(1)

The Laplace transform method for the Solution of Voltera Integral Equation of first kind

Through out in this course we are finding the solution of integral equations. In this regard, we have used different techniques. In this paper we will talk about the solution of voltera integral equation of first kind by using laplace transform method.

Laplace transform

The laplace transform of a function $f(x)$ is defined by on $(0, \infty)$ as

$$F(s) = L(f(x)) = \int_0^\infty e^{-sx} f(x)\, dx.$$

Properties of Laplace Transform :-

(i) Laplace of Derivative

$$L\left(\frac{df}{dx}\right) = s F(s) - f(0).$$

$$L\left(\frac{d^2f}{dx^2}\right) = s^2F(s) - f'(0) - s\,f(0).$$

Similarly in General

$$L\left[\frac{d^nf}{dx^n}\right] = s^nF(s) - f^{n-1}(0) - sf^{n-2}(0) - s^2f^{n-3}(0) - \cdots - s^{n-1}f(0).$$

(ii) Laplace of Integral

$$L\left\{\int_0^\infty f(z)\,dz\right\} = \int_0^\infty e^{-sx}\left(\int_0^\infty f(z)\,dz\right)dx.$$

$$= \frac{F(s)}{s}.$$

(iii) $$L\{x^n f(x)\} = (-1)^n \frac{d^n}{ds^n}(F(s)).$$

(iv) $$L(x^v) = \frac{\Gamma(v+1)}{s^{v+1}}.$$

Now the voltera Integral equation of first kind is defined as

$$f(x) = \int_0^x k(x,t)\,u(t)\,dt.$$

(3) If the kernal in above equation is difference kernal than above equation can be written as

$$f(x) = \int_0^x K(x-t)\, u(t)\, dt. \rightarrow (1)$$

Consider two functions $f_1(x)$ and $f_2(x)$ that posses the conditions needed for the existence of Laplace transform. Let the Laplace transform of the function $f_1(x)$ and $f_2(x)$ are given by

$$L(f_1(x)) = F_1(s)$$
$$L(f_2(x)) = F_2(s).$$

The laplace convolution product of these two functions is defined by.

$$(f_1 * f_2)x = \int_0^x f_1(x-t)\, f_2(t)\, dt.$$

$$(f_2 * f_1)x = \int_0^x f_2(x-t)\, f_1(t)\, dt.$$

Now the Laplace transform of this convolution product

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is given by

$$L(f_1 * f_2)(x) = L\left\{\int_0^x f_1(x-t)\, f_2(t)\, dt\right\} = ?$$

$$L(f_1 * f_2)(x) = F_1(S)\, F_2(S) \rightarrow (2)$$

Based on this summery we will examine specific voltera integral equation of the first kind where the kernel is difference kernel. By using the laplace transform on the both side of the equation

$$f(x) = \int_0^x k(x-t)\, u(t)\, dt.$$

$$L(f(x)) = L\left(\int_0^x k(x-t)\, u(t)\, dt\right.$$

$$F(S) = K(S)\, U(S) \longrightarrow (3)$$

Where

$$L(f(x)) = F(S), \quad L(k(x)) = k(S)$$

$$L(U(x)) = U(S).$$

(3) $\Rightarrow$

$$U(S) = \frac{F(S)}{k(S)} \longrightarrow (4)$$

Where $k(S) \neq 0$.

(5)

The solution $u(x)$ is obtained by taking the inverse laplace transform of (4)

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left[\frac{F(s)}{K(s)}\right].$$

$$u(x) = L^{-1}\left[\frac{F(s)}{K(s)}\right].$$

Which is the solution of voltera integral equation of first kind

Examples:-

Solve the voltera integral equation of the first kind by using the Laplace transform method.

i) $e^x - \sin x - \cos x = \int_0^x 2e^{x-t} u(t)\,dt$

Solution:-

$$e^x - \sin x - \cos x = \int_0^x 2e^{x-t} u(t)\,dt$$

Taking LaPlace on both side

$$\mathcal{L}[e^x] - \mathcal{L}[\sin x] - \mathcal{L}[\cos x] = \mathcal{L}\left[\int_0^x 2e^{x-t} u(t)\,dt\right]$$

$$\frac{1}{s-1} - \frac{1}{s^2+1} - \frac{s}{s^2+1} = 2 \cdot \mathcal{L}[e^x] \cdot \mathcal{L}[u(x)]$$

(using Convolution Theorem).

$$\frac{1}{s-1} - \frac{1}{s^2+1} - \frac{s}{s^2+1} = 2 \cdot \frac{1}{s-1} U(s)$$

$$\frac{\cancel{s^2}+1-\cancel{s}+1-\cancel{s^2}+\cancel{s}}{(s-1)(s^2+1)} = \frac{2}{s-1} \cdot U(s)$$

$$\frac{\cancel{2}}{\cancel{(s-1)}(s^2+1)} = \frac{\cancel{2}}{\cancel{(s-1)}} U(s)$$

$$U(s) = \frac{1}{s^2+1}$$

Taking LaPlace inverse of both side

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$\boxed{U(x) = \sin x}$$

ii) $1+x-e^x = \int_0^x (t-x)\, u(t)\, dt$

Solution:-

$$1+x-e^x = \int_0^x (t-x)\, u(t)\, dt$$

where the kernel is $(t-x) = -(x-t)$

Taking the Laplace Transform of both side

$$\mathcal{L}[1] + \mathcal{L}[x] - \mathcal{L}[e^x] = -\mathcal{L}\left[\int_0^x (x-t)\, u(t)\, dt\right]$$

$$\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s-1} = -\mathcal{L}[x] \cdot \mathcal{L}[u(x)]$$ (using convolution Theorem).

$$\frac{s(s-1)+s-1-s^2}{s^2(s-1)} = -\frac{1}{s^2} \cdot U(s)$$

$$\frac{\cancel{s^2}-\cancel{s}+\cancel{s}-1-\cancel{s^2}}{s^2(s-1)} = -\frac{1}{s^2}\, U(s)$$

$$\frac{-1}{\cancel{s^2}(s-1)} = -\frac{1}{\cancel{s^2}} \cdot U(s)$$

$$U(s) = \frac{1}{s-1}$$

Taking the inverse Laplace Transform of both side

$$\mathcal{L}^{-1}[U(s)] = \mathcal{L}^{-1}\left[\frac{1}{s-1}\right]$$

$$\boxed{u(x) = e^x}$$

iii) $-1+x^2+\frac{1}{6}x^3+2\sinh x+\cosh x=\int_0^x (x-t+2)\,u(t)\,dt$

Solution:-

$$-1+x^2+\frac{1}{6}x^3+2\sinh x+\cosh x=\int_0^x (x-t+2)\,u(t)\,dt$$

Taking Laplace Transform of both side

$$\mathcal{L}[-1]+\mathcal{L}[x^2]+\frac{1}{6}\mathcal{L}[x^3]+2\,\mathcal{L}[\sinh x]+\mathcal{L}[\cosh x]=\mathcal{L}\left[\int_0^x (x-t+2)\,u(t)\,dt\right]$$

$$\frac{-1}{s}+\frac{2!}{s^3}+\frac{3!}{6s^4}+2\cdot\frac{1}{s^2-1}+\frac{s}{s^2-1}=\mathcal{L}[x+2]\cdot\mathcal{L}[u(x)]$$

Here we use two formulas L.H.S use

$\mathcal{L}[x^n]=\frac{n!}{s^{n+1}}$ and R.H.S use Convolution Theorem

$$\frac{-1}{s}+\frac{2}{s^3}+\frac{1}{s^4}+\frac{(s+2)}{s^2-1}=\left[\frac{1}{s^2}+\frac{2}{s}\right]\cdot U(s)$$

$$\frac{-s^3+2s+1}{s^4}+\frac{(s+2)}{s^2-1}=\frac{(2s+1)}{s^2}\cdot U(s)$$

$$\frac{-\cancel{s^5}+s^3+2s^3-2s+s^2-1+\cancel{s^5}+2s^4}{s^4(s^2-1)}=\frac{(2s+1)}{s^2}\cdot U(s)$$

$$\frac{2s^4+2s^3-2s+s^3+s^2-1}{s^2(s^2-1)}=(1+2s)\cdot U(s)$$

$$U(s)=\frac{2s(s^3+s^2-1)+1(s^3+s^2-1)}{(1+2s)\,s^2(s^2-1)}$$

$$U(s)=\frac{\cancel{(2s+1)}\,(s^3+s^2-1)}{\cancel{(2s+1)}\,s^2(s^2-1)}$$

$$U(s)=\frac{s^3}{s^2(s^2-1)}+\frac{\cancel{(s^2-1)}}{s^2\cancel{(s^2-1)}}$$

$$U(s)=\frac{s}{(s^2-1)}+\frac{1}{s^2}$$

Taking Laplace inverse on both side

$$\mathcal{L}^{-1}[U(s)] = \mathcal{L}^{-1}\left[\frac{s}{s^2-1}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]$$

$$\boxed{U(x) = \cosh x + x}$$

Exercise

Date:

Use laplace Transform to solve volterra integral equation of 1st kind.

1 $$x - \sin x = \int_0^x (x-t)\, u(t)\, dt$$

Sol: $$x - \sin x = \int_0^x (x-t)\, u(t)\, dt$$

Taking laplace on both sides

$$L[x] - L[\sin x] = L[x] \cdot L[u(x)]$$

$$\frac{1}{s^2} - \frac{1}{s^2+1} = \frac{1}{s^2} U(s)$$

$$\frac{s^2+1-s^2}{s^2(s^2+1)} = \frac{1}{s^2} U(s)$$

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} U(s)$$

$$U(s) = \frac{1}{s^2+1}$$

Taking inverse laplace

$$L^{-1}[U(s)] = L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$u(x) = \sin x$$ Ans

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Date:

2 $e^x + \sin x - \cos x = \int_0^x 2e^{x-t} u(t)\,dt$

Sol: Taking laplace on both sides

$$\frac{1}{s-1} + \frac{1}{s^2+1} - \frac{s}{s^2+1} = \frac{2}{s-1} u(s)$$

$$\frac{1}{s-1} + \frac{1-s}{s^2+1} = \frac{2}{s-1} u(s)$$

$$\frac{s^2+1+s-1-s^2+s}{(s-1)(s^2+1)} = \frac{2}{s-1} u(s)$$

$$\frac{2s}{(s-1)(s^2+1)} = \frac{2}{s-1} u(s)$$

$$u(s) = \frac{s}{s^2+1}$$

Taking inverse laplace

$$L^{-1}[u(s)] = L^{-1}[s/s^2+1]$$

$u(x) = \cos x$ Answer

Date:

3 $\quad 1+\frac{1}{3!}x^3-\cos x=\int_0^x (x-t)\,u(t)\,dt$

Sol. Taking laplace on both sides

$$\frac{1}{s}+\frac{1}{s^4}-\frac{s}{s^2+1}=\frac{1}{s^2}u(s)$$

$$\frac{s^3+1}{s^4}-\frac{s}{s^2+1}=\frac{1}{s^2}u(s)$$

$$L[t^n]=\frac{n!}{s^{n+1}}$$

$$L[x^3]=\frac{3!}{s^4}$$

$$\frac{L[x^3]}{3!}=\frac{1}{s^4}$$

$$\frac{(s^3+1)(s^2+1)-s(s^4)}{s^4(s^2+1)}=\frac{1}{s^2}u(s)$$

$$u(s)=\frac{s^5+s^3+s^2+1-s^5}{s^2(s^2+1)}$$

$$u(s)=\frac{s^3+s^2+1}{s^2(s^2+1)}$$

$$u(s)=\frac{s^{\cancel{3}}}{\cancel{s^2}(s^2+1)},\quad u(s)=\frac{s^2+1}{s^2(s^2+1)}$$

$$u(s)=\frac{s}{s^2+1},\quad u(s)=\frac{1}{s^2}$$

Taking laplace inverse

Date:

$u(x) = \cos x, \quad u(x) = x$

4 $1 + x - \sin x - \cos x = \int_0^x (x-t)\, u(t)\, dt$

Sol: Taking laplace on both sides

$$\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^2+1} - \frac{s}{s^2+1} = \frac{1}{s^2} u(s)$$

$$\frac{s+1}{s^2} - \left[\frac{1+s}{s^2+1}\right] = \frac{1}{s^2} u(s)$$

$$\frac{(s+1)(s^2+1) - (1+s)s^2}{\cancel{s^2}(s^2+1)} = \frac{1}{\cancel{s^2}} u(s)$$

$$u(s) = \frac{s^3 + s + s^2 + 1 - s^2 - s^3}{s^2+1}$$

$$u(s) = \frac{s+1}{s^2+1}$$

$$u(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

Taking laplace inverse

$$u(x) = \cos x + \sin x$$

Exercise 3.3.2

Use the Laplace transform method to solve the Volterra integral equations of first kind:

(5) $$x = \int_0^x (1+2(x-t))\, u(t)\, dt$$

Solution:

$$x = \int_0^x (1+2(x-t))\, u(t)\, dt$$

$$x = \int_0^x u(t)\, dt + 2\int_0^x (x-t)\, u(t)\, dt$$

Taking Laplace transform

$$L[x] = L[\int_0^x u(t)\, dt] + 2L[\int_0^x (x-t)\, u(t)\, dt]$$

By convolution theorem

$$\frac{1}{s^2} = \frac{U(s)}{s} + 2L[x]\,L[u(x)] \qquad (\because L[\int_0^x F(t)dt] = \frac{F(s)}{s})$$

$$\frac{1}{s^2} = \frac{U(s)}{s} + \frac{2U(s)}{s^2}$$

$$\frac{1}{s^2} = \frac{U(s)}{s^2}(s+2)$$

$$U(s) = \frac{1}{s+2} = \frac{1}{s-(-2)}$$

Taking inverse Laplace transform

$$L^{-1}[U(s)] = L^{-1}[\frac{1}{s-(-2)}]$$

$$u(x) = e^{-2x}$$

The solution is completed.

(6) $$\sinh x = \int_0^x e^{x-t}\, u(t)\, dt$$

Solution:

$$\sinh x = \int_0^x e^{x-t}\, u(t)\, dt$$

Taking Laplace transform

$$L[\sinh x] = L[\int_0^x e^{x-t}\, u(t)\, dt]$$

RHS solved by convolution theorem, i.e

$$\frac{1}{s^2-1} = L[e^x]\,L[u(x)]$$

$$\frac{1}{s^2-1} = \frac{1}{s-1}\,u(s)$$

$$u(s) = \frac{s-1}{s^2-1}$$

$$u(s) = \frac{s-1}{(s+1)(s-1)} = \frac{1}{s+1}$$

$$u(s) = \frac{1}{s+1}$$

Taking inverse Laplace transform

$$L^{-1}[u(s)] = L^{-1}\left[\frac{1}{s+1}\right]$$

$$u(x) = e^{-x}$$

This completes the solutions

7. $x = \int_0^x (x-t+1)\,u(t)\,dt$

Solution:-

$$x = \int_0^x (x-t+1)\,u(t)\,dt$$

$$x = \int_0^x (x-t)\,u(t)\,dt + \int_0^x u(t)\,dt$$

Taking Laplace transform

$$L[x] = L\left[\int_0^x (x-t)\,u(t)\,dt\right] + L\left[\int_0^x u(t)\,dt\right]$$

$$L[x] = L[x]\,L[u(x)] + L\left[\int_0^x u(t)\,dt\right]$$

$$\frac{1}{s^2} = \frac{1}{s^2}\,u(s) + \frac{u(s)}{s}$$

$$\frac{1}{s^2} = \frac{u(s)}{s^2}\,(s+1)$$

$$u(s) = \frac{1}{s+1}$$

Taking inverse Laplace transform

$$L^{-1}[u(s)] = L^{-1}\left[\frac{1}{s+1}\right]$$

$$u(x) = e^{-x}$$

This completes the solution.

(8) $1 - x - e^{-x} = \int_0^x (t-x)\, u(t)\, dt$

Solution.

$$1 - x - e^{-x} = \int_0^x (t-x)\, u(t)\, dt$$

$$1 - x - e^{-x} = -\int_0^x (x-t)\, u(t)\, dt$$

$$-1 + x + e^{-x} = \int_0^x (x-t)\, u(t)\, dt$$

Taking Laplace transform

$$L[-1] + L[x] + L[e^{-x}] = L\left[\int_0^x (x-t)\, u(t)\, dt\right]$$

$$-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1} = L[x]\, L[u(x)]$$

$$-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1} = \frac{1}{s^2}\, u(s)$$

$$u(s) = -s + 1 + \frac{s^2}{s+1}$$

$$u(s) = -s + 1 + \frac{s^2 - 1 + 1}{s+1}$$

$$u(s) = -s + 1 + \frac{s^2 - 1}{s+1} + \frac{1}{s+1}$$

$$u(s) = -s + 1 + \frac{(s-1)\cancel{(s+1)}}{\cancel{s+1}} + \frac{1}{s+1}$$

$$u(s) = -s + 1 + s - 1 + \frac{1}{s+1}$$

$$u(s) = \frac{1}{s+1}$$

Taking inverse Laplace transform

$$L^{-1}[u(s)] = L^{-1}\left[\frac{1}{s+1}\right]$$

$$u(x) = e^{-x}$$

This completes the solution

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Q9:- $1+x-\frac{x^3}{3!}-e^x = \int_0^x (t-x)\,u(t)\,dt$

Solution:-

$$1+x-\frac{x^3}{3!}-e^x = -\int_0^x (x-t)\,u(t)\,dt$$

Taking Laplace transform

$$L\left[1+x-\frac{x^3}{3!}-e^x\right] = -L\left[\int_0^x (x-t)\,u(t)\,dt\right]$$

Since we know that the convolution theorem is

$$L\left[\int_0^x f(t-u)\,g(u)\,du\right] = L[f(u)]\,L[g(u)]$$

There fore

$$\frac{1}{s}+\frac{1}{s^2}-\frac{1}{s^4}-\frac{1}{s-1} = -L[t]\,L[u(t)]$$

$$\Rightarrow \frac{1}{s}+\frac{1}{s^2}-\frac{1}{s^4}-\frac{1}{s-1} = -\frac{1}{s^2}U(s)$$

$$\Rightarrow U(s) = -s^2\left[\frac{1}{s}+\frac{1}{s^2}-\frac{1}{s^4}-\frac{1}{s-1}\right]$$

$$U(s) = -s-1+\frac{1}{s^2}+\frac{s^2}{s-1}$$

$$U(s) = -(s+1)+\frac{(s-1+s^4)}{s^2(s-1)}$$

$$U(s) = \frac{(s^4+s-1)}{s^2(s-1)}-(s+1)$$

$$U(s) = \frac{(s^4+s-1)-s^2(s+1)(s-1)}{s^2(s-1)}$$

$$U(s) = \frac{s^4+s-1-s^2(s^2-1)}{s^2(s-1)}$$

$$U(s) = \frac{\cancel{s^4} + s - 1 - \cancel{s^4} + s^2}{s^2(s-1)}$$

$$U(s) = \frac{s^2 + s - 1}{s^2(s-1)} = \frac{\cancel{s^2}}{\cancel{s^2}(s-1)} + \frac{\cancel{(s-1)}}{s^2\cancel{(s-1)}}$$

$$U(s) = \frac{1}{s^2} + \frac{1}{s-1}$$

Taking Inverse laplace transform

$$\Rightarrow L^{-1}[U(s)] = L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s-1}\right]$$

$$\Rightarrow \boxed{u(x) = x + e^x}$$

is the solution of Volterra integral equation of first kind.

Q10:- $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \sin x - \cos x = \int_0^x (x-t+1)u(t)dt$

Solution:- Taking Laplace transform

$$\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} - \frac{1}{s^2+1} - \frac{s}{s^2+1} = L\left[\int_0^x (x-t)u(t)dt\right] + L\left[\int_0^x u(t)dt\right]$$

$$\Rightarrow \frac{s^2+s+1}{s^3} + \frac{1}{s^4} - \frac{(s+1)}{(s^2+1)} = \frac{1}{s^2}U(s) + \frac{U(s)}{s} \qquad \because L\left[\int_0^x f(t)dt\right] = \frac{F(s)}{s}$$

$$\Rightarrow \frac{(s^2+s+1)s^4+s^3}{s^7} - \frac{(s+1)}{(s^2+1)} = \frac{U(s)(s+1)}{s^2}$$

$$\frac{s^3[(s^2+s+1)s+1]}{s^{7}_{4}} - \frac{(s+1)}{(s^2+1)} = \frac{(s+1)}{s^2}U(s)$$

$$\frac{s^3+s^2+s+1}{s^4} - \frac{(s+1)}{(s^2+1)} = \frac{(s+1)}{s^2}U(s)$$

$$\Rightarrow \frac{(s^3+s^2+s+1)(s^2+1) - s^4(s+1)}{s^{4}_{2}(s^2+1)} \times \frac{s^2}{s+1} = U(s)$$

$$U(s) = \frac{\cancel{s^5} + \cancel{s^4} + s^3 + s^2 + s^3 + s^2 + s + 1 - \cancel{s^5} - \cancel{s^4}}{s^2(s+1)(s^2+1)}$$

$$U(s) = \frac{2s^3 + 2s^2 + s + 1}{s^2(s+1)(s^2+1)}$$

$$U(s) = \frac{(2s^2+1) + s(2s^2+1)}{s^2(s+1)(s^2+1)} = \frac{\cancel{(s+1)}(2s^2+1)}{s^2\cancel{(s+1)}(s^2+1)}$$

$$U(s) = \frac{s^2+s^2+1}{s^2(s^2+1)} = \frac{\cancel{s^2}}{\cancel{s^2}(s^2+1)} + \frac{\cancel{(s^2+1)}}{s^2\cancel{(s^2+1)}}$$

$$U(s) = \frac{1}{s^2} + \frac{1}{s^2+1}$$

Taking inverse laplace transform

$$\boxed{u(x) = x + \sin x}$$

is the solution of volterra integral equation of first kind

Q11:- $+3 - 7x + x^2 + \sinh x - 3\cosh x = \int_0^x (x-t-3)u(t)\,dt$

Solution:-

$$3 - 7x + x^2 + \sinh x - 3\cosh x = \int_0^x (x-t)u(t)\,dt - 3\int_0^x u(t)\,dt$$

Taking Laplace transform

$$\frac{3}{s} - \frac{7}{s^2} + \frac{2}{s^3} + \frac{1}{s^2-1} - \frac{3s}{s^2-1} = L[t]\,L[u(t)] - \frac{3}{s}U(s)$$

$$\frac{1-3s}{s^2-1} + \frac{2}{s^3} + \frac{3s^2-7s}{s^3} = \left(\frac{1}{s^2} - \frac{3}{s}\right)U(s)$$

$$\frac{1-3s}{s^2-1} + \frac{3s^2-7s+2}{s^3} = \frac{(1-3s)}{s^2}U(s)$$

$$\frac{s^3(1-3s) + (s^2-1)(3s^2-7s+2)}{\cancel{s^3}s(s^2-1)} = \frac{U(s)(1-3s)}{\cancel{s^2}}$$

$$\Rightarrow U(s) = \frac{s^3 - \cancel{3s^4} + \cancel{3s^4} - 7s^3 + 2s^2 - 3s^2 + 7s - 2}{s(1-3s)(s^2-1)}$$

$$\Rightarrow U(s) = \frac{-6s^3 - s^2 + 7s - 2}{s(s^2-1)(1-3s)}$$

$$U(s) = \frac{-6s^3 + 7s - s^2 - 2 + 2s^2 - 2s^2 + s - s}{s(s^2-1)(1-3s)}$$

$$U(s) = \frac{(2s^2+s-2) - 6s^3 - 3s^2 + 6s}{s(s^2-1)(1-3s)}$$

$$U(s) = \frac{(2s^2+s-2) - 3s(2s^2+s-2)}{s(s^2-1)(1-3s)}$$

$$U(s) = \frac{\cancel{(1-3s)}(2s^2+s-2)}{\cancel{(1-3s)}\, s(s^2-1)}$$

$$U(s) = \frac{2(s^2-1)+s}{s(s^2-1)} = \frac{2\cancel{(s^2-1)}}{s\cancel{(s^2-1)}} + \frac{\cancel{s}}{\cancel{s}(s^2-1)}$$

$$U(s) = \frac{2}{s} + \frac{1}{s^2-1}$$

Taking Inverse laplace transform

$$\Rightarrow \boxed{u(x) = 2 + \sinh x}$$

is the solution of Volterra integral equation of first kind.

Q12:- $1 - \cos x = \int_0^x \cos(x-t)\, u(t)\, dt$

Solution:- Taking Laplace transform

$$\Rightarrow \frac{1}{s} - \frac{s}{s^2+1} = L[\cos t]\, L[u(t)] \quad \text{(Convolution Theorem)}$$

$$\frac{1}{s} - \frac{s}{s^2+1} = \left(\frac{s}{s^2+1}\right) U(s)$$

$$\Rightarrow \frac{1}{s} = \frac{s}{s^2+1} + \left(\frac{s}{s^2+1}\right) U(s)$$

$$\frac{1}{s} = \{1+U(s)\} \frac{s}{s^2+1}$$

$$\Rightarrow 1+U(s) = \frac{s^2+1}{s^2} \Rightarrow U(s) = \frac{s^2+1}{s^2} - 1$$

$$U(s) = \frac{s^2+1-s^2}{s^2}$$

$$\Rightarrow U(s) = \frac{1}{s^2}$$

Taking inverse laplace transform

$$\Rightarrow \boxed{u(x) = x}$$

is the solution of volterra integral equation of 1st kind