solution and satisfy the given volterra integral equation.

Question #09

$$U(x) = \sec^2 x - \tan x + \int_0^x u(t)\,dt$$

Solution:

The standard Adomian method gives the recurrence relation

$$U_0(x) = \sec^2 x - \tan x$$
$$U_{k+1}(x) = \int_0^x U_k(t)\,dt \;;\; k \geq 0$$

This gives

$$U_0(x) = \sec^2 x - \tan x$$
$$U_1(x) = \int_0^x U_0(t)\,dt$$
$$= \int_0^x (\sec^2 t - \tan t)\,dt$$
$$= \tan t\Big|_0^x + \ln|\cos t|\Big|_0^x$$
$$U_1(x) = \tan x + \ln|\cos x|$$

The noise term appear $\pm \tan x$ in $U_0(x)$ and $U_1(x)$. Canceling this term from zeroth component $U_0(x)$ gives

$$U(x) = \sec^2 x$$

Verification:

R.H.S of given integral equation is

$$= \sec^2 x - \tan x + \int_0^x u(t)\,dt$$

$$= \sec^2 x - \tan x + \int_0^x \sec^2 t\,dt$$

$$= \sec^2 x - \tan x + \left|\tan t\right|_0^x$$

$$= \sec^2 x - \cancel{\tan x} + \cancel{\tan x}$$

$$= \sec^2 x$$

$$= U(x)$$

$$= L.H.S$$

This is exact solution and satisfy the given integral equation.

Question #10

$$U(x) = -\frac{1}{2}x - \frac{1}{4}\sin(2x) + \cos^2 x + \int_0^x u(t)\,dt$$

Solution:

The standard Adomian method gives the recurrence relation

$$U_0(x) = -\frac{1}{2}x - \frac{1}{4}\sin(2x) + \cos^2 x$$

$$U_{k+1}(x) = \int_0^x U_k(t)\,dt \;;\; k \geq 0$$

This gives

$$U_0(x) = -\frac{1}{2}x - \frac{1}{4}\sin(2x) + \cos^2 x$$

$$U_1(x) = \int_0^x U_0(t)\,dt$$

$$= \int_0^x \left(-\frac{1}{2}t - \frac{1}{4}\sin(2t) + \cos^2 t\right)dt$$

$$= -\frac{1}{2}\frac{t^2}{2}\Big|_0^x + \frac{\cos(2t)}{8}\Big|_0^x + \int_0^x \frac{1+\cos 2t\,dt}{2}$$

$$= -\frac{x^2}{4} + \frac{\cos(2x)}{8} - \frac{1}{8} + \frac{1}{2}x + \frac{\sin 2t}{4}\Big|_0^x$$

$$U_1(x) = -\frac{x^2}{4} + \frac{\cos(2x)}{8} - \frac{1}{8} + \frac{1}{2}x + \frac{\sin(2x)}{4}$$

The noise terms $\pm\frac{1}{2}x$, $\pm\frac{1}{4}\sin(2x)$ appear in $U_0(x)$ and $U_1(x)$. Canceling these terms from Zeroth Component $U_0(x)$ gives

$$U(x) = \cos^2 x$$

Verification:

R.H.S of given Volterra integral equation

$$= -\frac{x}{2}x - \frac{1}{4}\sin(2x) + \cos^2 x + \int_0^x U(t)\,dt$$

$$= -\frac{x}{2} - \frac{\sin(2x)}{4} + \cos^2 x + \int_0^x \cos^2 t\,dt$$

$$= -\frac{x}{2} - \frac{\sin(2x)}{4} + \cos^2 x + \int_0^x \left(\frac{1+\cos 2t}{2}\right)dt$$

$$= -\frac{x}{2} - \frac{\sin(2x)}{4} + \cos^2 x + \frac{1}{2}\Big|t\Big|_0^x + \frac{\sin 2t}{4}\Big|_0^x$$

$$= \cancel{-\frac{x}{2}} - \cancel{\frac{\sin(2x)}{4}} + \cos^2 x + \cancel{\frac{x}{2}} + \cancel{\frac{\sin(2x)}{4}}$$

$$= \cos^2 x$$

$$= U(x) = L.H.S$$

Hence $u(x) = \cos^2 x$ satisfy the given integral equation so it is exact solution.

Question # 11

$$U(x) = -\frac{1}{2}x + \frac{1}{4}\sin 2x + \sin^2 x + \int_0^x u(t)\,dt$$

Solution:-

The standard Adomian method gives the recurrence relation

$$U_0(x) = -\frac{1}{2}x + \frac{1}{4}\sin(2x) + \sin^2 x$$

$$U_{k+1}(x) = \int_0^x U_k(t)\,dt \quad ; k \geq 0$$

This gives

$$U_0(x) = -\frac{1}{2}x + \frac{1}{4}\sin(2x) + \sin^2 x$$

$$U_1(x)=\int_0^x U_0(t)\,dt$$

$$=\int_0^x\left(-\frac{1}{2}t+\frac{1}{4}\sin(2t)+\sin^2 t\right)dt$$

$$=-\frac{1}{2}\frac{t^2}{2}\Big|_0^x-\frac{\cos(2t)}{8}\Big|_0^x+\int_0^x\left(\frac{1-\cos(2t)}{2}\right)dt$$

$$U_1(x)=-\frac{x^2}{4}-\frac{\cos 2x}{8}+\frac{1}{8}+\frac{1}{2}x-\frac{\sin 2x}{4}$$

The noise term $\pm\frac{1}{2}x$, $\pm\frac{\sin 2x}{4}$ appear in $U_0(x)$ and $U_1(x)$. canceling this term from Zeroth component. $U_0(x)$ gives

$$U(x)=\sin^2 x$$

Verification:-

R.H.S of given integral equation is

$$=-\frac{1}{2}x+\frac{1}{4}\sin(2x)+\sin^2 x+\int_0^x U(t)\,dt$$

$$=-\frac{1}{2}x+\frac{1}{4}\sin(2x)+\sin^2 x+\int_0^x \sin^2 t\,dt$$

$$=-\frac{1}{2}x+\frac{1}{4}\sin(2x)+\sin^2 x+\int_0^x\left(\frac{1-\cos 2t}{2}\right)dt$$

$$=\cancel{-\frac{1}{2}x}+\cancel{\frac{1}{4}\sin(2x)}+\sin^2 x+\cancel{\frac{1}{2}x}-\cancel{\frac{\sin(2x)}{4}}$$

$$=\sin^2 x$$

$$=U(x)$$

$$=L.H.S$$

So $U(x)=\sin^2 x$ is exact solution as

it satisfy the given Volterra integral equation.

Question #12

$$U(x) = -x + \tan x + \tan^2 x - \int_0^x u(t)\,dt$$

Solution:-

The standard Adomian method ~~of~~ gives the recurrence relation

$$U_0(x) = -x + \tan x + \tan^2 x$$

$$U_{k+1}(x) = -\int_0^x u_k(t)\,dt \; ; \; k \geq 0$$

This gives

$$U_0(x) = -x + \tan x + \tan^2 x$$

$$u_1(x) = -\int_0^x U_0(t)\,dt$$

$$u_1(x) = -\int_0^x (-t + \tan t + \tan^2 t)\,dt$$

$$= -\left[-\frac{t^2}{2}\Big|_0^x + \int_0^x \frac{\sin t}{\cos t}\,dt + \int_0^x (\sec^2 t - 1)\,dt \right]$$

$$= -\left[\frac{-x^2}{2} - \ln \cos t\Big|_0^x + \tan t\Big|_0^x - t\Big|_0^x \right]$$

$$= -\left[\frac{-x^2}{2} - \ln \cos(x) + \ln(1) + \tan x - x \right]$$

$$U_1(x) = \frac{x^2}{2} + \ln \cos x - \tan x + x$$

The noise terms $\pm x$, $\pm \tan x$ appear in $U_0(x)$ and $U_1(x)$. Canceling these terms from zeroth component. $U_0(x)$ gives

$$U(x) = \tan^2 x$$

Verification:-

R.H.S of given Volterra integral equation is

$$= -x + \tan x + \tan^2 x - \int_0^x u(t)\,dt$$

$$= -x + \tan x + \tan^2 x - \int_0^x \tan^2 t\,dt$$

$$= -x + \tan x + \tan^2 x - \int_0^x (\sec^2 t - 1)\,dt$$

$$= -x + \tan x + \tan^2 x - \tan t\Big|_0^x + t\Big|_0^x$$

$$= \cancel{-x} + \cancel{\tan x} + \tan^2 x - \cancel{\tan x} + \cancel{x}$$

$$= \tan^2 x$$

$$= U(x)$$

$$= L.H.S$$

This satisfy the given integral equation so $U(x) = \tan^2 x$ is the exact solution.