Integral Equations

Date: ___/___/ 2020

M T W T F S

The Adomian Decomposition Method For Volterra Integral Equations.

The Adomian Decomposition Method was presented by G. Adomian

* The Adomian Decomposition Method Consist of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series as

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad \text{or} \quad \rightarrow (1)$$

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \dots,$$

where the components $u_n(x), n \geq 0$ are to be determine in a recursive manner. This method consist of finding u_0, u_1, u_2 — individually which can be achieved by a recurrence relation that usually

Parent's Sig: ______ Teacher's Sig: ______

involves simple integrals that can easily be evaluated

As the Voltera Integral Equ is

$$u(x) = f(x) + \lambda \int_0^x K(x,t)\, u(t)\,dt \rightarrow ②$$

Using ① in ②

$$\sum_0^\infty u_n(x) = f(x) + \lambda \int_0^x K(x,t) \sum_0^\infty u_n(t)\,dt$$

or

$$u_0(x) + u_1(x) + \dots = f(x) + \lambda \int_0^x \left(K(x,t) \{ u_0(t) + u_1(t) + \dots \} \right) dt$$

by Comparison

The zeroth component $u_0(x)$ is identified all the terms that are not included under the integral sign so.

$u_0(x) = f(x)$ and

recurrance relation is

$$u_{n+1}(x) = \lambda \int_0^x K(x,t)\, u_n(t)\,dt \quad , n \geq 0$$

so

$$u_1 = \lambda \int_0^x K(x,t)\, u_0(t)\,dt \quad , \lambda \neq 0$$

$$u_2 = \lambda \int_0^x K(x,t)\, u_1(t)\,dt \quad \lambda \in \mathbb{R}^*$$

and so on.

Now in view of these components that are completely determined as a result the solution $u(x)$ of Voltera Integral Equation in a series form is readily obtained by using assumption in equation 1.

3

★ It was formally shown by many researchers that if an exact solution exist for the problem, then the obtained infinite series convergent very rapidly to that solution. However, for concreate problems where a closed form solution is not obtained, a truncated number of terms is usually used for numerical purposes. The more components we use the higher accuracy we obtain.

Example 3.1

Solve the following voltera integral Equation.

$$u(x) = 1 - \int_0^x u(t)\,dt \quad \text{—} (1)$$

Solution:

By using Adomian Decomposition method, eq.(1) can be written as

$$\sum_{n=0}^{\infty} u_n(x) = 1 - \int_0^x \sum_{n=0}^{\infty} u_n(t)\,dt$$

or

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 - \int_0^x [u_0(t) + u_1(t) + \cdots]\,dt$$

We identify the zeroth component by all terms that are not included under integral sign. Therefore we obtain the following recurrence relation:

$$u_0(x) = 1, \quad u_{k+1}(x) = -\int_0^x u_k(t)\,dt, \quad k \geq 0$$

so that

$$u_0(x) = 1$$

$$u_1(x) = -\int_0^x u_0(t)\,dt = -\int_0^x 1\,dt = -|t|_0^x = -x$$

$$u_2(x) = -\int_0^x u_1(t)\,dt$$

$$= -\int_0^x (-t)\,dt$$

$$= + \left| \frac{t^2}{2} \right|_0^x$$

$$= \frac{x^2}{2} - 0$$

$$u_2(x) = \frac{x^2}{2} = \frac{x^2}{2!}$$

Example 1

$$u_3(x) = -\int_0^x u_2(t)\,dt$$

$$= -\int_0^x \frac{t^2}{2}\,dt = \frac{-1}{2}\left|\frac{t^3}{3}\right|_0^x$$

$$= -\frac{1}{2}\left[\frac{x^3}{3} - 0\right]$$

$$u_3(x) = -\frac{x^3}{6} = \frac{-x^3}{3!}$$

Similarly,

$$u_4(x) = \frac{x^4}{4!}$$

and so on.

By using series solution, we have

$$u(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \quad (2)$$

$$\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Replace 'x' with '-x' in above equation, we have

$$e^{-x} = 1 - x + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

Hence eq.(2) =>

$$\boxed{u(x) = e^{-x}}$$

① example 2

example 2

Solve the voltera integral equation

$$u(x) = 1 + \int_0^x (t-x)u(t)\,dt \quad —①$$

Since $f(x) = 1$

$K(x,t) = t - x$

applying decomposition series on both side of eq ①

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x (t-x) \sum_{n=0}^{\infty} u_n(t)\,dt$$

$$u_0(x) + u_1(x) + u_2(x) + \dots = 1 + \int_0^x (t-x)(u_0(t) + u_1(t) + \dots)\,dt$$

from above we get the following recurrence relation

$$u_0(x) = 1$$

$$u_{k+1}(x) = \int_0^x (t-x)u_k(t)\,dt \qquad k \geq 0$$

$k = 0$ $\quad u_1(x) = \int_0^x (t-x)u_0(t)\,dt$

$$u_1(x) = \int_0^x (t-x)\,dt \qquad \text{where } u_0(t) = 1$$

$$u_1(x) = \left| \frac{t^2}{2} - xt \right|_0^x = \frac{x^2}{2} - x^2 = -\frac{x^2}{2!}$$

$k = 1$ $\quad u_2(x) = \int_0^x (t-x)u_1(t)\,dt$

$$u_2(x) = \int_0^x (t-x)(-t^2/2!)\,dt$$

$$u_2(x) = -\frac{1}{2!}\int_0^x (t^3 - xt^2)\,dt$$

$$u_2(x) = -\frac{1}{2!}\left[\frac{t^4}{4} - \frac{xt^3}{3}\right]_0^x$$

$$u_2(x) = -\frac{1}{2!}\left[\frac{x^4}{4} - \frac{x^4}{3}\right]$$

$$u_2(x) = -\frac{1}{2!}\left[\frac{3x^4 - 4x^4}{12}\right]$$

$$u_2(x) = -\frac{1}{2!}\left[\frac{-x^4}{12}\right] = \frac{x^4}{24} = \frac{x^4}{4!}$$

$k = 2$ $\quad u_3(x) = \int_0^x (t-x)u_2(t)\,dt$

$$u_3(x) = \int_0^x (t-x)\frac{t^4}{4!}\,dt$$

$$u_3(x) = \frac{1}{4!}\int_0^x (t-x)t^4\,dt = \frac{1}{24}\int_0^x (t^5 - xt^4)\,dt$$

$$U_3(x) = \frac{1}{24}\left[\frac{t^6}{6} - \frac{xt^5}{5}\right]_0^x = \frac{1}{24}\left[\frac{x^6}{6} - \frac{x^6}{5}\right]$$

$$U_3(x) = \frac{1}{24}\left[\frac{5x^6 - 6x^6}{30}\right] = \frac{-x^6}{720} = \frac{-x^6}{6!}$$

Similary

$U_4(x) = \frac{x^8}{8!}$ and so on

the series solution will be

$$u(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

and is close form

$u(x) = \cos x$

above solution is obtain upon by taylor series for cosx

Example 3.3

Solve Volterra integral equation by ADM

$$u(x) = 1 - x - \frac{1}{2}x^2 - \int_0^x (t-x)\,u(t)\,dt$$

Solution:

Here

$f(x) = 1 - x - \frac{1}{2}x^2$, $\lambda = -1$, $K(x,t) = t - x$

Substituting

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$ in given VIE

So,

$$\sum_{n=0}^{\infty} u_n(x) = 1 - x - \frac{1}{2}x^2 - \int_0^x \sum_{n=0}^{\infty} (t-x)\,u_n(t)\,dt$$

or

$$u_0(x) + u_1(x) + \cdots = 1 - x - \frac{1}{2}x^2 - \int_0^x (t-x)\big(u_0(t) + u_1(t) + u_2(t) + \cdots\big)\,dt$$

we define a recurrence relation

$$u_0(x) = 1 - x - \frac{1}{2}x^2$$

$$u_{k+1}(x) = -\int_0^x (t-x)\,u_k(t)\,dt, \quad k \geq 0$$

For $k = 0$

$$u_1(x) = -\int_0^x (t-x)\,u_0(t)\,dt$$

using value of $u_0(t)$

$$= -\int_0^x \underset{\mathrm{I}}{(t-x)}\,\underset{\mathrm{II}}{\left(1 - t - \frac{1}{2}t^2\right)}\,dt$$

By integration by parts

$$= -\left[(t-x)\left(t - \frac{t^2}{2} - \frac{1}{2}\cdot\frac{t^3}{3}\right)\Big|_0^x - \int_0^x \left(t - \frac{t^2}{2} - \frac{1}{2}\cdot\frac{t^3}{3}\right) dt \right]$$

$$= -\left[0 - \left(\frac{t^2}{2} - \frac{1}{2}\cdot\frac{t^3}{3} - \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{t^4}{4} \right)\Big|_0^x \right]$$

$$u_1(x) = \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!}$$

For $k = 1$

$$u_2(x) = -\int_0^x (t-x)\,u_1(t)\,dt$$

$$U_2(x) = \frac{x^2}{2\cdot 1} - \frac{x^4}{4\cdot 3\cdot 4}$$

For $k = 2$

$$U_3(x) = \int_0^x U_2(t)\,dt$$

$$= \int_0^x \left(\frac{t^2}{2\cdot 1} - \frac{t^4}{4\cdot 3\cdot 2}\right)dt$$

$$= \left|\frac{t^3}{3\cdot 2\cdot 1} - \frac{t^5}{5\cdot 4\cdot 3\cdot 2}\right|_0^x$$

$$U_3(x) = \frac{x^3}{3\cdot 2\cdot 1} - \frac{x^5}{5\cdot 4\cdot 3\cdot 2}$$

$$\vdots$$

$$U_n(x) = \int_0^x U_{n-1}(t)\,dt$$

$$= \frac{x^n}{n!} - \frac{x^{n+2}}{(n+2)!}$$

So,

$$U(x) = \sum_{n=0}^{\infty}\left[\frac{x^n}{n!} - \frac{x^{n+2}}{(n+2)!}\right]$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}$$

$$= 1 + x + \sum_{n=0}^{\infty}\cancel{\frac{x^{n+2}}{(n+2)!}} - \sum_{n=0}^{\infty}\cancel{\frac{x^{n+2}}{(n+2)!}}$$

$$U(x) = 1 + x$$

is solution of required V.I.Eq. by ADM

Example 3.4

Solve the following volterra integral equation:

$$u(x) = 5x^3 - x^5 + \int_0^x t\,u(t)\,dt.$$

Solution:

First we identify the zeroth component $u_0(x)$ by the first two terms that are not included under the integral sign.

$$u_0(x) = 5x^3 - x^5$$

Now by using Adomian decomposition method we set a recurrence relation as;

$$u_{k+1}(x) = \int_0^x t\,u_k(t)\,dt, \quad k \geq 0$$

$$\Rightarrow u_1(x) = \int_0^x t\,u_0(t)\,dt$$

$$= \int_0^x t(5t^3 - t^5)\,dt$$

$$= \left(5\frac{t^5}{5} - \frac{t^7}{7}\right)\Big|_0^x = x^5 - \frac{x^7}{7}$$

and $u_2(x) = \int_0^x t\,u_1(t)\,dt$

$$= \int_0^x t\left(t^5 - \frac{t^7}{7}\right)dt$$

$$u_2(x) = \left(\frac{t^7}{7} - \frac{t^9}{7\times 9}\right)\Big|_0^x = \frac{x^7}{7} - \frac{x^9}{63}$$

$$u_3(x) = \int_0^x t\,u_2(t)\,dt$$

$$= \int_0^x t\left(\frac{t^7}{7} - \frac{t^9}{63}\right)dt$$

$$U_3(x) = \frac{t^9}{7\times 9} - \frac{t^{11}}{63\times 11} = \frac{t^9}{63} - \frac{t^{11}}{693}$$

The solution in the series form is given by

$$u(x) = (5x^3 - x^5) + \left(x^5 - \frac{x^7}{7}\right) + \left(\frac{x^7}{7} - \frac{x^9}{63}\right) +$$

$$\left(\frac{x^9}{63} - \frac{x^{11}}{693}\right) + \cdots$$

Cenceling the identical terms with opposit sign givens the exact Solution as

$$u(x) = 5x^3.$$

That satisfies the Voltera integral equation.

Example # 3.5

Consider the equation

$$u(x) = x + x^4 + \frac{1}{2}x^2 + \frac{1}{5}x^5 - \int_0^x u(t)\,dt$$

choosing the zeroth components as by method i.e

$$u_0(x) = x + x^4 + \frac{1}{2}x^2 + \frac{1}{5}x^5$$

The recurrance relation here is

$$u_{n+1}(x) = -\int_0^x u_n(t)\,dt \qquad n \geq 0$$

so,

$$u_1(x) = -\int_0^x \left(t + t^4 + \frac{1}{2}t^2 + \frac{1}{5}t^5\right)dt$$

$$u_1(x) = -\left\{\int_0^x t\,dt + \int_0^x t^4\,dt + \frac{1}{2}\int_0^x t^2\,dt + \frac{1}{5}\int_0^x t^5\,dt\right\}$$

$$u_1(x) = -\left[\frac{t^2}{2} + \frac{1}{5}t^5 + \frac{1}{6}t^3 + \frac{1}{30}t^6\right]_0^x$$

$$u_1(x) = -\left[\frac{x^2}{2} + \frac{x^5}{5} + \frac{x^3}{6} + \frac{x^6}{30}\right]$$

and

$$u_2(x) = +\int_0^x \left(\frac{t^2}{2} + \frac{t^5}{5} + \frac{t^3}{6} + \frac{t^6}{30}\right)dt$$

$$u_2(x) = \frac{1}{2}\int_0^x t^2\,dt + \frac{1}{5}\int_0^x t^5\,dt + \frac{1}{6}\int_0^x t^3\,dt + \frac{1}{30}\int_0^x t^6\,dt$$

$$u_2(x) = \left[\frac{1}{6}t^3 + \frac{1}{30}t^6 + \frac{1}{24}t^4 + \frac{1}{120}t^7\right]_0^x$$

$$U_2(x) = \frac{1}{6}x^3 + \frac{1}{30}x^6 + \frac{1}{24}x^4 + \frac{1}{210}x^7$$

Similary

$$U_3 = -\left[\frac{1}{24}x^4 + \frac{1}{210}x^7 + \cdots\cdots\right]$$

Now For series solution

$$U(x) = \sum_{n=0}^{\infty} U_n(x)$$

$$U(x) = U_0 + U_1 + U_2 + \cdots$$

$$\Rightarrow U(x) = x + x^4 + \cancel{\frac{1}{2}x^2} + \cancel{\frac{1}{5}x^5} - \cancel{\frac{1}{2}x^2} - \cancel{\frac{1}{5}x^5} - \cancel{\frac{1}{6}x^3} - \cancel{\frac{1}{30}x^6}$$
$$+ \cancel{\frac{1}{6}x^3} + \cancel{\frac{1}{30}x^6} + \cancel{\frac{1}{24}x^4} + \cancel{\frac{1}{210}x^7} - \cancel{\frac{1}{24}x^4} - \cancel{\frac{1}{210}x^7} \cdots$$

Which are noise terms so cancel out so we get

$$\boxed{U(x) = x + x^4}$$

Example:-3.6

$$u(x) = 2 + \frac{1}{3}\int_0^x x t^3 u(t)dt. \quad -(1)$$

Sol:- By using Adomian decomposition method, the equation(1) can be written as

$$\sum_{n=0}^{\infty} u_n(x) = 2 + \frac{1}{3}\int_0^x x t^3 \left[\sum_{n=0}^{\infty} u_n(t)\right]dt$$

Or

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 2 + \frac{1}{3}\int_0^x x t^3 [u_0(t) + u_1(t) + u_2(t) + \cdots]dt.$$

Now we identify the zeroth component by all terms that are not under the integral sign. we obtain the following recurrence relation

$$u_0(x) = 2, \quad u_{k+1}(x) = \frac{1}{3}\int_0^x x t^3 u_k(t)dt \quad k \geq 0$$

consequently, we obtain

$$u_1(x) = \frac{1}{3}\int_0^x x t^3 u_0(t)dt.$$

$$= \frac{1}{3}\int_0^x x t^3 (2)dt = \frac{2}{3}x\int_0^x t^3 dt$$

$$= \frac{2}{3}x \left|\frac{t^4}{4}\right|_0^x = \frac{x^5}{6}$$

$$u_2(x) = \frac{1}{3}\int_0^x x t^3 \left(\frac{t^5}{6}\right)dt$$

$$= \frac{1}{3.6}x\int_0^x t^8 dt$$

$$= \frac{1}{3.6}x \left|\frac{t^9}{9}\right|_0^x$$

$$= \frac{x^{10}}{6.3^3}.$$

$$u_3(x) = \frac{1}{3}\int_0^x x t^3 \left(\frac{t^{10}}{6.3^3}\right)dt$$

$$= \frac{1}{6.3^4}\, x \int_0^x t^{13}\, dt$$

$$= \frac{1}{6.3^4}\, x \left| \frac{t^{14}}{14} \right|_0^x$$

$$= \frac{1}{6.3^4.14}\, x^{15}.$$

and so on.
The solution in series form is given by

$$u(x) = 2 + \frac{1}{6}x^5 + \frac{1}{6.3^3}x^{10} + \frac{1}{6.3^4}x^{15} + \cdots$$

which is the required solution of Voltera integral of second kind.

Exercise 3.2.1

Solve the following voltera integral equations by using Adomian decomposition method.

(1) $u(x) = 6x - 3x^2 + \int_0^x u(t)\,dt$ ——— (1)

solution:-

By using Adomian Decomposition method, eq (1) can be written as

$$\sum_{n=0}^{\infty} u_n(x) = 6x - 3x^2 + \int_0^x \sum_{n=0}^{\infty} u_n(t)\,dt$$

or

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 6x - 3x^2 + \int_0^x [u_0(t) + u_1(t) + \cdots]\,dt$$

we identify the zeroth component by all terms that are not included under integral sign. Therefore we obtain the following recurrence relation

$u_0(x) = 6x - 3x^2$, $u_{k+1}(x) = \int_0^x u_k(t)\,dt$, $k \geq 0$

so that

$$u_0(x) = 6x - 3x^2$$

$$u_1(x) = \int_0^x u_0(t)\,dt = \int_0^x (6t - 3t^2)\,dt$$

$$= \left| 6\frac{t^2}{2} - 3\frac{t^3}{3} \right|_0^x$$

$$u_1(x) = 3x^2 - x^3$$

$$u_2(x) = \int_0^x u_1(t)\,dt = \int_0^x (3t^2 - t^3)\,dt$$

Q1

$$u_2(x) = \left| 3\frac{t^3}{3} - \frac{t^4}{4} \right|_0^x = x^3 - \frac{x^4}{4}$$

$$u_3 = \int_0^x u_2(t)\,dt = \int_0^x \left(t^3 - \frac{t^4}{4}\right) dt$$

$$u_3 = \left| \frac{t^4}{4} - \frac{t^5}{4.5} \right|_0^x = \frac{x^4}{4} - \frac{x^5}{20}$$

$$u_4 = \int_0^x u_3(t)\,dt$$

$$= \int_0^x \left(\frac{t^4}{4} - \frac{t^5}{20}\right) dt = \left| \frac{t^5}{4.5} - \frac{t^6}{20.6} \right|_0^x$$

$$= \frac{x^5}{20} - \frac{x^6}{120}$$

and so on.

By using series solution, we have

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots$$

$$u(x) = 6x - \cancel{3x^2} + \cancel{3x^2} - \cancel{x^3} + \cancel{x^3} - \cancel{\frac{x^4}{4}} + \cancel{\frac{x^4}{4}}$$

$$- \cancel{\frac{x^5}{20}} + \cancel{\frac{x^5}{20}} - \frac{x^6}{120} + \cdots$$

$$\boxed{u(x) = 6x}$$

Question No: 2

$$u(x) = 6x - x^3 + \int_0^x (x-t)\, u(t)\,dt \quad \text{--- (1)}$$

By using adomian decomposition method equation (1) can be written as

$$\sum_{n=0}^{\infty} u_n(x) = 6x - x^3 + \int_0^x (x-t)\left[\sum_{n=0}^{\infty} u_n(t)\right]dt.$$

or

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 6x - x^3 + \int_0^x (x-t)[u_0(t) + u_1(t) + \cdots]\,dt$$

we identify the zeroth component by all the terms that are not included under the integral sign. so we obtain the following relation

$$u_0(x) = 6x - x^3$$

$$u_{k+1}(x) = \int_0^x (x-t)\, u_k(t)\,dt \quad k \geq 0$$

$$u_1(x) = \int_0^x (x-t)\, u_0(t)\,dt$$

$$= \int_0^x (x-t)(6t - t^3)\,dt$$

Integrating by parts, we get

$$u_1(x) = (x-t)\left(6\frac{t^2}{2} - \frac{t^4}{4}\right)\Big|_0^x - \int_0^x (-1)\left(6\frac{t^2}{2} - \frac{t^4}{4}\right)dt$$

$$= 0 + \int_0^x \left(6\frac{t^2}{2} - \frac{t^4}{4}\right)dt$$

$$= \left|6\frac{t^3}{6} - \frac{t^5}{5 \cdot 4}\right|_0^x$$

$$u_1(x) = x^3 - \frac{x^5}{5 \cdot 4}$$

Now $u_2(x) = \int_0^x (x-t)\, u_1(t)\, dt$

$$= \int_0^x (x-t)\left(t^3 - \frac{t^5}{5\cdot 4}\right) dt$$

$$= (x-t)\left(\frac{t^4}{4} - \frac{t^6}{6\cdot 5\cdot 4}\right)\Big|_0^x - \int_0^x \left(\frac{t^4}{4} - \frac{t^6}{6\cdot 5\cdot 4}\right)(-1)\,dt$$

$$= 0 + \int_0^x \left(\frac{t^4}{4} - \frac{t^6}{6\cdot 5\cdot 4}\right) dt$$

$$= \left|\frac{t^5}{5\cdot 4} - \frac{t^7}{7\cdot 6\cdot 5\cdot 4}\right|_0^x$$

$$= \frac{x^5}{5\cdot 4} - \frac{x^7}{7\cdot 6\cdot 5\cdot 4}$$

$u_3(x) = \int_0^x (x-t)\, u_2(t)\, dt$

$$= \int_0^x (x-t)\left[\frac{t^5}{5\cdot 4} - \frac{t^7}{7\cdot 6\cdot 5\cdot 4}\right] dt$$

$$= \left|(x-t)\left[\frac{t^5}{5\cdot 4} - \frac{t^7}{7\cdot 6\cdot 5}\right]\right|_0^x - \int_0^x \left(\frac{t^6}{6\cdot 5\cdot 4} - \frac{t^8}{8\cdot 7\cdot 6\cdot 5\cdot 4}\right)(-1)\,dt$$

$$= 0 + \int_0^x \left[\frac{t^6}{6\cdot 5\cdot 4} - \frac{t^8}{8\cdot 7\cdot 6\cdot 5\cdot 4}\right] dt$$

$$\left|\frac{t^7}{7\cdot 6\cdot 5\cdot 4} - \frac{t^9}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}\right|_0^x = \frac{x^7}{7\cdot 6\cdot 5\cdot 4} - \frac{x^9}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}$$

So the solution in series form is

$$u(x) = 6x - x^3 + x^3 - \frac{x^5}{5\cdot 4} + \frac{x^5}{5\cdot 4} - \frac{x^7}{7\cdot 6\cdot 5\cdot 4} + \frac{x^7}{7\cdot 6\cdot 5\cdot 4} - \frac{x^9}{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4} + \cdots$$

Canceling the identical terms with opposite sign, gives $u(x) = 6x$.

Question 3

$$u(x) = 1 - \frac{1}{2}x^2 + \int_0^x u(t)\,dt$$

Solution:

Let $u(x)$ is decomposed in infinite number of components by

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Putting into given V.I. Eq

$$\sum_{n=0}^{\infty} u_n(x) = 1 - \frac{1}{2}x^2 + \int_0^x \sum_{n=0}^{\infty} u_n(t)\,dt$$

or

$$u_0(x) + u_1(x) + \cdots = 1 - \frac{1}{2}x^2 + \int_0^x (u_0(t) + u_1(t) + \cdots)\,dt$$

Here,

$$f(x) = 1 - \frac{1}{2}x^2, \quad \lambda = 1, \quad K(x,t) = 1$$

we define a recurrence relation

$$u_0(x) = 1 - \frac{1}{2}x^2$$

$$u_{k+1}(x) = \lambda \int_0^x u_k(t)\,dt, \quad k \geq 0$$

For $k = 0$

$$u_1(x) = \int_0^x u_0(t)\,dt$$

$$= \int_0^x \left(1 - \frac{1}{2}t^2\right)dt$$

$$= \left| t - \frac{1}{2} \cdot \frac{t^3}{3} \right|_0^x$$

$$u_1(x) = x - \frac{x^3}{3 \cdot 2}$$

For $k = 1$

$$u_2(x) = \int_0^x u_1(t)\,dt$$

$$= \int_0^x \left(t - \frac{t^3}{3 \cdot 2}\right)dt$$

$$= \left| \frac{t^2}{2} - \frac{t^4}{4 \cdot 3 \cdot 2} \right|_0^x$$

$$U_2(x) = \frac{x^2}{2\cdot 1} - \frac{x^4}{4\cdot 3\cdot 4}$$

For $K = 2$

$$U_3(x) = \int_0^x U_2(t)\,dt$$

$$= \int_0^x \left(\frac{t^2}{2\cdot 1} - \frac{t^4}{4\cdot 3\cdot 2}\right)dt$$

$$= \left|\frac{t^3}{3\cdot 2\cdot 1} - \frac{t^5}{5\cdot 4\cdot 3\cdot 2}\right|_0^x$$

$$U_3(x) = \frac{x^3}{3\cdot 2\cdot 1} - \frac{x^5}{5\cdot 4\cdot 3\cdot 2}$$

$\vdots$

$$U_n(x) = \int_0^x U_{n-1}(t)\,dt$$

$$= \frac{x^n}{n!} - \frac{x^{n+2}}{(n+2)!}$$

So,

$$U(x) = \sum_{n=0}^{\infty}\left[\frac{x^n}{n!} - \frac{x^{n+2}}{(n+2)!}\right]$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}$$

$$= 1 + x + \cancel{\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}} - \cancel{\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}}$$

$$U(x) = 1 + x$$

is solution of required V.I.Eq. by ADM

4 $u(x) = x - \frac{2}{3}x^3 - 2\int_0^x u(t)\,dt$

By using adomian decomposition method the above equation can be written as

$$\sum_{n=0}^{\infty} U_n(x) = x - \frac{2}{3}x^3 - 2\int_0^x \sum_{n=0}^{\infty} U_n(t)\,dt$$

on expending we get,

$$u_0(x) + u_1(x) + u_2(x) + \cdots = x - \frac{2}{3}x^3 - 2\int_0^x (u_0(t) + u_1(t) + \cdots)\,dt$$

First we identify the zeroth component $u_0(x)$ by the first two term that are not included under the integral sign.

$$u_0(x) = x - \frac{2}{3}x^3$$

Therefore the recurrence relation is written as;

$$U_{k+1}(x) = -2\int_0^x U_k(t)\,dt, \quad k \geq 0.$$

$$U_1(x) = -2\int_0^x U_0(t)\,dt$$

$$= -2\int_0^x \left(t - \frac{2}{3}t^3\right)dt$$

$$= -2\left(\frac{t^2}{2} - \frac{2}{3}\frac{t^4}{4}\right)\Big|_0^x$$

$$u_1(x) = -2\left(\frac{x^2}{2} - \frac{x^4}{6}\right) = \frac{x^4}{3} - x^2$$

$$U_2(x) = -2\int_0^x u_1(t)\,dt$$

$$u_2(x) = -2\int_0^x \left(\frac{t^4}{3} - t^2\right) dt$$

$$= -2 \left(\frac{t^5}{15} - \frac{t^3}{3}\right)\Big|_0^x$$

$$= -2\left(\frac{x^5}{15} - \frac{x^3}{3}\right)$$

$$u_3(x) = (-2)^2 \int_0^x \left(\frac{t^5}{15} - \frac{t^3}{3}\right) dt$$

$$= (-2)^2 \left(\frac{t^6}{90} - \frac{t^4}{12}\right)\Big|_0^x$$

$$= (-2)^2 \left(\frac{x^6}{90} - \frac{x^4}{12}\right)$$ and so on.

All values substitute in equation we get

$$u(x) = x - \cancel{\frac{2}{3}x^3} + \cancel{\frac{x^4}{3}} - x^2 - \frac{2x^5}{15} + \cancel{\frac{2x^3}{3}} + \frac{4x^6}{90} - \cancel{\frac{x^4}{3}} + \cdots\cdots + \infty$$

$\Rightarrow$

$$u(x) = x - x^2$$

③
Q⑤

Question ⑤

$u(x) = 1+x + \int_0^x (x-t)u(t)dt$ — ①

assume that $u(x)$ is decomposed into infinite number of component i.e

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad -Ⓐ$$

Using decomposition series on both sides of equation ①

$$\sum_{n=0}^{\infty} u_n(x) = 1+x + \int_0^x (x-t) \sum_{n=0}^{\infty} u_n(t)dt$$

$$u_0(x)+u_1(x)+\ldots = 1+x+\int_0^x (x-t)(u_0(t)+u_1(t)+\ldots)dt$$

setting $u_0(x) = 1+x$ from above we get the following recurrence relation

$$u_{k+1}(x) = \int_0^x (x-t)u_k(t)dt \quad k \geq 0$$

$k=0$ $\quad u_1(x) = \int_0^x (x-t)u_0(t)dt$

$u_1(x) = \int_0^x (x-t)(1+t)dt$

$u_1(x) = (x-t)\left(t+\frac{t^2}{2}\right)\Big|_0^x - \int_0^x \left(t+\frac{t^2}{2}\right)(0-1)dt$

$u_1(x) = 0-0+\frac{t^2}{2}+\frac{t^3}{6}\Big|_0^x = \frac{x^2}{2}+\frac{x^3}{6}$

$k=1$ $\quad u_2(x) = \int_0^x (x-t)u_1(t)dt$

$u_2(x) = \int_0^x (x-t)\left(\frac{t^2}{2}+\frac{t^3}{6}\right)dt$

$u_2(x) = (x-t)\left(\frac{t^3}{6}+\frac{t^4}{24}\right)\Big|_0^x - \int_0^x \left(\frac{t^3}{6}+\frac{t^4}{24}\right)(-1)dt$

$u_2(x) = 0-0+\left|\frac{t^4}{24}+\frac{t^5}{120}\right|_0^x$

$u_2(x) = \frac{x^4}{24}+\frac{x^5}{120}$

$k=2$ $\quad u_3(x) = \int_0^x (x-t)u_2(t)dt$

$u_3(x) = \int_0^x (x-t)\left(\frac{t^4}{24}+\frac{t^5}{120}\right)dt$

$u_3(x) = (x-t)\left(\frac{t^5}{120}+\frac{t^6}{720}\right)\Big|_0^x - \int_0^x \left(\frac{t^5}{120}+\frac{t^6}{720}\right)(-1)dt$

$u_3(x) = 0-0+\frac{t^6}{720}+\frac{t^7}{5040}\Big|_0^x$

$$U_3(x) = \frac{x^6}{720} + \frac{x^7}{5040}$$

and so on

(A) implies

$$u(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \cdots$$

$$u(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

the above series is an taylor series of e^x so

$$u(x) = e^x$$

(6) $$u(x) = 1 - x + \int_0^x (x-t)\, u(t)\, dt \quad \text{---(1)}$$

Solution:-

By using Adomean decomposition method, eq.(1) can be written as

$$\sum_{n=0}^{\infty} u_n(x) = 1 - x + \int_0^x (x-t) \sum_{n=0}^{\infty} u_n(t)\, dt$$

or

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 - x + \int_0^x (x-t)[u_0(t) + u_1(t) + \cdots]\, dt$$

We identify the zeroth component by all terms that are not included under integral sign. Therefore we obtain the following recurrence relation

$$u_0(x) = 1 - x, \quad u_{k+1}(x) = \int_0^x (x-t)\, u_k(t)\, dt, \quad k \geq 0$$

so that

$$u_0(x) = 1 - x$$

$$u_1(x) = \int_0^x (x-t)\, u_0(t)\, dt = \int_0^x (x-t)(1-t)\, dt$$

$$= \int_0^x (x - xt - t + t^2)\, dt$$

$$= \left| xt - x\frac{t^2}{2} - \frac{t^2}{2} + \frac{t^3}{3} \right|_0^x$$

$$= x^2 - \frac{x^3}{2} - \frac{x^2}{2} + \frac{x^3}{3}$$

$$u_1(x) = \frac{x^2}{2} - \frac{x^3}{6}$$

Q6

$$u_2(x) = \int_0^x (x-t)\,u_1(t)\,dt = \int_0^x (x-t)\left(\frac{t^2}{2} - \frac{t^3}{6}\right)dt$$

$$= \int_0^x \left(\frac{xt^2}{2} - \frac{xt^3}{6} - \frac{t^3}{2} + \frac{t^4}{6}\right)dt$$

$$= \left| \frac{x}{2}\frac{t^3}{3} - \frac{x}{6}\frac{t^4}{4} - \frac{t^4}{2\,4} + \frac{t^5}{6\,5} \right|_0^x$$

$$= \frac{x^4}{6} - \frac{x^5}{24} - \frac{x^4}{8} + \frac{x^5}{30}$$

$$= \frac{4x^4 - 3x^4}{24} + \frac{4x^5 - 5x^5}{120}$$

$$u_2(x) = \frac{x^4}{24} - \frac{x^5}{120}$$

$$u_3(x) = \int_0^x (x-t)\,u_2(t)\,dt = \int_0^x (x-t)\left(\frac{t^4}{24} - \frac{t^5}{120}\right)dt$$

$$= \int_0^x \left(\frac{xt^4}{24} - \frac{xt^5}{120} - \frac{t^5}{24} + \frac{t^6}{120}\right)dt$$

$$= \left| \frac{x}{24}\frac{t^5}{5} - \frac{x}{120}\frac{t^6}{6} - \frac{t^6}{24.6} + \frac{t^7}{120.7} \right|_0^x$$

$$= \frac{x^6}{120} - \frac{x^7}{720} - \frac{x^6}{144} + \frac{x^7}{840}$$

$$= \frac{6x^6 - 5x^6}{720} + \frac{6x^7 - 7x^7}{5040}$$

$$u_3(x) = \frac{x^6}{720} - \frac{x^7}{5040} \quad \text{and so on}$$

By using series solution, we have

$$u(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040} + \cdots$$

$$u(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \cdots$$

$$\boxed{u(x) = e^{-x}}$$

Question 7
$$u(x) = 1 + x - \int_0^x (x-t)\, u(t)\, dt$$

Solution:

Let $u(x)$ be decomposed into an infinite number of components by
$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute above series in given V.I.Eq
$$\sum_{n=0}^{\infty} u_n(x) = 1 + x - \int_0^x (x-t) \sum_{n=0}^{\infty} u_n(t)\, dt$$

or
$$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 + x - \int_0^x (x-t)(u_0(t) + u_1(t) + \cdots)\, dt$$

Here
$$f(x) = 1 + x, \quad K(x,t) = x - t, \quad \lambda = -1$$

we define a recurrence relation
$$u_0(x) = 1 + x$$
$$u_{k+1}(x) = -\int_0^x (x-t)\, u_k(t)\, dt$$

For $k = 0$
$$u_1(x) = -\int_0^x (x-t)\, u_0(t)\, dt$$
$$= -\int_0^x \underset{I}{(x-t)} \underset{II}{(1+t)}\, dt$$
$$= -\left[(x-t)\left(t + \frac{t^2}{2}\right)\Big|_0^x - \int_0^x \left(t + \frac{t^2}{2}\right)(-1)\, dt \right]$$
$$= -\left[0 + \left(\frac{t^2}{2} + \frac{t^3}{3\cdot 2}\right)\Big|_0^x \right]$$
$$= -\left(\frac{x^2}{2} + \frac{x^3}{3\cdot 2} \right)$$

For $k = 1$
$$u_2(x) = (-1)^2 \int_0^x \underset{I}{(x-t)} \underset{II}{\left(\frac{t^2}{2} + \frac{t^3}{3\cdot 2}\right)} dt$$
$$= (-1)^2 \left[(x-t)\left(\frac{t^3}{3\cdot 2} + \frac{t^4}{4\cdot 3\cdot 2}\right)\Big|_0^x + \int_0^x \left(\frac{t^3}{3\cdot 2} + \frac{t^4}{4\cdot 3\cdot 2}\right) dt \right.$$

$$= (-1)^2 \left(\left| \frac{t^4}{4\cdot3\cdot2\cdot1} + \frac{t^5}{5\cdot4\cdot3\cdot2} \right|_0^x \right)$$

$$= (-1)^2 \left(\frac{x^4}{4\cdot3\cdot2\cdot1} + \frac{x^5}{5\cdot4\cdot3\cdot2} \right)$$

For $k = 2$:

$$u_3(x) = -\int_0^x (x-t)\, u_2(t)\, dt$$

$$= (-1)^3 \int_0^x \underset{\mathrm{I}}{(x-t)} \underset{\mathrm{II}}{\left(\frac{t^4}{4\cdot3\cdot2\cdot1} + \frac{t^5}{5\cdot4\cdot3\cdot2} \right)} dt$$

$$= (-1)^3 \left| (x-t)\left(\frac{t^5}{5\cdot4\cdot3\cdot2\cdot1} + \frac{t^6}{6\cdot5\cdot4\cdot3\cdot2} \right) \right|_0^x + \int_0^x \left(\frac{t^5}{5\cdot4\cdot3\cdot2\cdot1} + \frac{t^6}{6\cdot5\cdot4\cdot3\cdot2} \right) dt$$

$$= (-1)^3 \left(\frac{t^6}{6\cdot5\cdot4\cdot3\cdot2\cdot1} + \frac{t^7}{7\cdot6\cdot5\cdot4\cdot3\cdot2} \right) \Big|_0^x$$

$$u_3(x) = (-1)^3 \left(\frac{x^6}{6\cdot5\cdot4\cdot3\cdot2\cdot1} + \frac{x^7}{7\cdot6\cdot5\cdot4\cdot3\cdot2} \right)$$

$\vdots$

$$u_n(x) = -\int_0^x (x-t)\, u_{n-1}(t)\, dt$$

$$= (-1)^n \left[\frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!} \right]$$

$$= (-1)^n \left[\frac{x^{2n}}{(2n)!} \right] + (-1)^n \left[\frac{x^{2n+1}}{(2n+1)!} \right]$$

So,

$$u(x) = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n}}{(2n)!} + \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$u(x) = \cos x + \sin x$$

is required solution of V.I. Eq. by ADM

Q 8): $u(x) = 1 - x - \int_0^x (x-t)u(t)dt.$

By using The Adomian Decomposition method The above equation can be written as

$$\sum_{n=0}^{\infty} u_n(x) = 1 - x - \int_0^x (x-t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt$$

on expending we get

$$u_0(x) + u_1(x) + \cdots = 1 - x - \int_0^x (x-t)\left(u_0(t) + u_1(t) + \cdots\right)dt$$

Now we identify the zeroth component of $u_0(x)$ by the first two terms that are not included under the integral Sign; $u_0(x) = 1 - x$

Now we form a recurrence relation as;

$$u_{k+1}(x) = -\int_0^x (x-t)\, u_k(t)\, dt \quad , k \geq 0.$$

$$u_1(x) = -\int_0^x (x-t)(1-t)\,dt$$

$$= -\int_0^x (x - t + t^2 - xt)\,dt.$$

$$= -\left(xt - \frac{t^2}{2} + \frac{t^3}{3} - \frac{xt^2}{2}\right)\Big|_0^x$$

$$= -\left(x^2 - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^3}{2}\right) = -\left(\frac{x^2}{2} - \frac{x^3}{6}\right).$$

$$u_2(x) = -\int_0^x -(x-t)\left(\frac{t^2}{2} - \frac{t^3}{6}\right)dt.$$

$$= \int_0^x \left(\frac{xt^2}{2} - \frac{xt^3}{6} - \frac{t^3}{2} + \frac{t^4}{6}\right)dt$$

$$= \left(\frac{xt^3}{6} - \frac{xt^4}{24} - \frac{t^4}{8} + \frac{t^5}{30}\right)\Big|_0^x$$

$$= \frac{x^4}{6} - \frac{x^5}{24} - \frac{x^4}{8} + \frac{x^5}{30} - 0$$

$$= \left(\frac{x^4}{24} - \frac{x^5}{120}\right).$$

Now $U_3(x) = -\int_0^x (x-t)\left(\frac{t^4}{24} - \frac{t^5}{120}\right)dt$

$$= -\int_0^x \left(\frac{xt^4}{24} - \frac{xt^5}{120} - \frac{t^5}{24} + \frac{t^6}{120}\right)dt.$$

$$= -\left(\frac{xt^5}{120} - \frac{xt^6}{720} - \frac{t^6}{144} + \frac{t^7}{840}\right)\Big|_0^x$$

$$= -\left(\frac{24x^6}{17280} - \frac{120x^7}{604800}\right) = -\frac{x^6}{720} + \frac{x^7}{5040}$$

$$U_3(x) = \frac{-x^6}{720} + \frac{x^7}{5040}.$$

and so on.

using all values in Series we get

$$u(x) = 1 - x - \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!} + \frac{x^7}{7!} - \cdots + \infty.$$

$$= \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots - \infty\right) -$$
$$\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \infty\right).$$

$$u(x) = \cos x - \sin x.$$

Date: ___/___/ 2020

M T W T F S

Q. No. 9

Consider the problem

$$u(x) = 1 - \int_0^x (x-t)\, dt$$

Taking the zeroth component

$$u_0(x) = 1$$

Then by recurrance relation we have here

$$u_{k+1}(x) = -\int_0^x (x-t)\, u_k(t)\, dt$$

$$u_1(x) = -\int_0^x (x-t)\, dt = \int_0^x t\, dt - x\int_0^x dt$$

$$u_1(x) = \frac{x^2}{2} - x^2 = -\frac{x^2}{2} \quad \text{or} = -\frac{x^2}{2!}$$

$$\Rightarrow u_2(x) = \int_0^x (x-t) \frac{t^2}{2}\, dt$$

$$u_2(x) = \frac{x}{2}\int_0^x t^2\, dt - \frac{1}{2}\int_0^x t^3\, dt$$

$$= \frac{1}{2}\left[\frac{x^4}{3} - \frac{x^4}{4}\right]$$

$$u_2(x) = \frac{1}{24}\left[x^4\right] = \frac{x^4}{4!}$$

Date: __/__/ 2020 M T W T F S

$$U_3(x) = -\frac{1}{24}\int_0^x (x-t)t^4\,dt$$

$$U_3(x) = -\frac{1}{24}\int_0^x xt^4\,dt - \frac{1}{24}\int_0^x t^5\,dt$$

$$U_3(x) = -\frac{1}{24}\left[\frac{xt^5}{5} - \frac{t^6}{6}\right]_0^x$$

$$U_3(x) = -\frac{1}{720}\left(x^6\right)$$

$$U_3(x) = -\frac{x^6}{6!}$$

Continuing this we get other components so

$U_0(x) = U_1 + U_2 + \cdots$ will be

$$U_0(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$\boxed{U(x) = \cos x}$ Required

Question 10

$$u(x) = 1 + \int_0^x (x-t)u(t)dt \quad — (1)$$

assume that $u(x)$ is decomposed into infinite number of component i.e

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad — (A)$$

Using decomposition series on both sides of eq (1)

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x (x-t)u_n(t)dt$$

$$u_0(x) + u_1(x) + \dots = 1 + \int_0^x (x-t)(u_0(t) + \dots)dt$$

setting $u_0(x) = 1$ from above we get the following recurrence relation

$$u_{k+1}(x) = \int_0^x (x-t)u_k(t)dt$$

$k=0$ $\quad u_1(x) = \int_0^x (x-t)u_0(t)dt$

$$u_1(x) = \int_0^x (x-t)dt \quad \text{where } u_0(t) = 1$$

$$u_1(x) = xt - \frac{t^2}{2}\Big|_0^x = x^2 - \frac{x^2}{2} = \frac{x^2}{2} = \frac{x^2}{2!}$$

$k=1$ $\quad u_2(x) = \int_0^x (x-t)u_1(t)dt = \int_0^x (x-t)\frac{t^2}{2}dt$

$$u_2(x) = \int_0^x \left(\frac{xt^2}{2} - \frac{t^3}{2}\right)dt = \frac{xt^3}{6} - \frac{t^4}{8}\Big|_0^x$$

$$u_2(x) = \frac{x^4}{6} - \frac{x^4}{8} = \frac{4x^4 - 3x^4}{24} = \frac{x^4}{24} = \frac{x^4}{4!}$$

$k=2$ $\quad u_3(x) = \int_0^x (x-t)u_2(t)dt = \int_0^x (x-t)\frac{t^4}{24}dt$

$$u_3(x) = \frac{1}{24}\int_0^x (xt^4 - t^5)dt = \frac{1}{24}\left(\frac{xt^5}{5} - \frac{t^6}{6}\right)\Big|_0^x$$

$$u_3(x) = \frac{1}{24}\left(\frac{x^6}{5} - \frac{x^6}{6}\right) = \frac{1}{24}\left(\frac{6x^6 - 5x^6}{30}\right)$$

$$= \frac{x^6}{720} = \frac{x^6}{6!}$$

and so on

(A) implies

$$u(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

the above series is a taylor series of Coshx so
$u(x) = \cosh x$

Date: __/__/ 2020 **Exercise # 3.2.1** M T W T F S

Q. No. 11

Consider the problem

$$u(x) = x - \int_0^x (x-t)\,u(t)\,dt$$

by def of method

Letting x as zeroth component

$$u_0(x) = x$$

So by $u_{k+1}(x) = \lambda \int_0^x k(x,t)\,u_k(t)\,dt$

we have here

$$u_{k+1}(x) = -\int_0^x (x-t)\,u_k(t)\,dt, \quad k \geq 0$$

So,

$$u_1 = -\int_0^x t(x-t)\,dt$$

$$= -x\int_0^x t\,dt + \int_0^x t^2\,dt$$

$$u_1 = -x\frac{x^2}{2} + \frac{x^3}{3} = -\frac{x^3}{6} = \frac{-x^3}{3!}$$

and

$$u_2 = \frac{1}{3!}\int_0^x t^3(x-t)\,dt = \frac{1}{3!}\left[x\int_0^x t^3\,dt - \int_0^x t^4\,dt\right]$$

$$u_2 = \frac{1}{3!}\left[\frac{x^5}{4} - \frac{x^5}{5}\right] = \frac{1}{6\times 20}\left[x^5\right]$$

$$u_2 = \frac{x^5}{5!}$$

So

$$u_3 = -\int_0^x \frac{t^5}{5!}(x-t)\,dt$$

$$U_3 = \frac{1}{120}\int_0^x t^5(t-x)\,dt$$

$$U_3 = \frac{1}{120}\left[\int_0^x t^6\,dt - x\int_0^x t^5\,dt\right]$$

$$U_3 = \frac{1}{120}\left[\frac{x^7}{7} - \frac{x^7}{6}\right]$$

$$U_3 = \frac{1}{120}\times\frac{1}{42}\left[6x^7 - 7x^7\right]$$

$$U_3 = -\frac{1}{7!}\,x^7$$

So we have Now

$$U_4 = \frac{x^9}{9!}$$

$$U_5 = -\frac{x^{11}}{\lfloor 11} \quad \cdots$$

So

$$U(x) = U_0 + U_1 + U_2 + \cdots$$

$$\Rightarrow U(x) = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \frac{x^9}{\lfloor 9}$$

$$\Rightarrow \boxed{U(x) = \sin x}$$ Required

(12) $u(x) = x + \int_0^x (x-t)\, u(t)\, dt.$ —(1)

Solution:

By using Adomian decomposition method, eq (1) can be written as

$$\sum_{n=0}^{\infty} u_n(x) = x + \int_0^x (x-t) \sum_{n=0}^{\infty} u_n(t)\, dt$$

or

$$u_0(x) + u_1(x) + \cdots = x + \int_0^x (x-t)\left(u_0(t) + u_1(t) + \cdots\right) dt$$

we identify the zeroth component by all terms that are not included under integral sign. Therefore we obtain the following recurrence relation.

$$u_0(x) = x \quad , \quad u_{k+1}(x) = \int_0^x (x-t)\, u_k(t)\, dt, \; k \geq 0$$

so that

$$u_0(x) = x$$

$$u_1(x) = \int_0^x (x-t)\, u_0(t) = \int_0^x (x-t)\, t\, dt$$

$$= \int_0^x (xt - t^2)\, dt$$

$$= \left| \frac{xt^2}{2} - \frac{t^3}{3} \right|_0^x$$

$$u_1(x) = \frac{x^3}{2} - \frac{x^3}{3}$$

$$u_1(x) = \frac{x^3}{6}$$

$$u_2(x) = \int_0^x (x-t)\, u_1(t)\, dt = \int_0^x (x-t) \frac{t^3}{6}\, dt$$

$$= \int_0^x \left(\frac{x t^3}{6} - \frac{t^4}{6} \right) dt$$

$$= \left| \frac{x}{6} \frac{t^4}{4} - \frac{t^5}{6 \cdot 5} \right|_0^x$$

$$= \frac{x^5}{24} - \frac{x^5}{30}$$

$$u_2(x) = \frac{5x^5 - 4x^5}{120} = \frac{x^5}{120}$$

$$u_3(x) = \int_0^x (x-t)\, u_2(t)\, dt = \int_0^x (x-t) \frac{t^5}{120}\, dt$$

$$= \int_0^x \left(\frac{x t^5}{120} - \frac{t^6}{120} \right) dt$$

$$= \left| \frac{x \cdot t^6}{120 \cdot 6} - \frac{t^7}{120 \cdot 7} \right|_0^x$$

$$= \frac{x^7}{720} - \frac{x^7}{840}$$

$$= \frac{7x^7 - 6x^7}{5040}$$

$$u_3(x) = \frac{x^7}{5040}$$

and so on.

By using series solution, we have

$$u(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\boxed{u(x) = \sinh x}$$

(E)

13. $u(x) = 1 + \int_0^x u(t)\,dt$

Sol: $u(x) = 1 + \int_0^x u(t)\,dt \quad -①$

using adomian decomposition method.

Eq ① can be written as

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x \sum_{n=0}^{\infty} u_n(t)\,dt$$

or

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 + \int_0^x [u_0(x) + u_1(x) + \cdots]\,dt$$

we identify the zeroth component by all the terms that are not under the integral sign. So we obtain

$$u_0(x) = 1, \qquad u_{k+1}(x) = \int_0^x u_k(t)\,dt \; ; \; k \geq 0$$

Now $u_1(x) = \int_0^x u_0(t)\,dt$

$$= \int_0^x (1)\,dt = |t|_0^x = x$$

$$u_2(x) = \int_0^x u_1(t)\,dt = \int_0^x t\,dt$$

$$= \left|\frac{t^2}{2}\right|_0^x = \frac{x^2}{2!}$$

$$u_3(x) = \int_0^x u_2(t)\,dt$$

$$= \int_0^x \frac{t^2}{2}\,dt = \frac{1}{2}\left|\frac{t^3}{3}\right|_0^x$$

$$= \frac{x^3}{6} = \frac{x^3}{3!}$$

So we obtain the solution

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$u(x) = e^x$: Ans:

Question 14

$$u(x) = 1 - \int_0^x u(t)\,dt$$

Solution:

Suppose that $u(x)$ is decomposed into infinite number of components by

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Putting into given V.I.Eq.

$$\sum_{n=0}^{\infty} u_n(x) = 1 - \int_0^x \sum_{n=0}^{\infty} u_n(t)\,dt$$

$$u_0(x) + u_1(x) + \cdots = 1 - \int_0^x (u_0(t) + u_1(t) + \cdots)\,dt$$

Here

$f(x) = 1$, $K(x,t) = 1$, $\lambda = -1$

we define a recurrence relation

$$u_0(x) = 1$$

$$u_{k+1}(x) = -\int_0^x u_k(t)\,dt, \quad k \geq 0$$

For $k = 0$

$$u_1(x) = -\int_0^x u_0(t)\,dt$$

$$= -\int_0^x 1\,dt$$

$$= -1\left|t\right|_0^x$$

$$= -x$$

For $k = 1$

$$u_2(x) = -\int_0^x u_1(t)\,dt$$

$$= (-1)^2 \int_0^x t\,dt$$

$$= (-1)^2 \left|\frac{t^2}{2}\right|_0^x$$

$$= (-1)^2 \frac{x^2}{2\cdot 1}$$

For $k=2$

$$U_3(x) = -\int_0^x U_2(t)\,dt$$

$$= (-1)^3\int_0^x \left(\frac{t^2}{2\cdot 1}\right) dt$$

$$= (-1)^3 \left[\left|\frac{t^3}{3\cdot 2\cdot 1}\right|_0^x\right]$$

$$U_3(x) = (-1)^3 \frac{x^3}{3\cdot 2\cdot 1}$$

$\vdots$

$$U_n(x) = -\int_0^x U_{n-1}(t)\,dt$$

$$= (-1)^n \frac{x^n}{n!}$$

$$U_n(x) = \frac{(-x)^n}{n!}$$

So,

$$U(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$U(x) = e^{-x} \quad (\because \text{By T.S. Expansion})$$

is required solution for given V.I. Eq.

P-7

Question (15)

$$u(x) = 1 + 2\int_0^x t\,u(t)\,dt \quad —(1)$$

assume that $u(x)$ is decomposed into infinite number of component i.e

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad —(A)$$

Use decomposition series on the both sides of equation (1)

$$\sum_{n=0}^{\infty} u_n(x) = 1 + 2\int_0^x t \sum_{n=0}^{\infty} u_n(t)\,dt$$

$$u_0(x) + u_1(x) + \dots = 1 + 2\int_0^x t\,(u_0(t) + u_1(t) + \dots)\,dt$$

setting $u_0(x) = 1$ and from above we get the following recurrence relation

$$u_{k+1}(x) = 2\int_0^x t\,u_k(t)\,dt \qquad k \geq 0$$

$k=0$ $\quad u_1(x) = 2\int_0^x t\,u_0(t)\,dt$

$$u_1(x) = 2\int_0^x t\,dt$$

$$u_1(x) = \cancel{2}\,\frac{t^2}{\cancel{2}}\Big|_0^x$$

$$u_1(x) = x^2$$

$k=1$ $\quad u_2(x) = 2\int_0^x t\,u_1(t)\,dt$

$$u_2(x) = 2\int_0^x t(t^2)\,dt$$

$$u_2(x) = 2\int_0^x t^3\,dt$$

$$u_2(x) = \cancel{2}\,\frac{t^4}{\cancel{2}4}\Big|_0^x = \frac{x^4}{2!}$$

$k=2$ $\quad u_3(x) = 2\int_0^x t\,u_2(t)\,dt$

$$u_3(x) = \cancel{2}\int_0^x t\left(\frac{t^4}{\cancel{2}}\right)dt = \int_0^x t^5\,dt$$

$$u_3(x) = \frac{t^6}{6}\Big|_0^x = \frac{x^6}{6} = \frac{x^6}{3!}$$

and so on

(A) implies

$$u(x) = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots$$

$$= 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \dots$$

the above series is a taylor series of e^{x^2} so

$u(x) = e^{x^2}$

16. $u(x) = 1 - 2\int_0^x t\,u(t)\,dt$

Sol:- $u(x) = 1 - 2\int_0^x t\,u(t)\,dt$ —①

Using Adomian decomposition method, Equation ① can be written as

$\sum_{n=0}^{\infty} u_n(x) = 1 - 2\int_0^x t\left(\sum_{n=0}^{\infty} u_n(t)\right)dt.$

Now we identify the zeroth component by all the terms that are not included under integral sign. So we get the following relation.

$u_0(x) = 1$, $u_{k+1}(x) = -2\int_0^x t\,u_k(t)\,dt$, $k \geq 0$

so $u_1(x) = -2\int_0^x t\,u_0(t)\,dt$

$= -2\int_0^x t\,(1)\,dt$

$= -2\left|\frac{t^2}{2}\right|_0^x$

$= -x^2$

$u_2(x) = -2\int_0^x t\,u_1(t)\,dt = -2\int_0^x t(-t^2)\,dt$

$= 2\int_0^x t^3\,dt$

$= 2\left|\frac{t^4}{4}\right|_0^x = \frac{x^4}{2!}$

$u_3(x) = -2\int_0^x t \cdot \frac{t^4}{2!}\,dt = -\int_0^x t^5\,dt$

$= -\left|\frac{t^6}{6}\right|_0^x = \frac{-x^6}{6} = \frac{-x^6}{3!}$

$u(x) = u_0(x) + u_1(x) + u_2(x) + u_3(x) + \cdots$

$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$

$u(x) = e^{-x^2}$

Q.No.17

Now consider the problem

$$U(x) = 1 - x^2 - \int_0^x (x-t)\, u(t)\, dt$$

Here the zeroth componet is

$$U_0(x) = 1 - x^2$$

by recurrance relation we have

$$U_{k+1}(x) = -\int_0^x (x-t)\, U_k(t)\, dt$$

So

$$U_1(x) = -\int_0^x (x-t)(1-t^2)\, dt$$

$$U_1(x) = -\left[\frac{1}{4}t^4 - \frac{1}{3}xt^3 - \frac{1}{2}t^2 + xt\right]_0^x$$

$$U_1(x) = -\frac{1}{2}x^2 + \frac{1}{12}x^4$$

$$U_2(x) = -\int_0^x (x-t)\left(\frac{t^4}{12} - \frac{t^2}{2}\right) dt$$

$$U_2(x) = \left[\frac{1}{72}t^6 - \frac{1}{60}xt^5 - \frac{1}{8}t^4 + \frac{1}{6}xt^3\right]_0^x$$

$$U_2(x) = \frac{1}{72}x^6 - \frac{1}{60}x^6 - \frac{1}{8}x^4 + \frac{1}{6}x^4$$

$$U_2(x) = \frac{1}{24}x^4 - \frac{1}{360}x^6$$

and

$$U_3(x) = -\int_0^x (x-t)\left(\frac{t^4}{24} - \frac{t^6}{360}\right) dt$$

$$U_3(x) = -x\int_0^x\left(\frac{t^4}{24}-\frac{t^6}{360}\right)dt + \int_0^x\left(\frac{t^5}{24}-\frac{t^7}{360}\right)dt$$

$$U_3(x) = \left(\frac{1}{2520}t^7 - \frac{t^5}{120}\right)x - \frac{t^8}{2880} + \frac{t^6}{144}\Big|_0^x$$

$$U_3(x) = -\frac{1}{720}x^6 + \frac{1}{20160}x^8$$

by similar method we have the series as

$$\Sigma U_n(x) = U(x) = U_1 + U_2 + \cdots$$

$$\Rightarrow U(x) = 1 - x^2 - \frac{x^2}{2} + \frac{1}{12}x^4 + \frac{1}{24}x^4 - \frac{1}{360}x^6 - \frac{1}{720}x^6 + \frac{1}{20160}x^8 + \frac{1}{40320}x^8 - \cdots$$

$$U(x) = 1 - \frac{3}{2}x^2 + \frac{3}{24}x^4 - \frac{3}{720}x^6 + \frac{3}{40320}x^8 - \cdots$$

$$U(x) = -2 + 3 - \frac{3}{2}x^2 + \frac{3}{24}x^4 - \frac{3}{720}x^6 + \cdots$$

$$U(x) = -2 + 3\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right]$$

$$U(x) = -2 + 3\cos x$$

so

$$\boxed{U(x) = 3\cos x - 2}$$ Required

(18) $u(x) = -2 + 3x - x^2 - \int_0^x (x-t)\, u(t)\, dt$ — (1)

Solution :-

By using Adomian decomposition method, eqn (1) can be written as

$$\sum_{n=0}^{\infty} u_n(x) = -2 + 3x - x^2 - \int_0^x (x-t) \sum_{n=0}^{\infty} u_n(t)\, dt$$

or

$$u_0(x) + u_1(x) + \cdots = -2 + 3x - x^2 - \int_0^x (x-t)\left[u_0(t) + u_1(t) + \cdots\right] dt$$

We identify zeroth component by all terms that are ~~under~~ not included under integral sign. Therefore, we obtain the following recurrence relation

$$u_0(x) = -2 + 3x - x^2, \quad u_{k+1}(x) = -\int_0^x (x-t)\, u_k(t)\, dt, \quad k \geq 0$$

so that $u_0(x) = -2 + 3x - x^2$

$$u_1(x) = -\int_0^x (x-t)\, u_0(t)\, dt = -\int_0^x (x-t)(-2 + 3t - t^2)\, dt$$

$$= -\int_0^x (-2x + 3tx - t^2 x + 2t - 3t^2 + t^3)\, dt$$

$$= \left| -\left(-2xt + 3x\frac{t^2}{2} - x\frac{t^3}{3} + 2\frac{t^2}{2} - 3\frac{t^3}{3} + \frac{t^4}{4}\right) \right|_0^x$$

$$= 2x^2 - 3\frac{x^3}{2} + \frac{x^4}{3} - x^2 + x^3 - \frac{x^4}{4}$$

$$u_1(x) = x^2 - \frac{x^3}{2} + \frac{x^4}{12}$$

now, $u_2(t) = -\int_0^x (x-t)\, u_1(t)\, dt$

$$= -\int_0^x (x-t)\left(t^2 - \frac{t^3}{2} + \frac{t^4}{12}\right) dt$$

$$= -\int_0^x \left(xt^2 - x\frac{t^3}{2} + x\frac{t^4}{12} - t^3 + \frac{t^4}{2} - \frac{t^5}{12}\right) dt$$

$$= \left| -\left(x\frac{t^3}{3} - x\frac{t^4}{2\cdot 4} + \frac{x}{12}\frac{t^5}{5} - \frac{t^4}{4} + \frac{t^5}{2\cdot 5} - \frac{t^6}{12\cdot 6}\right) \right|_0^x$$

$$= \frac{-x^4}{3} + \frac{x^5}{8} - \frac{x^6}{60} + \frac{x^4}{4} - \frac{x^5}{10} + \frac{x^6}{72}$$

$$u_2(x) = \frac{-x^4}{12} + \frac{x^5}{40} - \frac{x^6}{360}$$

Now, $u_3(x) = -\int_0^x (x-t)\left(-\frac{t^4}{12} + \frac{t^5}{40} - \frac{t^6}{360}\right)dt$

$$= -\int_0^x \left(-\frac{xt^4}{12} + \frac{xt^5}{40} - \frac{xt^6}{360} + \frac{t^5}{12} - \frac{t^6}{40} + \frac{t^7}{360}\right)dt$$

$$= \left| -\left(\frac{-x}{12}\frac{t^5}{5} + \frac{x}{40}\frac{t^6}{6} - \frac{x}{360}\frac{t^7}{7} + \frac{t^6}{12\,6} - \frac{t^7}{40\,7} + \frac{t^8}{360\,8}\right)\right|_0^x$$

$$= \frac{x^6}{60} - \frac{x^7}{240} + \frac{x^8}{2520} - \frac{x^6}{72} + \frac{x^7}{280} - \frac{x^8}{2880}$$

$$u_3(x) = \frac{x^6}{360} - \frac{x^7}{1680} + \frac{x^8}{20160}$$

and so on

By using series solution, we have

$$u(x) = -2 + 3x - x^2 + x^2 - \frac{x^3}{2} + \frac{x^4}{12} - \frac{x^4}{12} + \frac{x^5}{40} - \frac{x^6}{360} + \frac{x^6}{360}$$

$$- \frac{x^7}{1680} + \frac{x^8}{20160} - \cdots$$

$$= -2 + 3x - \frac{x^3}{2} + \frac{x^5}{40} - \frac{x^7}{1680} + \cdots$$

$$= -2 + 3\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots\right)$$

$$= -2 + 3\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$\boxed{u(x) = -2 + 3\sin x}$$

19) $u(x) = x^2 + \int_0^x (x-t)u(t)\,dt$

By using the Adomian Decomposition method the above volterra integral equation can be written as

$$\sum_{n=0}^{\infty} U_n(x) = x^2 + \int_0^x (x-t)\left(\sum_{n=0}^{\infty} U_n(t)\right)dt$$

on expending

$$U_0(x) + U_1(x) + \cdots = x^2 + \int_0^x (x-t)\left(U_0(t) + U_1(t) + \cdots\right)dt$$

Now we identify the zeroth component $U_0(x)$ which are not incuded under the integral sign.

$$U_0(x) = x^2$$

Now the recurrence relation can be written as

$$U_{k+1}(x) = \int_0^x (x-t)\,U_k(t)\,dt \quad , k \geq 0.$$

$$U_1(x) = \int_0^x (x-t)\,U_0(t)\,dt$$

$$= \int_0^x (x-t)t^2\,dt$$

$$= \int_0^x (xt^2 - t^3)\,dt$$

$$= \left(\frac{xt^3}{3} - \frac{t^4}{4}\right)\Big|_0^x = \frac{x^4}{3} - \frac{x^4}{4} = \frac{x^4}{12}$$

$$U_2(x) = \int_0^x (x-t)\frac{t^4}{12}\,dx$$

$$U_2(x) = \frac{1}{12}\int_0^x (xt^4 - t^5)\,dt$$

$$= \frac{1}{12}\left(\frac{xt^5}{5} - \frac{t^6}{6}\right)\Big|_0^x = \frac{1}{360}x^6$$

$$U_3(x) = \frac{1}{360}\int_0^x (x-t)\,t^6\,dt$$

$$= \frac{1}{360}\int_0^x (xt^6 - t^7)\,dt$$

$$= \frac{1}{360}\left(\frac{xt^7}{7} - \frac{t^8}{8}\right)\Big|_0^x$$

$$= \frac{1}{360}\left(\frac{x^8}{7} - \frac{x^8}{8}\right) = \frac{1}{360}\left(\frac{x^8}{56}\right) = \frac{x^8}{20160}$$

and so on using all values in series we get

$$U(x) = x^2 + \frac{x^4}{12} + \frac{x^6}{360} + \frac{x^8}{20160} + \cdots + \infty$$

$$= 2\left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots + \infty\right)$$

$$U(x) = 2(\cosh x - 1)$$ Ans

Q. No. 20

Consider the problem

$$u(x) = x^2+2x-2+\int_0^x (x-t)\,u(t)\,dt$$

So the zeroth component is

$$u_0(x) = x^2+2x-2 = -2+2x+x^2$$

And recurrance relation is

$$u_{k+1}(x) = \int_0^x (x-t)\,u_k(t)\,dt \qquad k \geq 0$$

So

$$u_1(x) = \int_0^x (x-t)(t^2+2t-2)\,dt$$

$$u_1(x) = x\left[\int_0^x (t^2+2t-2)\,dt\right] - \int_0^x (t^3+2t^2-2t)\,dt$$

$$u_1(x) = x\left[\frac{1}{3}t^3+t^2-2t\right]_0^x - \left[\frac{t^4}{4}+\frac{2t^3}{3}-t^2\right]_0^x$$

$$u_1(x) = \frac{1}{12}x^4+\frac{1}{3}x^3-x^2$$

$$u_2 = \int_0^x (x-t)\left(\frac{1}{12}t^4+\frac{1}{3}t^3-t^2\right)dt$$

$$u_2 = x\int_0^x \left(\frac{1}{12}t^4+\frac{1}{3}t^3-t^2\right)dt - \int_0^x \left(\frac{1}{12}t^5+\frac{1}{3}t^4-t^3\right)dt$$

$$u_2 = x\left[\frac{t^5}{60}+\frac{t^4}{12}-\frac{t^3}{3}\right]_0^x - \left[\frac{t^6}{72}+\frac{t^5}{15}-\frac{t^4}{4}\right]_0^x$$

Parent's Sig: ________ Teacher's Sig: ________

$$u_2(x) = x\left[\frac{x^5}{60}+\frac{x^4}{12}-\frac{x^3}{3}\right]-\left[\frac{x^6}{72}+\frac{x^5}{5}-\frac{x^4}{4}\right]$$

on simplification.

$$u_2(x) = \frac{x^6}{60}-\frac{x^6}{72}+\frac{x^5}{12}-\frac{x^5}{5}+\frac{x^4}{4}-\frac{x^4}{3}$$

$$u_2(x) = \frac{1}{360}x^6+\frac{1}{60}x^5-\frac{1}{12}x^4$$

$$u_3(x) = \int_0^x (x-t)\left(\frac{t^6}{360}+\frac{t^5}{60}-\frac{t^4}{12}\right)dt$$

$$u_3(x) = x\int_0^x\left(\frac{t^6}{360}+\frac{t^5}{60}-\frac{t^4}{12}\right)dt-\int_0^x\left(\frac{t^7}{360}+\frac{t^6}{60}-\frac{t^5}{12}\right)dt$$

$$u_3(x) = x\left[\frac{t^7}{2520}+\frac{t^6}{360}-\frac{t^5}{60}\right]_0^x-\left[\frac{t^8}{2880}+\frac{t^7}{420}-\frac{t^6}{72}\right]_0^x$$

$$u_3(x) = \cancel{x}\left[\frac{x^8}{2520}+\frac{x^7}{360}-\frac{x^6}{60}-\frac{x^8}{2880}-\frac{x^7}{420}+\frac{x^6}{72}\right]$$

$$u_3(x) = \frac{1}{20160}x^8+\frac{1}{2520}x^7-\frac{1}{360}x^6$$

So by similarity we have

$u(x) = u_1+u_2+u_3+\dots$ as

$$u(x) = -2+2x+\cancel{x^2}+\frac{1}{12}\cancel{x^4}+\frac{1}{3}x^3-\cancel{x^2}+\frac{1}{360}\cancel{x^6}+\frac{1}{60}x^5$$
$$-\frac{1}{12}\cancel{x^4}+\frac{1}{20160}\cancel{x^8}+\frac{1}{2520}x^7-\frac{1}{360}\cancel{x^6}-$$
$$-\frac{1}{20160}\cancel{x^8}+\frac{x^9}{181440}-\dots$$

$$u(x) = -2+2x+\frac{1}{3}x^3+\frac{1}{60}x^5+\frac{1}{2520}x^7+\frac{1}{181440}x^9+\dots$$

~~So No we multiply~~

$$U(x) = -2 + \left(2x + \frac{x^3}{3} + \frac{x^5}{60} + \frac{x^7}{2520} + \cdots\right)$$

$$U(x) = -2 + 2\left[x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots\right]$$

$$U(x) = -2 + 2\left[\frac{x}{\lfloor 1} + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \frac{x^7}{\lfloor 7} + \frac{x^9}{\lfloor 9} + \cdots\right]$$

$$U(x) = -2 + 2[\sinh x]$$

$$\boxed{U(x) = 2\sinh x - 2} \quad \text{OR}$$

$$\boxed{U(x) = 2(\sinh x - 1)} \quad \text{Required}$$

Question 21

$$u(x) = 1+2x+4\int_0^x (x-t)\,u(t)\,dt$$

Solution:

Consider $u(x)$ is decomposed into infine Components s.t

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Putting in above V.I.Eq

$$\sum_{n=0}^{\infty} u_n(x) = 1+2x+4\int_0^x (x-t)\sum_{n=0}^{\infty} u_n(t)\,dt$$

or

$$u_0(x)+u_1(x)+\cdots = 1+2x+4\int_0^x (x-t)(u_0(t)+u_1(t)+\cdots)\,dt$$

Here

$f(x) = 1+2x$, $K(x,t) = x-t$, $\lambda = 4$

we define a recurrence relation

$$u_0(x) = 1+2x$$

$$u_{k+1}(x) = 4\int_0^x (x-t)\,u_k(t)\,dt, \quad k\geq 0$$

For $k=0$

$$u_1(x) = 4\int_0^x (x-t)\,u_0(t)\,dt$$

$$= 4\int_0^x \underset{I}{(x-t)}\underset{II}{(1+2t)}\,dt$$

$$= 4\left\{\left[(x-t)\left(t+2\frac{t^2}{2}\right)\right]_0^x + \int_0^x \left(t+2\frac{t^2}{2}\right)dt\right\}$$

$$= 4\left\{0 + \frac{t^2}{2} + \frac{2t^3}{3\cdot 2}\right\}_0^x$$

$$u_1(x) = 4\left(\frac{x^2}{2} + \frac{2x^3}{3\cdot 2}\right)$$

For $k=1$

$$u_2(x) = 4\int_0^x (x-t)\,u_1(t)\,dt$$

$$= (4)^2\int_0^x \underset{I}{(x-t)}\underset{II}{\left(\frac{t^2}{2} + \frac{2t^3}{3\cdot 2}\right)}dt$$

$$= (4)^2 \left[(x-t)\left(\frac{t^3}{3\cdot 2} + \frac{2t^4}{4\cdot 3\cdot 2}\right)\Big|_0^x + \int_0^x \left(\frac{t^3}{3\cdot 2} + \frac{2t^4}{4\cdot 3\cdot 2}\right) dt \right]$$

$$= (4)^2 \left[0 + \left| \frac{t^4}{4\cdot 3\cdot 2\cdot 1} + \frac{2t^5}{5\cdot 4\cdot 3\cdot 2} \right|_0^x \right]$$

$$U_2(x) = (4)^2 \left[\frac{x^4}{4\cdot 3\cdot 2\cdot 1} + \frac{2x^5}{5\cdot 4\cdot 3\cdot 2} \right]$$

For $K = 2$

$$U_3(x) = 4\int_0^x (x-t)\, U_2(t)\, dt$$

$$= (4)^3 \int_0^x \underset{\mathrm{I}}{(x-t)} \underset{\mathrm{II}}{\left(\frac{t^4}{4\cdot 3\cdot 2\cdot 1} + \frac{2t^5}{5\cdot 4\cdot 3\cdot 2}\right)} dt$$

$$= (4)^3 \left[0 + \int_0^x \left| \frac{t^5}{5\cdot 4\cdot 3\cdot 2\cdot 1} + \frac{2t^6}{6\cdot 5\cdot 4\cdot 3\cdot 2} \right| dt \right]$$

$$= (4)^3 \left[\left| \frac{t^6}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} + \frac{2t^7}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} \right|_0^x \right]$$

$$U_3(x) = (4)^3 \left[\frac{x^6}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} + \frac{2x^7}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} \right]$$

$\vdots$

$$U_n(x) = 4\int_0^x (x-t)\, U_{n-1}(t)\, dt$$

$$U_n(x) = 4^n \left[\frac{x^{2n}}{(2n)!} + \frac{2x^{2n+1}}{(2n+1)!} \right]$$

$$U_n(x) = \frac{(2x)^{2n}}{(2n)!} + \frac{(2x)^{2n+1}}{(2n+1)!} \quad (\because 4^n = 2^{2n})$$

So,

$$U(x) = \sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(2x)^{2n+1}}{(2n+1)!}$$

$$U(x) = \cosh 2x + \sinh 2x$$

22 $u(x) = 5+2x^2 - \int_0^x (x-t)\,u(t)\,dt.$

using Adomian decomposition method above equation can be written as

$$\sum_{n=0}^{\infty} u_n(x) = 5+2x^2 - \int_0^x (x-t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt.$$

on expending we get.

$$u_0(x)+u_1(x)+u_2(x)+\cdots = 5+2x^2 - \int_0^x (x-t)\left(u_0(t)+u_1(t)+\cdots\right)dt$$

we identified the zero components by all the terms that are not included under the integral sign.

Therefore we obtained the following recurrence relation.

$$u_0(x) = 5+2x^2$$

$$u_{k+1}(x) = -\int_0^x (x-t)\,u_k(t)\,dt.$$

where $k \geq 0$.

$$u_1(x) = -\int_0^x (x-t)(5+2t^2)\,dt$$

$$= -\int_0^x (5x+2xt^2 - 5t - 2t^3)\,dt$$

$$= -\left(5xt + \frac{2xt^3}{3} - \frac{5t^2}{2} - \frac{2t^4}{4}\right)\Bigg|_0^x$$

$$= -\left(5x^2 + \frac{2x^4}{3} - \frac{5x^2}{2} - \frac{2x^4}{4}\right) - 0$$

$$u_1(x) = -\left(\frac{5x^2}{2} + \frac{2x^4}{12}\right)$$

$$u_2(x) = -\int_0^x -(x-t)\left(\frac{5t^2}{2} + \frac{\cancel{x}t^4}{6\,\cancel{12}}\right)dt$$

$$= \int_0^x \left(\frac{5xt^2}{2} + \frac{xt^4}{6} - \frac{5t^3}{2} + \frac{t^5}{6}\right)dt$$

$$= \left(\frac{5}{2}x\frac{t^3}{3} + \frac{xt^5}{30} - \frac{5t^4}{8} + \frac{t^6}{36}\right)\Big|_0^x$$

$$= \frac{5}{6}x^4 + \frac{x^6}{30} - \frac{5x^4}{8} + \frac{x^6}{36} = \frac{10x^4}{48} + \frac{6x^6}{1080}$$

$$= \frac{5x^4}{24} + \frac{x^6}{180}$$

$$u_3(x) = -\int_0^x (x-t)\left(\frac{5t^4}{24} + \frac{t^6}{180}\right)dt$$

$$= -\int_0^x \left(\frac{5xt^4}{24} + \frac{xt^6}{180} - \frac{5t^5}{24} + \frac{t^7}{180}\right)dt$$

$$= -\left(\frac{5x}{24}\cdot\frac{t^5}{5} + \frac{xt^7}{180\cdot 7} - \frac{5t^6}{6\cdot 24} + \frac{t^8}{8\cdot 180}\right)\Big|_0^8$$

$$= -\left(\frac{x^6}{24} + \frac{x^8}{1260} + \frac{5x^6}{144} + \frac{x^8}{1440}\right) - 0$$

$$= -\left(\frac{5x^6}{6!} + \frac{2x^8}{20160}\right) \text{ and so on.}$$

using Series we have

$$u(x) = 5 + 2x^2 - \frac{5x^2}{2} + \frac{2x^4}{12} + \frac{5x^4}{24} + \frac{x^6}{180} - \frac{5x^6}{6!} - \frac{2x^8}{20160} + \cdots + \infty$$

$$u(x) = 5\left[1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \infty\right] + 6\left(\frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} - \frac{x^8}{40320} + \cdots\right]$$

$$= 5\cos x - 4\left(-\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right)$$

$$= 5\cos x - 4\left[\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \infty\right) - 1\right]$$

$$u(x) = 5\cos x - 4\left[\cos x - 1\right) = 5\cos x - 4\cos x + 4 = \cos x + 4 \text{ Ans}$$

23. $u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$

Sol:- $u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt$ — ①

Using Adomian decomposition method, Equation ① can be witten as

$$\sum_{n=0}^{\infty} u_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 \left[\sum_{n=0}^{\infty} u_n(t)\right] dt$$

or we can write

$u_0(x) + u_1(x) + u_2(x) + \cdots$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 [u_0(t) + u_1(t) + \cdots]\,dt$$

we idenitify the zeroth component as

$u_0(x) = 1 + x + \frac{1}{2}x^2$

$u_{k+1}(x) = \frac{1}{2}\int_0^x (x-t)^2 u_k(t)\,dt \quad k \geq 0.$

$$u_1(x) = \frac{1}{2}\int_0^x (x-t)^2 u_0(t)\,dt$$

$$= \frac{1}{2}\int_0^x (x-t)^2 \left[1 + t + \frac{1}{2}t^2\right] dt.$$

Integrating by parts.

$$= \frac{1}{2}\left[(x-t)^2\left[t + \frac{t^2}{2} + \frac{t^3}{3.2}\right]\Big|_0^x - \int_0^x \left[t + \frac{t^2}{2} + \frac{t^3}{6}\right] 2(x-t)\,dt\right]$$

$$= 0 + \frac{2}{2}\int_0^x (x-t)\left[t + \frac{t^2}{2} + \frac{t^3}{3.2}\right] dt$$

$$= \left|(x-t)\left[\frac{t^2}{2} + \frac{t^3}{3.2} + \frac{t^4}{4.3.2}\right]\right|_0^x - \int_0^x \left[\frac{t^2}{2} + \frac{t^3}{3.2} + \frac{t^4}{4.3.2}\right](-1)\,dt$$

$$= \int_0^x \left[\frac{t^2}{2} + \frac{t^3}{3.2} + \frac{t^4}{4.3.2}\right] dt$$

$$= \left|\frac{t^3}{3.2} + \frac{t^4}{4.3.2} + \frac{t^5}{5.4.3.2}\right|_0^x = \frac{x^3}{3.2} + \frac{x^4}{4.3.2} + \frac{x^5}{5.4.3.2}$$

⑧

$$u_2(x) = \frac{1}{2}\int_0^x (x-t)^2 u_1(t)dt$$

$$= \frac{1}{2}\int_0^x (x-t)^2 \left[\frac{t^3}{3.2} + \frac{t^4}{4.3.2} + \frac{t^5}{5.4.3.2}\right]dt$$

$$= \frac{1}{2}\left[(x-t)^2\left[\frac{t^4}{4.3.2} + \frac{t^5}{5.4.3.2} + \frac{t^6}{6.5.4.3.2}\right]\Big|_0^x - \int_0^x \left[\frac{t^4}{4.3.2} + \frac{t^5}{5.4.3.2} + \frac{t^6}{6.5.4.3.2}\right] 2(x-t)(-1)dt\right]$$

$$= 0 + \frac{2}{2}\int_0^x (x-t)\left[\frac{t^4}{4.3.2.1} + \frac{t^5}{5.4.3.2.1} + \frac{t^6}{6.5.4.3.2.1}\right]dt$$

$$= \Big|(x-t)\left(\frac{t^5}{5.4.3.2} + \frac{t^6}{6!} + \frac{t^7}{7!}\right]\Big|_0^x - \int_0^x \left[\frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!}\right](-1)dt$$

$$= 0 + \int_0^x \left[\frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!}\right]dt$$

$$= \Big|\frac{t^6}{6!} + \frac{t^7}{7!} + \frac{t^8}{8!}\Big|_0^x$$

$$= \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!}$$

and so on.

So we obtain the solution

$$u(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \cdots$$

$$u(x) = e^x.$$

(24) $u(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)\,dt$ ——(1)

Solution:-

By using Adomian decomposition method, eq(1) can be written as

$$\sum_{n=0}^{\infty} u_n(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 \sum_{n=0}^{\infty} u_n(t)\,dt$$

or

$$u_0(x) + u_1(x) + \cdots = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 [u_0(t) + u_1(t) + \cdots]\,dt$$

we identify the zeroth component by all Terms that are not included under integral sign. Therefore we obtain the following recurrence relation

$$u_0(x) = 1 - \frac{1}{2}x^2, \quad u_{k+1}(x) = \frac{1}{6}\int_0^x (x-t)^3 u_k(t)\,dt, \quad k \geq 0$$

so that $u_0(x) = 1 - \frac{1}{2}x^2$.

$$u_1(x) = \frac{1}{6}\int_0^x (x-t)^3 u_0(t)\,dt = \frac{1}{6}\int_0^x (x-t)^3 \left(1 - \frac{1}{2}t^2\right) dt$$

$$= \frac{1}{6}\int_0^x (x^3 - t^3 - 3x^2t + 3xt^2)\left(1 - \frac{1}{2}t^2\right) dt$$

$$= \frac{1}{6}\int_0^x \left(x^3 - \frac{x^3t^2}{2} - t^3 + \frac{t^5}{2} - 3x^2t + \frac{3x^2t^3}{2} + 3xt^2 - \frac{3xt^4}{2}\right) dt$$

$$= \frac{1}{6}\left[x^3t - \frac{x^3}{2}\frac{t^3}{3} - \frac{t^4}{4} + \frac{t^6}{12} - 3x^2\frac{t^2}{2} + \frac{3}{2}x^2\frac{t^4}{4} + 3x\frac{t^3}{3} - \frac{3}{2}x\frac{t^5}{5}\right]_0^x$$

$$= \frac{x^4}{6} - \frac{x^6}{36} - \frac{x^4}{24} + \frac{x^6}{72} - \frac{3x^4}{12} + \frac{3x^6}{48} + \frac{x^4}{6} - \frac{3x^6}{60}$$

Q24

$$u_1(x) = \frac{x^4}{24} - \frac{x^6}{720}$$

Now, $$u_2(x) = \frac{1}{6}\int_0^x (x-t)^3 u_1(t)\,dt$$

$$= \frac{1}{6}\int_0^x (x^3 - t^3 - 3x^2 t + 3x t^2)\left(\frac{t^4}{24} - \frac{t^6}{720}\right) dt$$

$$= \frac{1}{6}\int_0^x \left(\frac{x^3 t^4}{24} - \frac{x^3 t^6}{720} - \frac{t^7}{24} + \frac{t^9}{720} - \frac{3x^2 t^5}{24} + \frac{3x^2 t^7}{720} + \frac{3x t^6}{24} - \frac{3x t^8}{720}\right) dt$$

$$= \frac{1}{6}\left[\frac{x^3}{24}\frac{t^5}{5} - \frac{x^3}{720}\frac{t^7}{7} - \frac{t^8}{24 \cdot 8} + \frac{t^{10}}{720 \cdot 10} - \frac{3x^2}{24}\frac{t^6}{6} + \frac{3x^2}{720}\frac{t^8}{8} + \frac{3x}{24 \cdot 7}t^7 - \frac{3x}{720 \cdot 9}t^9\right]_0^x$$

$$= \frac{x^8}{720} - \frac{x^{10}}{30240} - \frac{x^8}{1152} + \frac{x^{10}}{43200} - \frac{3x^8}{864} + \frac{3x^{10}}{34560} + \frac{3x^8}{1008} - \frac{3x^{10}}{38880}$$

$$u_2(x) = \frac{x^8}{40320} - \frac{x^{10}}{3628800}$$

and so on

By using series solution, we have

$$u(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots$$

$$\boxed{u(x) = \cos x}$$

Question 25

$$u(x) = 1 + \frac{1}{2}x + \frac{1}{2}\int_0^x (x-t+1)u(t)\,dt \quad -(1)$$

assume that $u(x)$ is decomposed into infinite number of components

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad -(A)$$

Using decomposition series on both sides of eq (1)

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \frac{1}{2}x + \frac{1}{2}\int_0^x (x-t+1)\sum_{n=0}^{\infty} u_n(t)\,dt$$

$$u_0(x) + u_1(x) + \dots = 1 + \frac{1}{2}x + \frac{1}{2}\int_0^x (x-t+1)(u_0(t) + u_1(t) + \dots)\,dt$$

setting $u_0(x) = 1 + \frac{1}{2}x$ and from above we get the following recurrence relation

$$u_{k+1}(x) = \frac{1}{2}\int_0^x (x-t+1)u_k(t)\,dt \text{ where } k \geq 0$$

$k=0$ $\quad u_1(x) = \frac{1}{2}\int_0^x (x-t+1)u_0(t)\,dt$

$$u_1(x) = \frac{1}{2}\int_0^x (x-t+1)\left(1 + \frac{1}{2}t\right)dt$$

$$u_1(x) = \frac{1}{2}\left[(x-t+1)\left(t + \frac{t^2}{4}\right)\Big|_0^x - \int_0^x \left(t + \frac{t^2}{4}\right)(-1)\,dt\right]$$

$$u_1(x) = \frac{1}{2}\left[x + \frac{x^2}{4} - 0 + \left|\frac{t^2}{2} + \frac{t^3}{12}\right|_0^x\right]$$

$$u_1(x) = \frac{1}{2}\left[x + \frac{x^2}{4} + \frac{x^2}{2} + \frac{x^3}{12}\right]$$

$$u_1(x) = \frac{1}{2}\left[x + \frac{3x^2}{4} + \frac{x^3}{12}\right]$$

$$u_1(x) = \frac{x}{2} + \frac{3x^2}{8} + \frac{x^3}{24}$$

$k=1$ $\quad u_2(x) = \frac{1}{2}\int_0^x (x-t+1)u_1(t)\,dt$

$$u_2(x) = \frac{1}{2}\int_0^x (x-t+1)\left(\frac{t}{2} + \frac{3t^2}{8} + \frac{t^3}{24}\right)dt$$

$$u_2(x) = \frac{1}{2}\left[(x-t+1)\left(\frac{t^2}{4} + \frac{t^3}{8} + \frac{t^4}{96}\right)\Bigg|_0^x - \int_0^x (-1)\left(\frac{t^2}{4} + \frac{t^3}{8} + \frac{t^4}{96}\right)dt\right]$$

$$u_2(x) = \frac{1}{2}\left[\frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{96} + \left[\frac{t^3}{12} + \frac{t^4}{32} + \frac{t^5}{480}\right]_0^x\right]$$

$$u_2(x) = \frac{1}{2}\left[\frac{x^2}{4} + \frac{x^3}{8} + \frac{x^4}{96} + \frac{x^3}{12} + \frac{x^4}{32} + \frac{x^5}{480}\right]$$

$$u_2(x) = \frac{1}{2}\left[\frac{x^2}{4} + \frac{5x^3}{24} + \frac{4x^4}{96} + \frac{x^5}{480}\right]$$

$$u_2(x) = \frac{x^2}{8} + \frac{5x^3}{48} + \frac{x^4}{48} + \frac{x^5}{960}$$

$k=2$ $$u_3(x) = \frac{1}{2}\int_0^x (x-t+1)\,u_2(t)\,dt$$

$$u_3(x) = \frac{1}{2}\int_0^x (x-t+1)\left(\frac{t^2}{8} + \frac{5t^3}{48} + \frac{t^4}{48} + \frac{t^5}{960}\right)dt$$

$$u_3(x) = \frac{1}{2}\left[(x-t+1)\left[\frac{t^3}{24} + \frac{5t^4}{192} + \frac{t^5}{240} + \frac{t^6}{5760}\right]_0^x - \int_0^x \left(\frac{t^3}{24} + \frac{5t^4}{192} + \frac{t^5}{240} + \frac{t^6}{5760}\right)(-1)\,dt\right]$$

$$u_3(x) = \frac{1}{2}\left[\frac{x^3}{24} + \frac{5x^4}{192} + \frac{x^5}{240} + \frac{x^6}{5760} + \left(\frac{t^4}{96} + \frac{t^5}{192} + \frac{t^6}{1440} + \frac{t^7}{40320}\right)\Bigg|_0^x\right]$$

$$u_3(x) = \frac{1}{2}\left[\frac{x^3}{24} + \frac{5x^4}{192}\,(2) + \frac{x^5}{240}\,(1) + \frac{x^6}{5760}\,(3) + \frac{x^4}{96}\,(2) + \frac{x^5}{192}\,(1) + \frac{x^6}{1440}\,(3) + \frac{x^7}{40320}\right]$$

$$u_3(x) = \frac{1}{2}\left[\frac{x^3}{24} + \frac{18x^5}{1920} + \frac{x^7}{40320} + \frac{7x^4}{192} + \frac{5x^6}{5760}\right]$$

rearranging the terms

$$u_3(x) = \frac{x^3}{48} + \frac{7x^4}{384} + \frac{3x^5}{640} + \frac{x^6}{2304} + \frac{x^7}{80640}$$

and so on

So (A) implies

$$u(x) = 1 + \frac{1}{2}x + \frac{x}{2} + \frac{3x^2}{8} + \frac{x^3}{24} + \frac{x^2}{8} + \frac{5x^3}{48}$$

$$+ \frac{x^4}{48} + \frac{x^5}{960} + \frac{x^3}{48} + \frac{7x^4}{384} + \frac{3x^5}{640}$$

$$+ \frac{x^6}{2304} + \frac{x^7}{80640} + \cdots$$

$$u(x) = 1 + x + \frac{x^2}{2} + \frac{8x^3}{486} + \cdots$$

$$u(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

the above series is taylor series of e^x So

$$u(x) = e^x$$

Date: ___/___/ 2020 M T W T F S

Q. No 26 :-

Consider the problem

$$u(x) = 1 + x^2 - \int_0^x (x-t+1) u^2 dt$$

so the zeroth component u_0 is given as

$$u_0(x) = 1 + x^2$$

and

$$u_{k+1}(x) = -\int_0^x (x-t+1)^2 u_k(t)\, dt$$

is recurrence relation so,

$$u_1(x) = -\int_0^x (x-t+1)^2 (1+t^2)\, dt$$

$$u_1(x) = -\int_0^x (t^2+1)(x^2+t^2+1-2xt-2t+2x)\, dt$$

$$u_1(x) = -\int_0^x \big(t^2x^2 + t^4 + t^2 - 2xt^3 - 2t^3 + 2xt^2 + x^2 + t^2 + 1 - 2xt - 2t + 2x\big)\, dt$$

$$u_1(x) = -\left[\frac{1}{5}t^5 - \frac{1}{2}t^4x - \frac{1}{2}t^4 + \frac{1}{3}t^3x^2 + \frac{2}{3}t^3x + \frac{2}{3}t^3 - t^2x - t^2 + tx^2 + 2tx + t\right]_0^x$$

$$u_1(x) = -\left[x + x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4 + \frac{1}{30}x^5\right]$$

$$u_2(x) = \int_0^x (x-t+1)^2 \left(t + t^2 + \frac{2}{3}t^3 + \frac{1}{6}t^4 + \frac{1}{30}t^5\right) dt$$

$$U_2(x) = \int_0^x (x^2+t^2+1+2x-2xt-2t)\left(t+t^2+\frac{2}{3}t^3+\frac{t^4}{6}+\frac{t^5}{30}\right)dt$$

$$U_2(x) = \int_0^x \Big(\frac{t^6}{10}+\frac{t^7}{30}+\frac{11}{30}t^6-\frac{t^4}{6}+tx^2-\frac{2}{3}t^3x+t^2x^2$$
$$-t^4x+\frac{2}{3}t^3x^2-\frac{4}{15}t^5x+\frac{1}{6}t^4x^2-\frac{1}{15}t^6x+$$
$$\frac{1}{30}t^5x^2-\frac{1}{3}t^3+2xt+t\Big)dt$$

$$U_2(x) = x^2\left[\frac{1}{180}t^6+\frac{1}{30}t^5+\frac{1}{6}t^4+\frac{1}{3}t^3+\frac{1}{2}t^2\right]_0^x$$
$$+x\left[t^2-\frac{1}{6}t^4-\frac{1}{5}t^5-\frac{2}{45}t^6-\frac{1}{105}t^7\right]_0^x$$
$$+\left[\frac{t^8}{240}+\frac{t^7}{70}+\frac{11t^6}{180}-\frac{t^5}{30}-\frac{t^4}{12}-\frac{t^3}{3}+\frac{t^2}{2}\right]_0^x$$

$$U_2(x) = \frac{1}{2}x^2+\frac{2}{3}x^3+\frac{5}{12}x^4+\frac{2}{15}x^5+\frac{1}{36}x^6$$
$$+\frac{x^7}{315}+\frac{x^8}{5040}$$

Similarly we have

$$U_3(x) = -\left[\frac{1}{6}x^3+\frac{1}{4}x^4+\frac{1}{6}x^5 \ldots\right]$$

So

$U(x) = U_1 + U_2 + U_3 + \cdots$

will become

Date: ___/___/ 2020 M T W T F S

$$U(x) = 1 + \cancel{x^2} - x - \cancel{\frac{x^2}{2}} - \cancel{\frac{2}{3}x^3} - \frac{1}{6}x^4 - \frac{1}{30}x^5$$

$$+ \frac{1}{2}x^2 + \cancel{\frac{2}{3}x^3} + \frac{5}{12}x^4 + \frac{2}{15}x^5 + \frac{1}{36}x^6 + \cdots$$

$$- \frac{1}{6}x^3 - \frac{1}{4}x^4 - \frac{1}{6}x^5 + \cdots$$

$$U(x) = 1 - x + \frac{x^2}{2} - \frac{1}{6}x^3 + x^4\cancelto{0}{\left(\frac{5}{12} - \frac{1}{6} - \frac{1}{4}\right)}$$

$$+ \frac{x^4}{24} \cdots$$

$$U(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} \cdots$$

$$U(x) = 1 - x + \frac{x^2}{\underline{|2}} - \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}}$$

which is

$$\boxed{U(x) = e^{-x}}$$ Required.

27) $u(x) = 3 + \frac{1}{4}\int_0^x x t^2 u(t)\,dt$.

By using adomian decomposition method the above equation can be written as

$$\sum_{n=0}^{\infty} U_n(x) = 3 + \frac{1}{4}\int_0^x x t^2 \left(\sum_{n=0}^{\infty} U_n(t)\right)dt$$

on expending we get

$$U_0(x) + U_1(x) + \cdots = 3 + \frac{1}{4}\int_0^x x t^2 (u_0(t) + u_1(t) + \cdots)\,dt$$

we identify the zeroth componentent $U_0(x)$ by all the terms that are not included under the integral sign.

Therefore we obtained the recurrence relation as

$$U_0(x) = 3$$

$$U_{k+1}(x) = \frac{1}{4}\int_0^x x t^2\, U_k(t)\,dt \quad \text{where } k \geq 0.$$

$$U_1(x) = \frac{1}{4}\int_0^x x t^2 (3)\,dt = \frac{3}{4}\int_0^x x t^2\,dt$$

$$= \frac{3}{4}\left(x\frac{t^3}{3}\right)\Big|_0^x = \frac{3}{4}\left(\frac{x^4}{3} - 0\right) = \frac{x^4}{4}.$$

$$U_2(x) = \frac{1}{4}\int_0^x x t^2 \frac{t^4}{4}\,dt = \frac{1}{16}\int_0^x x t^6\,dt$$

$$= \frac{1}{16}x\left(\frac{t^7}{7}\right)\Big|_0^x = \frac{x^8}{112}$$

$$U_3(x) = \frac{1}{4}\int_0^x x t^2 \frac{t^8}{112} = \frac{1}{448}\int_0^x x t^{10}\,dt$$

27

$$= \frac{1}{448}\left(\frac{x t^{11}}{11}\right)\Big|_0^x = \frac{x^{12}}{4928}$$

and So on.

using Series we have,

$$u(x) = 3 + \frac{1}{4}x^4 + \frac{1}{112}x^8 + \frac{1}{4928}x^{12} + \cdots$$

Ans

Question 28

$$u(x) = 3 + \frac{1}{4}\int_0^x (x+t^2)u(t)dt$$

Solution:

Consider $u(x)$ be decomposed into an infinite number of components

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute series into V.I. Eq

$$\sum_{n=0}^{\infty} u_n(x) = 3 + \frac{1}{4}\int_0^x (x+t^2)\sum_{n=0}^{\infty} u_n(t)dt$$

or

$$u_0(x) + u_1(x) + \cdots = 3 + \frac{1}{4}\int_0^x (x+t^2)(u_0(t) + u_1(t) + \cdots)dt$$

Here

$f(x) = 3$, $K(x,t) = x+t^2$, $\lambda = \frac{1}{4}$

So, we define a recurrence relation

$$u_0(x) = 3$$

$$u_{k+1}(x) = \frac{1}{4}\int_0^x (x+t^2)u_k(t)dt, \quad k \geq 0$$

For $k=0$

$$u_1(x) = \frac{1}{4}\int_0^x (x+t^2)3\,dt$$

$$= \frac{3}{4}\int_0^x (x+t^2)dt$$

$$= \frac{3}{4}\left[x\,|t|_0^x + \left|\frac{t^3}{3}\right|_0^x\right]$$

$$u_1(x) = \frac{3}{4}x^2 + \frac{1}{4}x^3$$

For $k=1$

$$u_2(x) = \frac{1}{4}\int_0^x (x+t^2)u_1(t)dt$$

$$= \frac{1}{4}\int_0^x \underset{\mathrm{I}}{(x+t^2)}\underset{\mathrm{II}}{\left(\frac{3}{4}t^2 + \frac{1}{4}t^3\right)}dt$$

$$= \frac{1}{4}\left\{\left|(x+t^2)\left(\frac{3}{4}\cdot\frac{t^3}{3} + \frac{1}{4}\cdot\frac{t^4}{4}\right)\right|_0^x - \int_0^x \left(\frac{3}{4}\cdot\frac{t^3}{3} + \frac{1}{4}\cdot\frac{t^4}{4}\right)2t\,dt\right.$$

$$= \frac{1}{4}\Big[(x+x^2)\left(\tfrac{1}{4}x^3 + \tfrac{1}{16}x^4\right) - 2\int^{x} \left(\tfrac{1}{4}t^3 + \tfrac{1}{16}t^4\right) t\,dt \Big]$$

$$= \frac{1}{4}\Big[\left(\tfrac{1}{4}x^4 + \tfrac{1}{16}x^5 + \tfrac{1}{4}x^5 + \tfrac{1}{16}x^6\right) - 2 \Big| (t)\left(\tfrac{1}{4}\cdot\tfrac{t^4}{4} + \tfrac{1}{16}\cdot\tfrac{t^5}{5}\right)\Big|_0^x - \int_0^x \left(\tfrac{t^4}{16} + \tfrac{t^5}{80}\right) dt \Big]$$

$$= \frac{1}{4}\Big[\tfrac{1}{4}x^4 + \tfrac{5}{16}x^5 + \tfrac{1}{16}x^6 - 2\Big[x\left(\tfrac{x^4}{16} + \tfrac{x^5}{80}\right) - \Big| \tfrac{t^5}{80} + \tfrac{t^6}{480}\Big|_0^x \Big]\Big]$$

$$= \frac{1}{4}\Big[\tfrac{1}{4}x^4 + \tfrac{5}{16}x^5 + \tfrac{1}{16}x^6 - 2\Big[\tfrac{x^5}{16} + \tfrac{x^6}{80} - \left(\tfrac{x^5}{80} + \tfrac{x^6}{480}\right)\Big]\Big]$$

$$= \frac{1}{4}\Big[\tfrac{1}{4}x^4 + \tfrac{5}{16}x^5 + \tfrac{1}{16}x^6 - \tfrac{x^5}{8} - \tfrac{x^6}{40} + \tfrac{x^5}{40} + \tfrac{x^6}{240} \Big]$$

$$= \frac{1}{4}\Big[\tfrac{1}{4}x^4 + \left(\tfrac{5}{16} - \tfrac{1}{8} + \tfrac{1}{4}\right)x^5 + \left(\tfrac{1}{16} - \tfrac{1}{40} + \tfrac{1}{240}\right)x^6 \Big]$$

$$= \frac{1}{4}\Big[\tfrac{1}{4}x^4 + \tfrac{17}{80}x^5 + \tfrac{1}{24}x^6 \Big]$$

$$= \tfrac{1}{16}x^4 + \tfrac{17}{320}x^5 + \tfrac{1}{96}x^6$$

and So on

So,

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

$$= u_0(x) + u_1(x) + u_2(x) + \cdots$$

$$u(x) = 3 + \tfrac{3}{2}x^2 + \tfrac{1}{4}x^3 + \tfrac{1}{16}x^4 + \cdots$$

is required solution for V.I. Eq.

29. $u(x) = 1 + \frac{1}{2}\int_0^x (x^2 - t^2)\, u(t)\,dt$

Sol:- $u(x) = 1 + \frac{1}{2}\int_0^x (x^2 - t^2) u(t)\,dt$ — (1)

Using Adomian decomposition method equation (1) can be written as

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \frac{1}{2}\int_0^x (x^2 - t^2)\left[\sum_{n=0}^{\infty} u_n(t)\right]dt$$

Or

$u_0(x) + u_1(x) + u_2(x) + \cdots$

$= 1 + \frac{1}{2}\int_0^x (x^2 - t^2)[u_0(t) + u_1(t) + \cdots]\,dt$

Now we identify the zeroth component by all the terms that are not included under the integral sign. we get following relation

$u_0(x) = 1$, $u_{k+1}(x) = \frac{1}{2}\int_0^x (x^2 - t^2) u_k(t)\,dt$, $k \geq 1$.

$u_1(x) = \frac{1}{2}\int_0^x (x^2 - t^2) u_0(t)\,dt$

$= \frac{1}{2}\int_0^x (x^2 - t^2)(1)\,dt = \frac{1}{2}\left\{x^2 |t|_0^x - \frac{t^3}{3}\Big|_0^x\right\}$

$= \frac{1}{2}\left[x^3 - \frac{x^3}{3}\right] = \frac{x^3}{3}$

$u_2(x) = \frac{1}{2}\int_0^x (x^2 - t^2) u_1(t)\,dt$

$= \frac{1}{2}\int_0^x (x^2 - t^2)\left(\frac{t^3}{3}\right)dt$

$= \frac{1}{2}\int_0^x \left(x^2\frac{t^3}{3} - \frac{t^5}{3}\right)dt$

$= \frac{1}{2}\left[x^2 \left|\frac{t^4}{12}\right|_0^x - \left|\frac{t^6}{18}\right|_0^x\right]$

$= \frac{1}{2}\left[\frac{x^6}{12} - \frac{x^6}{18}\right]$

10.

$$= \frac{1}{12}\left[\frac{x^6}{2} - \frac{x^6}{3}\right] = \frac{x^6}{72}$$

$$u_3(x) = \frac{1}{2}\int_0^x (x^2 - t^2)\, u_2(t)\, dt$$

$$= \frac{1}{2}\int_0^x (x^2 - t^2)\frac{t^6}{72}\, dt$$

$$= \frac{1}{144}\int_0^x t^6 (x^2 - t^2)\, dt$$

$$= \frac{1}{144}\left[\int_0^x (x^2 t^6 - t^8)\, dt\right]$$

$$= \frac{1}{144}\left[x^2 \frac{t^7}{7}\Big|_0^x - \frac{t^9}{9}\Big|_0^x\right]$$

$$= \frac{1}{144}\left[\frac{x^9}{7} - \frac{x^9}{9}\right] = \frac{1}{144}\left[\frac{2x^9}{63}\right]$$

$$= \frac{x^9}{4536}$$

and so on.

Add these components, we get

$$u(x) = 1 + \frac{x^3}{3} + \frac{x^6}{72} + \frac{x^9}{4536} + \cdots$$

Use Adomian decomposition method to find the series solution

(30) $U(x) = 1 + \frac{1}{2}\int_0^x x^2 u(t)\,dt$ —(1)

Assume that $u(x)$ is decomposed into infinite number of components i.e

$$U(x) = \sum_{n=0}^{\infty} U_n(x) \quad \text{—(A)}$$

using decomposition series on both sides of eq (1)

$$\sum_{n=0}^{\infty} U_n(x) = 1 + \frac{1}{2}\int_0^x x^2 \sum_{n=0}^{\infty} U_n(t)\,dt$$

$$U_0(x) + U_1(x) + \dots = 1 + \frac{1}{2}\int_0^x x^2 (U_0(t) + U_1(t) + \dots)\,dt$$

setting $U_0(x) = 1$ and from above we get the following recurrence relation

$$U_{k+1}(x) = \frac{1}{2}\int_0^x x^2 U_k(t)\,dt \qquad k \geq 0$$

$k=0$ $\quad U_1(x) = \frac{1}{2}\int_0^x x^2 U_0(t)\,dt = \frac{1}{2}\int_0^x x^2\,dt = \frac{1}{2} x^2 t\Big|_0^x$

$U_1(x) = \frac{x^3}{2}$

$k=1$ $\quad U_2(x) = \frac{1}{2}\int_0^x x^2 U_1(t)\,dt = \frac{1}{2}\int_0^x x^2 \frac{t^3}{2}\,dt = \frac{1}{4} x^2 \frac{t^4}{4}\Big|_0^x$

$U_2(x) = \frac{x^6}{16}$

$k=2$ $\quad U_3(x) = \frac{1}{2}\int_0^x x^2 U_2(t)\,dt = \frac{1}{2}\int_0^x x^2 \frac{t^6}{16}\,dt$

$U_3(x) = \frac{1}{32} \cdot x^2 \cdot \frac{t^7}{7}\Big|_0^x = \frac{x^9}{224}$

and so on, so (A) implies

$$U(x) = 1 + \frac{x^3}{2} + \frac{x^6}{12} + \frac{x^9}{224} + \dots$$

is the required series solution