Day:________ Date:________

Conversion to a Volterra Equation of the Second Kind

In this section we will present a method that will convert Volterra integral equations of the first kind, to volterra equation of the second kind. The conversion technique works effectively only if $K(x,x) \neq 0$.

Differentiating both sides of the Volterra integral equation of the first kind

$$f(x) = \int_0^x K(x,t)\, u(t)\, dt.$$

with respect to x, and using Leibnitz rules, we find

$$f'(x) = K(x,x)\, u(x) + \int_0^x K_x(x,t)\, u(t)\, dt$$

Solving for $u(x)$, provided that $K(x,x) \neq 0$, we obtain the Volterra integral equation

GM Paper's

Day: ______ Date: ______

of the Second Kind given by

$$u(x) = \frac{f'(x)}{K(x,x)} - \int_0^x \frac{1}{K(x,x)} K_x(x,t)\,u(t)\,dt$$

Notice that the non-homogenous term and the kernel have changed to

$$g(x) = \frac{f'(x)}{K(x,x)}$$

$$G(x,t) = -\frac{1}{K(x,x)} K_x(x,t).$$

respectively.

Having converted the volterra integral equation of the first kind to the volterra integral equation of the Second Kind. We than can use any method that was presented

GM Paper's

Day:________ Date:________

before. Because we solved the Volterra integral eq^n s. of the second kind by many methods we will ~~slect~~ Select distinct methods for solving the volterra integral equation of the first kind after reducing it to a second kind Volterra integral equation.

Remarks:-

1. It was stated before that if $K(x,x) = 0$ than the conversion of the first kind to the second kind fails. However if $K(x,x) = 0$ and $K_x(x,x) \neq 0$ Then by differentiating the volterra integral equation of the first kind As many times as needed,

Day: Date:

Provided that $K(x,t)$ is differentiable then the equation will be reduced to the volterra integral equation of the Second kind.

2. The function $f(x)$ must satisfied specific conditions to guarantee a unique continuous solution for $u(x)$. The determination of these special conditions will be left as an exercise.

However for the first remark where $K(x,x)=0$ but $K_x(x,x)\neq 0$ we will differentiate twice by using Leibnitz rule as will be shown by the following illustrative example.

NCH

INTEGRAL EQUATION:-

EXAMPLE # 3.37

Convert the voltera integral equation of the 1st kind to the 2nd kind and solve the resulting equation

$$\sinh x = \int_0^x e^{x-t} u(t)\,dt \cdot \to (i)$$

Solution:-

Differentiating both sides of (i) and using Leibnitz rule we obtain

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left[\int_0^x e^{x-t} u(t)\,dt\right]$$

$$\cosh x = \int_0^x \frac{\partial}{\partial x}\left(e^{x-t}\right)u(t)\,dt + e^{x-x} u(x)\frac{dx}{dx} - e^{x-0} u(0)\frac{d(0)}{dx}$$

$$= u(x) + \int_0^x e^{x-t} u(t)\,dt$$

Now,

$$u(x) = \cosh x - \int_0^x e^{x-t} u(t)\,dt$$

Taking Laplace on both sides

$$L\{u(x)\} = L\{\cosh x\} - L\left\{\int_0^x e^{x-t} u(t)\,dt\right\}$$

$$U(S) = \frac{S}{(S^2-1)} - \frac{1}{S-1} U(S)$$

$$U(S) + \frac{U(S)}{S-1} = \frac{S}{S^2-1}$$

$$U(S)\left[1 + \frac{1}{S-1}\right] = \frac{S}{S^2-1}$$

$$u(s)\left[\frac{s-\cancel{1}+\cancel{1}}{s-1}\right] = \frac{s}{s^2-1}$$

$$u(s) = \frac{\cancel{s}}{s^2-1} \times \frac{(s-1)}{\cancel{s}}$$

$$= \frac{s-1}{s^2-1}$$

$$u(s) = \frac{\cancel{(s-1)}}{\cancel{(s-1)}(s+1)}$$

$$u(s) = \frac{1}{s+1}$$

Now Taking laplace inverse on both sides

$$L^{-1}\{u(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\boxed{u(x) = e^{-x}}$$

<u>VERIFICATION:</u>

using $u(x) = e^{-x}$ in (i)

$$\sinh x = \int_0^x e^{x-t} e^{-t}\, dt$$

Consider

$$RHS = e^{x}\int_0^x e^{-2t}\, dt$$

$$= e^{x}\left[\frac{e^{-2t}}{-2}\Big|_0^x\right]$$

$$RHS = \frac{e^{x}}{-2}\left[e^{-2x} - e^{0}\right]$$

$$= \left[\frac{e^{x-2x} - e^{x}}{-2}\right]$$

$$= \frac{e^{-x} - e^{x}}{-2}$$

$$R.H.S = \frac{e^{x} - e^{-x}}{2}$$

which is $\sinh x$.

EXAMPLE# 3.38

Convert vollera integral equation of the 1st kind to the 2nd kind and also solve the resulting equation

$$1 + \sin x - \cos x = \int_0^x (x - t + 1)u(t)dt \rightarrow (i)$$

Sol.

Differentiating both sides of (i) and using Leibnitz Rule we obtain

$$\frac{d}{dx}(1 + \sin x - \cos x) = \frac{d}{dx}\left[\int_0^x (x - t + 1)u(t)dt\right]$$

$$0 + \cos x + \sin x = \int_0^x \frac{\partial}{\partial x}(x - t + 1)u(t)dt + (x - x + 1)u(x) - (x - 0 + 1)\frac{d(0)}{dx}$$

$$+\cos x + \sin x = \int_0^x u(t)dt + u(x) + 0$$

$$u(x) = \cos x + \sin x - \int_0^x u(t)dt$$

(4)

$$u(x) = \cos x + \sin x - \int_0^x u(t)\,dt$$

Here

$$f(x) = \cos x \cdot \sin x$$
$$= f_1(x) + f_2(x)$$

$$u_0(x) = f_1(x) = \cos x$$
$$f_2(x) = \sin x$$

So, Recurrence Relation is

$$u_{n+1}(x) = f_2(x) - \int_0^x u_n(t)\,dt$$

So,

$$u_1(x) = \sin x - \int_0^x u_0(t)\,dt$$

$$= \sin x - \int_0^x \cos t\,dt$$

$$= \sin x - \sin t\Big|_0^x$$
$$= \sin x - \sin x$$

$$\boxed{u_1(x) = 0}$$

So upto so on $u_2(x) = 0$, $u_3(x) = 0$

So, Exact solution will be

$$\boxed{u(x) = \cos x}$$

EXAMPLE # 3.39

Convert voltera integral equation of 1st kind to the second kind and Solve the resulting equation

$$9x^2+5x^3 = \int_0^x (10x-10t+6)\,u(t)\,dt \rightarrow (i)$$

Sol:

Differentiating both sides of (i) and using Leibnitz rule we obtain

$$18x+15x^2 = \int_0^x \frac{\partial}{\partial x}(10x-10t+6)\,u(t)\,dt + (10x-10x+6)\,u(x)\frac{dx}{dx} + 0$$

$$18x+15x^2 = \int_0^x 10u(t)\,dt + 6u(x)$$

$$= 6u(x) + \int_0^x 10\,u(t)\,dt$$

$$18x+15x^2 = 6u(x) + 10\int_0^x u(t)\,dt$$

$$6u(x) = 18x+15x^2 - 10\int_0^x u(t)\,dt$$

$$u(x) = \frac{5}{2}x^2 + 3x - \frac{5}{3}\int_0^x u(t)\,dt$$

using Adomian decomposition Method

$$u_0(x) = \frac{5}{2}x^2 + 3x$$

So Recurrence Relation will be

$$u_{k+1}(x) = -\frac{5}{3}\int_0^x u_k(t)dt$$

$$u_1(x) = -\frac{5}{3}\int_0^x u_0(t)dt$$

$$u_1(x) = -\frac{5}{3}\left[\int_0^x\left(\frac{5}{2}t^2+3t\right)dt\right]$$

$$= -\frac{5}{3}\left[\frac{5}{2}\left|\frac{t^3}{3}\right|_0^x + \frac{3}{2}\left|t^2\right|_0^x\right]$$

$$= -\frac{5}{3}\left(\frac{5}{6}x^3 + \frac{3}{2}x^2\right)$$

$$= \frac{-25}{18}x^3 - \frac{5}{2}x^2$$

$$u_1(x) = -\frac{5}{2}x^2 - \frac{25}{18}x^3$$

Here $\pm\frac{5}{2}x^2$ is the noise term in $u_0(x)$ and $u_1(x)$. So cancell it from $u_0(x)$. Then we have exact solution as follow

$$\boxed{u(x) = 3x}$$

EXAMPLE#3.40

Convert the voltera integral equation of 1st kind into 2nd kind and solve the resulting equation

$$x\sinh x = 2\int_0^x \sinh(x-t)u(t)dt \rightarrow (i)$$

Solution:

Differentiating both sides of (i) and using Leibnitz rule, we obtain,

$$\frac{d}{dx}(x\sinh x) = 2\frac{d}{dx}\left[\int_0^x \sinh(x-t)u(t)dt\right]$$

$$x\cosh x + \sinh x = 2\left[\int_0^x \frac{\partial}{\partial x}(\sinh(x-t))u(t)dt + \sinh(x-x)u(x) + 0\right]$$

$$= 2\int_0^x \cosh(x-t)\,u(t)dt$$

which is still volterra integral equation of Ist kind.

So, again Differentiating and using Leibnitz Rule,

$$x\sinh x + \cosh x + \cosh x = 2\left[\int_0^x \sinh(x-t)u(t)dt + \cosh(0)u(x) + 0\right]$$

$$x\sinh x + 2\cosh x = 2\int_0^x \sinh(x-t)\,u(t)dt + 2u(x)$$

$$2u(x) = x\sinh x + 2\cosh x - 2\int_0^x \sinh(x-t)u(t)dt$$

$$u(x) = \frac{x}{2}\sinh x + \cosh x - \int_0^x \sinh(x-t)u(t)dt$$

Now using Modified decomposition Method

(8)

let $f(x) = \frac{x}{2}\sinh x + \cosh x$

$f_1(x) = \cosh x$, $f_2(x) = \frac{x}{2}\sinh x$

$u_0(x) = f_1(x) = \cosh x$

So Reccurrence Relation will be

$$u_{k+1}(x) = f_2(x) - \int_0^x \sinh(x-t)\, u_k(t)\,dt$$

$$u_1(x) = \frac{x}{2}\sinh x - \int_0^x \sinh(x-t)\, u_0(t)\,dt$$

$$u_1(x) = \frac{x}{2}\sinh x - \int_0^x \sinh(x-t)\cosh t\,dt$$

$$= \frac{x}{2}\sinh x - \int_0^x \{\sinh x\cosh t - \cosh x\sinh t\}\cosh t\,dt$$

$$= \frac{x}{2}\sinh x - \sinh x\int_0^x \cosh^2 t\,dt + \int_0^x \cosh x\sinh t\cosh t\,dt$$

$$= \frac{x}{2}\sinh x - \sinh x\int_0^x \left(\frac{\cosh 2t + 1}{2}\right)dt + \cosh x\int_0^x \sinh t\cosh t\,dt$$

$$= \frac{x}{2}\sinh x - \frac{\sinh x}{2}\left[\frac{\sinh 2t}{2}\Big|_0^x + t\Big|_0^x\right] + \frac{\cosh x}{2}\left|\frac{\sinh^2 t}{2}\right|_0^x$$

$$= \cancel{\frac{x}{2}\sinh x} - \frac{\sinh x \sinh 2x}{4} - \cancel{\frac{x\sinh x}{2}} + \frac{\cosh x \sinh^2 x}{2}$$

$$= \frac{-\sinh x\, 2\sinh x\cosh x}{4} + \frac{\cosh x \sinh^2 x}{2}$$

$$= -\frac{\cosh x \sinh^2 x}{2} + \frac{\cosh x \sinh^2 x}{2} = 0$$

i.e $\qquad u_4(x) = 0$

So upto so on $u_2(x) = 0$, $u_3(x) = 0$, —

Hence the exact solution will be

$$u(x) = \cosh x$$

Q#01 $e^x + \sin x - \cos x = \int_0^x 2e^{x-t} u(t)\, dt$

differentiat w.r.t 'x' eq by using Laibeiz Rule.

$$\frac{d}{dx}\int_{G(x)}^{H(x)} F(x,t)\, dt = \int_{G(x)}^{H(x)} \frac{\partial}{\partial x} F(x,t)\,dt + F(x, H(x))\frac{dH(x)}{dx} - F(x, G(x))\frac{dG(x)}{dx}$$

$$\frac{d}{dx}e^x + \frac{d}{dx}\sin x - \frac{d}{dx}\cos x = \frac{d}{dx}\int_0^x 2e^{x-t}u(t)\,dt$$

$$e^x + \cos x + \sin x = 2\int_0^x \frac{\partial}{\partial x} e^x \cdot e^{-t} u(t)\,dt + 2e^{x-x}u(x) - 0$$

$$e^x + \cos x + \sin x = 2\int_0^x e^{x-t}u(t)\,dt + 2u(x)$$

$$\Rightarrow 2u(x) = e^x + \cos x + \sin x - 2\int_0^x e^{x-t}u(t)\,dt$$

$$u(x) = \frac{1}{2}(e^x + \cos x + \sin x) - \int_0^x e^{x-t}u(t)\,dt \quad -①$$

Here ~~u(x)~~ we will find the Solution Volteran integral eq of Second Kind

Here we will use successive approximation to solve it.

let zeroth approximation is

$$u_0(x) = \cos x$$

$$u_1(x) = \frac{1}{2}(e^x + \cos x + \sin x) - \int_0^x e^{x-t}\cos t\,dt$$

$$= \frac{1}{2}(e^x + \cos x + \sin x) - e^x\int_0^x e^{-t}\cos t\,dt \quad -(2)$$

Here

$$I = \int_0^x e^{-t}\cos t\,dt$$

$$= -\cos t\, e^{-t}\Big|_0^x - \int_0^x e^{-t}\sin t\,dt$$

$$I = [-\cos x\, e^{-x} + 1] - \left[-\sin t\, e^{-t}\Big|_0^x + \int_0^x e^{-t}\cos t\, dt\right]$$

$$I = 1 - \cos x\, e^{-x} + \sin x\, e^{-t} - \int_0^x e^{-t}\cos t\, dt$$

$$I + I = 1 - \cos x\, e^{-x} + \sin x\, e^{-x}$$

$$2I = \frac{e^x}{e^x} - \frac{\cos x}{e^x} + \frac{\sin x}{e^x}$$

$$I = \frac{1}{2}\frac{1}{e^x}\left(e^x - \cos x + \sin x\right)$$

put in ② we get

$$u_1(x) = \frac{1}{2}(e^x + \sin x + \cos x) - \frac{1}{2}e^x \cdot \frac{1}{e^x}\left(e^x - \cos x + \sin x\right)$$

$$= \frac{1}{2}\left(\cancel{e^x} + \cancel{\sin x} + \cos x - \cancel{e^x} + \cos x - \cancel{\sin x}\right) = \cos x$$

$$u_1(x) = \cos x$$

Similarly $u_2(x) = \cos x$, $u_3(x) = \cos x$

$\vdots$

$$u_n(x) = \cos x \quad (\text{constant})$$

$$\Rightarrow u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \cos x = \cos x$$

Q#02

$$e^x - \cos x = \int_0^x e^{x-t} u(t)\,dt$$

differentiate w.r.t x and using Lebneize rule.

$$e^x + \sin x = \int_0^x \frac{\partial}{\partial x} e^x \cdot e^{-t} u(t)\,dt + e^{x-x} u(x) - 0$$

$$e^x + \sin x = \int_0^x e^{x-t} u(t)\,dt + u(x)$$

$$\Rightarrow u(x) = e^x + \sin x - \int_0^x e^{x-t} u(t)\,dt \quad — ①$$

this Voltera Integral eq of 2nd kind. we will use Leplace to find solution.

taking Leplace of ①

$$U(s) = \frac{1}{s-1} + \frac{1}{s^2+1} - \mathcal{L}\left\{\int_0^x e^{x-t} u(t)\,dt\right\}$$

$$U(s) = \frac{1}{s-1} + \frac{1}{s^2+1} - \mathcal{L}\{e^x\}\mathcal{L}\{u(t)\} \quad \because \text{by convolution theoram}$$

$$U(s) = \frac{1}{s-1} + \frac{1}{s^2+1} - \frac{U(s)}{s-1}$$

$$\Rightarrow U(s) + \frac{U(s)}{s-1} = \frac{1}{s-1} + \frac{1}{s^2+1}$$

$$U(s)\left(1 + \frac{1}{s-1}\right) = \frac{s^2+\cancel{1}+s-\cancel{1}}{(s-1)(s^2+1)} = \frac{s(s+1)}{(s-1)(s^2+1)}$$

$$U(s)\left(\frac{s-\cancel{1}+\cancel{1}}{s-1}\right) = \frac{s(s+1)}{(s^2+1)(s-1)}$$

$$\frac{\cancel{s}\,U(s)}{(\cancel{s-1})} = \frac{\cancel{s}(s+1)}{(\cancel{s-1})(s^2+1)} \quad \Rightarrow U(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1} \quad — ②$$

By taking inverse Leplace of ②

$$u(x) = \cos x + \sin x$$

Q#03 $x = \int_0^x (x-t+1)\,u(t)\,dt$

differentiate w.r.t x Eq using Lebneiz rule

$1 = \frac{d}{dx}\int_0^x (x-t+1)\,u(t)$

$1 = \int_0^x \frac{\partial}{\partial x}(x-t+1)\,u(t)\,dt + (x-x+1)u(x) - 0$

$1 = \int_0^x (1-0)\,u(t)\,dt + u(x)$

$\Rightarrow u(x) = 1 - \int_0^x u(t)\,dt \quad -①$

① is Voltera integral eq of 2nd kind

Now taking Laplace of ①

$\mathcal{L}\{u(x)\} = \mathcal{L}\{1\} - \mathcal{L}\{\int_0^x (1)\,u(t)\,dt$

$u(s) = \frac{1}{s} - \mathcal{L}\{1\}\,\mathcal{L}\{u(t)\}$ ∵ by convolution theorem

$u(s) = \frac{1}{s} - \frac{u(s)}{s}$.

$u(s) + \frac{u(s)}{s} = \frac{1}{s}$

$\Rightarrow u(s)\left(1+\frac{1}{s}\right) = \frac{1}{s}$

$u(s)\left(\frac{s+1}{s}\right) = \frac{1}{s}$

$\Rightarrow u(s) = \frac{1}{s+1}$

taking inverse Laplace

$u(x) = e^{-x}$

Q#04 $e^x + \sin x - \cos x = \int_0^x 2\cos(x-t)u(t)\,dt$

differentiate w.r.t x and using Lebneize rule.

$e^x + \cos x + \sin x = 2\int_0^x -\sin(x-t)u(t) + 2\cos(x-x)u(x) - 0$

$e^x + \cos x + \sin x = -2\int_0^x \sin(x-t)u(t) + 2(1)u(x)$

$\Rightarrow u(x) = \frac{1}{2}(e^x + \cos x + \sin x) + \int_0^x \sin(x-t)u(t)\,dt$ — ①

① is V.I.Eq of 2nd kind

Now we will use successive approximation to solve ① Eq let

$u_0(x) = e^x$ (zero approximation)

$\Rightarrow u_1(x) = \frac{1}{2}(e^x + \cos x + \sin x) + \int_0^x \sin(x-t)e^t\,dt$ — ②

Here

$I = \int_0^x \sin(x-t)e^t\,dt = \sin(x-t)e^t\Big|_0^x + \int_0^x \cos(x-t)e^t\,dt$

$= 0 - \sin x e^0 + \left[\cos(x-t)e^t\Big|_0^x - \int_0^x \sin(x-t)e^t\,dt\right]$

$= -\sin x + (1)e^x - \cos x(e^0) - I$

$2I = -\sin x + e^x - \cos x$

$I = \frac{1}{2}(-\sin x + e^x - \cos x)$

② $\Rightarrow$

$u_1(x) = \frac{1}{2}(e^x + \cos x + \sin x) + \frac{1}{2}(-\sin x + e^x - \cos x)$

$= \frac{1}{2}(2e^x) = e^x$

Similarly $u_2(x) = e^x$, $u_3(x) = e^x$

$\vdots$

$u_n(x) = e^x$

So $u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} e^x = e^x$

$u(x) = e^x$

Q#05 $e^x - x - 1 = \int_0^x (x-t+1)\,u(t)\,dt$

differentiat w.r.t x Eq using Lebniz rule

$e^x - 1 - 0 = \int_0^x \frac{\partial}{\partial x}(x-t+1)\,u(t) + (x-x+1)u(x) - 0$

$e^x - 1 = \int_0^x u(t) + u(x)$

$u(x) = e^x - 1 - \int_0^x u(t)$ —①

① is volterra I.Eq of 2nd kind.

taking Laplace of ① we get

$u(s) = \frac{1}{s-1} - \frac{1}{s} - \mathcal{L}\{1\}\mathcal{L}\{u(x)\}$

∵ using convolution theoram

$u(s) = \frac{1}{s-1} - \frac{1}{s} - \frac{u(s)}{s}$

$u(s)\left(1 + \frac{1}{s}\right) = \frac{1}{s-1} - \frac{1}{s}$

$u(s)\left(\frac{s+1}{s}\right) = \frac{s-(s-1)}{s(s-1)}$

$u(s)(s+1) = \frac{s-s+1}{s-1}$

$u(s) = \frac{1}{(s+1)(s-1)} = \frac{1}{s^2-1}$

taking inverse Laplace:

$\mathcal{L}^{-1}\{u(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\}$

$u(x) = \sinh x$

Q#06 $\frac{1}{2}x^2e^x = \int_0^x e^{x-t}u(t)dt$

differentiat w.r.t x & using Lebneize theorem

$$\frac{1}{2}\frac{d}{dx}(x^2e^x) = \frac{d}{dx}\int_0^x e^{x-t}u(t)dt$$

$$\frac{1}{2}(2xe^x + x^2e^x) = \int_0^x e^{x-t}u(t)dt + e^{x-x}u(x) - 0$$

$$xe^x + \frac{x^2e^x}{2} = \int_0^x e^{x-t}u(t)dt + u(x)$$

$$\Rightarrow u(x) = xe^x + \frac{1}{2}x^2e^x - \int_0^x e^{x-t}u(t)dt \quad -①$$

① is Valtera integral eq of 2nd kind

for this Let $u_0(x) = xe^x$ (zeroth Approx)

$$u_1(x) = xe^x + \frac{1}{2}e^x x^2 - \int_0^x e^{x-t}te^t dt$$

$$= xe^x + \frac{1}{2}x^2e^x - \int_0^x e^{x-t+t}\cdot t\,dt$$

$$= xe^x + \frac{1}{2}e^x x^2 - e^x\int_0^x t\,dt$$

$$u_1(x) = xe^x + \frac{1}{2}x^2e^x - e^x\frac{t^2}{2}\Big|_0^x$$

$$u_1(x) = xe^x + \cancel{\frac{1}{2}x^2e^x} - \cancel{\frac{x^2e^x}{2}}$$

$$\Rightarrow u_1(x) = xe^x$$

Similarly $u_2(x) = xe^x$, $u_3(x) = xe^x$

$\vdots$

$$u_n(x) = xe^x$$

So $\lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} xe^x$ (Constant w.r.t n)

$$u(x) = xe^x$$

Q#07 $e^x - 1 = \int_0^x (x-t+1)\,u(t)\,dt$

diff w.r.t x and using Lebneiz Rule.

$$e^x - 0 = \frac{d}{dx}\int_0^x (x-t+1)\,u(t)\,dt$$

$$e^x = \int_0^x (1-0+0)\,u(t)\,dt + (x-x+1)\,u(x)$$

$$e^x = \int_0^x u(t)\,dt + u(x)$$

$$u(x) = e^x - \int_0^x u(t)\,dt \quad \text{—} ①$$

① is Volterra I.Eq of 2nd kind

Now

taking Laplace of ①

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{e^x\} - \mathcal{L}\left\{\int_0^x u(t)\,dt\right\}$$

$$U(s) = \frac{1}{s-1} - \mathcal{L}\{1\}\,\mathcal{L}\{u(x)\}$$ ∴ By using convolution theoram

$$U(s) = \frac{1}{s-1} - \frac{U(s)}{s}$$

$$U(s)\left(1+\frac{1}{s}\right) = \frac{1}{s-1}$$

$$U(s)\left(\frac{s+1}{s}\right) = \frac{1}{s-1}$$

$$\Rightarrow U(s) = \frac{s}{(s+1)(s-1)} = \frac{s}{s^2-1}$$

taking inverse Laplace

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\}$$

$$u(x) = \cosh x$$

Q# 08

$$\sin x - \cos x + 1 = \int_0^x (x-t+1)u(t)\,dt$$

diff w.r.t x Eq using Lebneize theoram

$$\frac{d}{dx}(\sin x) - \frac{d}{dx}(\cos x) + 0 = \frac{d}{dx}\int_0^x (x-t+1)u(t)\,dt$$

$$\cos x + \sin x = \int_0^x (1-0+0)u(t)\,dt + (x-x+1)u(x)$$

$$\cos x + \sin x = \int_0^x u(t)\,dt + u(x)$$

$$\Rightarrow u(x) = \cos x + \sin x - \int_0^x u(t)\,dt \quad \text{—} ①$$

① is Voltera I.Eq of 2nd Kind

taking Laplace of ①

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{\cos x\} + \mathcal{L}\{\sin x\} - \mathcal{L}\left\{\int_0^x u(t)\,dt\right\}$$

$$u(s) = \frac{s}{s^2+1} + \frac{1}{s^2+1} - \mathcal{L}\{1\}\mathcal{L}\{u(x)\}$$

$\because$ By Convolution theoram

$$u(s) + \frac{u(s)}{s} = \frac{s+1}{s^2+1}$$

$$\Rightarrow u(s)\left(1 + \frac{1}{s}\right) = \frac{s+1}{s^2+1}$$

$$u(s)\left(\frac{\cancel{s+1}}{s}\right) = \frac{\cancel{s+1}}{s^2+1}$$

$$u(s) = \frac{s}{s^2+1}$$

taking inverse Laplace.

$$\mathcal{L}^{-1}\{u(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$u(x) = \cos x$$

Q#09 $5x^4 + x^5 = \int_0^x (x-t+1)u(t)\,dt$

diff w.r.t "x" eq using Lebneizs theorem

$$5.4x^3 + 5x^4 = \int_0^x \frac{\partial}{\partial x}(x-t+1)u(t)\,dt + (x-x+1)u(x) \quad -0$$

$$20x^3 + 5x^4 = \int_0^x (1)u(t)\,dt + u(x)$$

$$\Rightarrow u(x) = 20x^3 + 5x^4 - \int_0^x u(t)\,dt \quad -①$$

① is Volterra I.Eq of 2nd Kind

taking Laplace of ①

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{20x^3\} + \mathcal{L}\{5x^4\} - \mathcal{L}\{\int_0^x u(t)\,dt\}$$

$$u(s) = \frac{20 \times 3!}{s^4} + \frac{4!}{s^5} - \mathcal{L}\{1\}\mathcal{L}\{u(x)\}$$

$$u(s) = \frac{20 \times 6}{s^4} + \frac{120}{s^5} - \frac{u(s)}{s} \quad \because \text{using convolution theorem}$$

$$u(s)\left(1 + \frac{1}{s}\right) = \frac{120}{s^4}\left(1 + \frac{1}{s}\right)$$

$$\Rightarrow u(s) = \frac{120}{s^4}$$

$$u(s) = \frac{20 \times 6}{s^4} = \frac{20 \times 3!}{s^4}$$

taking inverse Laplace

$$\mathcal{L}^{-1}\{u(s)\} = 20\,\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$$

$$u(x) = 20x^3$$

Q#10 $4+x-4e^x+3xe^x=\int_0^x (x-t+2)u(t)dt$

diff w.r.t x Eq using Leibniz theorem

$0+1-4e^x+(3e^x+3xe^x)=\int_0^x \frac{\partial}{\partial x}(x-t+2)u(t)dt + (x-x+2)u(x)-0$

$1-4e^x+3e^x+3xe^x=\int_0^x u(t)dt+2u(x)$

$\Rightarrow u(x)=\frac{1}{2}[1-e^x+3xe^x]-\frac{1}{2}\int_0^x u(t)dt$ —①

① is Voltera I.Eq of 2nd Kind.

taking Laplace of ①

$u(s)=\frac{1}{2}\left[\frac{1}{s}-\frac{1}{s-1}-3\frac{d}{ds}\frac{1}{(s-1)}\right]-\frac{1}{2}\mathcal{L}\{1\}\mathcal{L}\{u(x)\}$

$u(s)=\frac{1}{2}\left[\frac{1}{s}-\frac{1}{s-1}+\frac{3}{(s-1)^2}\right]-\frac{u(s)}{2s}$

$u(s)\left(1+\frac{1}{2s}\right)=\frac{1}{2}\left[\frac{(s-1)^2-s(s-1)+3s}{s(s-1)^2}\right]$

$u(s)\left(\frac{2s+1}{\cancel{2s}}\right)=\frac{1}{\cancel{2}}\left[\frac{\cancel{s^2}-2s+1-\cancel{s^2}+s+3s}{\cancel{s}(s-1)^2}\right]$

$u(s)\cancel{(2s+1)}=\frac{\cancel{(2s+1)}}{(s-1)^2}$

$u(s)=\frac{1}{(s-1)^2}$

taking inverse Laplace.

$\mathcal{L}^{-1}\{u(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$

$u(x)=xe^x$

Q#11 $-3-x+x^2+\frac{x^3}{3!}+3e^x=\int_0^x (x-t+2)u(t)\,dt$

diff w.r.t x Eq using Lebneiz theorm

$-0-1+2x+\frac{3x^2}{6}+3e^x=\int_0^x (1-0+0)u(t)+(x-x+2)u(x)$

$-1+2x+\frac{x^2}{2}+3e^x=\int_0^x u(t)\,dt+2u(x)$

$\Rightarrow u(x)=\frac{1}{2}\left[-1+2x+\frac{x^2}{2}+3e^x\right]-\frac{1}{2}\int_0^x u(t)\,dt$ — ①

① is Valtera I.Eq of 2nd kind.

taking Laplace of ①

$u(s)=\frac{1}{2}\left[\frac{-1}{s}+\frac{2}{s^2}+\frac{2!}{2s^3}+\frac{3}{s-1}\right]-\frac{\mathcal{L}\{1\}\mathcal{L}\{u(x)\}}{2}$

$u(s)=\frac{1}{2}\left(\frac{-s^2(s-1)+2s(s-1)+s-1+3s^3}{s^3(s-1)}\right)-\frac{u(s)}{2s}$

$u(s)\left(1+\frac{1}{2s}\right)=\frac{1}{2}\left(\frac{-s^3+s^2+2s^2-2s+s-1+3s^3}{s^3(s-1)}\right)$

$u(s)\left(\frac{2s+1}{2s}\right)=\frac{1}{2}\left(\frac{2s^3+3s^2-s-1}{s^3(s-1)}\right)$

$u(s)(2s+1)=\frac{(2s^3+3s^2-s-1)}{s^2(s-1)}$

$2s^3+3s^2-s-1$ (By synthetic division)

$$\begin{array}{r|rrrr} & 2 & 3 & -1 & -1 \\ -\frac{1}{2} & \downarrow & -\frac{1}{2}\cdot 2 & -\frac{1}{2}\cdot 2 & -\frac{1}{2}\cdot(-2) \\ \hline & 2 & 2 & -2 & 0 \end{array}$$

$(2s^3+3s^2-s-1)=(s+\frac{1}{2})(2s^2+2s-2)$

$=\frac{(2s+1)}{2}2(s^2+s-1)=(2s+1)(s^2+s-1)$

13

$$\Rightarrow (2s^3+3s^2-s-1) = (2s+1)(s^2+s-1)$$

$$\Rightarrow u(s)(2s+1) = \frac{(2s+1)(s^2+s-1)}{s^2(s-1)}$$

$$\Rightarrow u(s) = \frac{s^2+(s-1)}{s^2(s-1)}$$

$$u(s) = \frac{s^2}{s^2(s-1)} + \frac{(s-1)}{s^2(s-1)}$$

$$u(s) = \frac{1}{s-1} + \frac{1}{s^2}.$$

taking inverse Laplace.

$$L^{-1}\{u(s)\} = L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$u(x) = e^{x} + x$$

$$u(x) = x + e^{x}$$

Q#12 $\tan x - \ln\cos x = \int_0^x (x-t+1)u(t)dt$

diff w.r.t x eq using Lebneize theoram ; $x<\pi/2$

$\frac{d}{dx}(\tan x) - \frac{d}{dx}(\ln\cos x) = \frac{d}{dx}\left\{\int_0^x (x-t+1)u(t)dt\right.$

$\sec^2 x - \frac{1}{\cos x}(-\sin x) = \int_0^x (1-0+0)u(t)dt + u(x)$

$\sec^2 x + \tan x = \int_0^x u(t)dt + u(x)$

$\Rightarrow u(x) = \sec^2 x + \tan x - \int_0^x u(t)dt$ —①

① is Volter I.Eq of 2nd kind.

we will use successive Approximation

Let $u_0(x) = \sec^2 x$ (zeroth approximation)

$u_1(x) = \sec^2 x + \tan x - \int_0^x \sec^2 t\,dt$

$= \sec^2 x + \tan x - \tan t\Big|_0^x$

$= \sec^2 x + \tan x - \tan x - \tan 0$

$u_1(x) = \sec^2 x$

Similarly: $u_2(x) = \sec^2 x$

$\vdots$

$u_n(x) = \sec^2 x$

$u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \sec^2 x$

$= \sec^2 x$

$u(x) = \sec^2 x.$

Question 13 :-

$$x\sin x = 2\int_0^x \sin(x-t)u(t)dt$$

Sol:-

Let

$$x\sin x = 2\int_0^x \sin(x-t)\,u(t)dt$$

Differentiate on both sides w.r.t "x"

$$x\cos x + \sin x = 2\Big[\int_0^x \cos(x-t)u(t)dt + \sin(x-x)u(x)$$
$$- \sin(x-0)u(0)\Big] \quad \because \text{Apply the Leibniz}$$

Rule.

$$x\cos x + \sin x = 2\int_0^x \cos(x-t)u(t)dt + \sin(0)u(x)$$

$$x\cos x + \sin x = 2\int_0^x \cos(x-t)u(t)dt \quad \because K(x,x) \neq 0$$

Again Differentiate and apply the Leibniz Rule.

So,

$$-x\sin x + \cos x + \cos x = 2\Big[-\int_0^x \sin(x-t)u(t)dt$$
$$+ \cos(x-x)u(x) - \cos(x-0)u(0)\Big]$$

$$-x\sin x + 2\cos x = -2\int_0^x \sin(x-t)u(t)dt + 2u(x)$$

$$2u(x) = -x\sin x + 2\cos x + 2\int_0^x \sin(x-t)u(t)dt$$

$$u(x) = -\frac{x\sin x}{2} + \cos x + \int_0^x \sin(x-t)u(t)dt \quad \text{--- (1)}$$

Applying the Laplace Equation.

$$\mathcal{L}\{u(x)\} = -\frac{1}{2}\mathcal{L}\{x\sin x\} + \mathcal{L}\{\cos x\} + \mathcal{L}\Big[\int_0^x \sin(x-t)u(t)dt\Big]$$

$$U(s) = -\frac{1}{2}\left(\frac{s}{(s^2+1)^2}\right) + \frac{s}{s^2+1} + \mathcal{L}\{\sin x\}\,\mathcal{L}\{u(x)\}$$

∴ By Laplace Convolution Product

$$U(s) = -\frac{s}{2(s^2+1)^2} + \frac{s}{s^2+1} + \frac{1}{s^2+1}.U(s)$$

$$U(s) - \frac{U(s)}{s^2+1} = \frac{-s}{2(s^2+1)^2} + \frac{s}{s^2+1}$$

$$\left[1 - \frac{1}{s^2+1}\right]U(s) = \frac{-s}{2(s^2+1)^2} + \frac{s}{s^2+1}$$

$$\frac{s^2}{(s^2+1)}\,U(s) = \frac{-s + 2s(s^2+1)}{2(s^2+1)^{2 1}}$$

$$U(s) = \frac{-s + 2s^3 + 2s}{2s^2(s^2+1)}$$

$$U(s) = \frac{2s^3+s}{2s^2(s^2+1)} \Rightarrow \frac{s(2s^2+1)}{2s^{2}(s^2+1)}$$

$$U(s) = \frac{2s^{2}}{2s(s^2+1)} + \frac{1}{2s(s^2+1)}$$

$$U(s) = \frac{s}{s^2+1} + \frac{1}{2}\,\frac{1}{s(s^2+1)} \quad \text{——— (2)}$$

Use partial fraction,

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \quad (2')$$

$1 = A(s^2+1) + (Bs+C)s$ — (3)

put $s=0$ in eq. (3)

So,

$\boxed{1 = A}$

Now

$1 = As^2 + A + Bs^2 + Cs$

$1 = (A+B)s^2 + Cs + A$

Comparing for co-efficient "s^2"

$0 = A+B$

$B = -A$ $\boxed{B=-1}$

Comparing co-efficient "s"

$\boxed{0 = C}$

Now Equation (2') Becomes,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

So, Equation (2) becomes,

$$U(s) = \frac{s}{s^2+1} + \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]$$

$$U(s) = \frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+1}$$

Applying the Laplace Inverse.

$$U(x) = \cos x + \frac{1}{2}x - \frac{1}{2}\cos x$$

$$U(x) = \frac{1}{2}x + \left(1 - \frac{1}{2}\right)\cos x$$

$$U(x) = \frac{1}{2}x + \frac{1}{2}\cos x \Rightarrow \boxed{U(x) = \frac{1}{2}(x + \cos x)}$$

Question 14:-

$$e^{x}-\sin x-\cos x=2\int_0^x \sin(x-t)u(t)dt$$

Sol:-

Let

$$e^{x}-\sin x-\cos x=2\int_0^x \sin(x-t)u(t)dt$$

Differentiate w.r.t "x" and applying the Leibniz Rule.

$$e^{x}-\cos x+\sin x=2\left[\int_0^x \cos(x-t)u(t)dt+\sin(x-x)u(x)-\sin(x-0)u(0)\right]$$

$$e^{x}-\cos x+\sin x=2\int_0^x \cos(x-t)u(t)dt \quad \because K(x,x)=0$$

Again Differentiate w.r.t "x" and applying the Leibniz Rule.

$$e^{x}+\sin x+\cos x=2\left[-\int_0^x \sin(x-t)u(t)dt+\cos(x-x)u(x)-\cos(x-0)u(0)\right]$$

$$e^{x}+\sin x+\cos x=-2\int_0^x \sin(x-t)u(t)dt+2u(x)$$

$$2u(x)=e^{x}+\sin x+\cos x+2\int_0^x \sin(x-t)u(t)dt$$

$$u(x)=\frac{e^{x}}{2}+\frac{\sin x}{2}+\frac{\cos x}{2}+\int_0^x \sin(x-t)u(t)dt$$

taking Laplace and apply Laplace convolution product.

$$U(s)=\frac{1}{2}\mathcal{L}\{e^{x}\}+\frac{1}{2}\mathcal{L}\{\sin x\}+\frac{1}{2}\mathcal{L}\{\cos x\}+\mathcal{L}\{\sin x\}\mathcal{L}\{u(x)\}$$

$$U(s)=\frac{1}{2}\frac{1}{s-1}+\frac{1}{2}\frac{1}{s^2+1}+\frac{1}{2}\cdot\frac{s}{s^2+1}+\frac{1}{s^2+1}U(s)$$

$$U(s) - \frac{U(s)}{s^2+1} = \frac{1}{2(s-1)} + \frac{1}{2(s^2+1)} + \frac{s}{2(s^2+1)}$$

$$U(s)\left[1 - \frac{1}{s^2+1}\right] = \frac{s^2+\cancel{1}+s-\cancel{1}+s(s-1)}{2(s-1)(s^2+1)}$$

$$U(s)\left[\frac{s^2}{\cancel{(s^2+1)}}\right] = \frac{s^2+\cancel{s}+s^2-\cancel{s}}{2(s-1)\cancel{(s^2+1)}}$$

$$s^2 U(s) = \frac{\cancel{2}s^2}{\cancel{2}(s-1)}$$

$$U(s) = \frac{1}{s-1}$$

Taking Laplace Inverse

$$\boxed{U(x) = e^{x}}$$

Q15: $\sin x - \cos x + e^{-x} = \int_0^x 2\sin(x-t)\,u(t)\,dt$

The volterra integral equation of first kind is given

Differentiating both sides w.r.t x and using Leibnitz Rule

$$\frac{d}{dx}(\sin x - \cos x + e^{-x}) = \frac{d}{dx}\left(2\int_0^x \sin(x-t)\,u(t)\,dt\right)$$

$$\cos x + \sin x - e^{-x} = 2\int_0^x \frac{\partial}{\partial x}[\sin(x-t)\,u(t)]\,dt + 2\sin(x-x)\,u(x)\frac{d}{dx}(x) - 2\sin(x-0)\,u(0)\frac{d}{dx}(0)$$

$$\cos x + \sin x - e^{-x} = 2\int_0^x \cos(x-t)\,u(t)\,dt$$

Which is still a Volter Integral equation of first kind

Differentiating again w.r.t x and using Leibnitz Rule

$$-\sin x + \cos x + e^{-x} = 2\int_0^x -\sin(x-t)\,u(t)\,dt + 2\cos(0)\,u(x)\,(1) - 2\cos x\,u(0)\,(0)$$

$$\Rightarrow -\sin x + \cos x + e^{-x} = 2u(x) - 2\int_0^x \sin(x-t)\,u(t)\,dt$$

or

$$2u(x) = e^{-x} + \cos x - \sin x + 2\int_0^x \sin(x-t)\,u(t)\,dt$$

$$\Rightarrow u(x) = \frac{1}{2}(e^{-x} + \cos x - \sin x) + \int_0^x \sin(x-t)u(t)\,dt \quad \text{--- (1)}$$

That is Voltera integral Equation of the second kind

Now we will solve this equation Using Laplace transform on (1)

$$L[u(x)] = L\left[\frac{1}{2}(e^{-x} + \cos x - \sin x)\right] + L\left[\int_0^x \sin(x-t)u(t)\,dt\right]$$

$$\Rightarrow U(s) = \frac{1}{2}\left\{L[e^{-x}] + L[\cos x] - L[\sin x]\right\} + L[\sin x]\,L[u(x)]$$

$$= \frac{1}{2}\left\{\frac{1}{s+1} + \frac{s}{s^2+1} - \frac{1}{s^2+1}\right\} + \frac{1}{s^2+1}U(s)$$

$$\Rightarrow U(s) + \frac{1}{s^2+1}U(s) = \frac{1}{2}\left\{\frac{s^2+1+s^2+s-s-1}{(s+1)(s^2+1)}\right\}$$

$$U(s)\left\{1 + \frac{1}{s^2+1}\right\} = \frac{1}{2}\left\{\frac{2s^2}{(s+1)(s^2+1)}\right\}$$

$$U(s)\left\{\frac{s^2}{s^2+1}\right\} = \frac{s^2}{(s+1)(s^2+1)}$$

$$\Rightarrow U(s) = \frac{1}{s+1}$$

Taking Laplace Inverse on both sides

$$\mathcal{L}^{-1}[U(s)] = \mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

$$\Rightarrow \quad u(x) = e^{-x}$$

Q16: $\sin x - x\cos x = \int_0^x 2\sinh(x-t)\,u(t)\,dt$

The Voltera Integral Equation of first kind is given

Differentiating both sides w.r.t x and using Leibnitz Rule

$$\frac{d}{dx}(\sin x - x\cos x) = \frac{d}{dx}\left(\int_0^x 2\sinh(x-t)u(t)dt\right)$$

$$\cos x - \cos x + x\sin x = \int_0^x \frac{\partial}{\partial x}(2\sinh(x-t)\,u(t))\,dt$$
$$+ 2\sinh(x-x)u(x)\frac{d}{dx}(x)$$
$$- 2\sinh(x-0)\,u(0)\,\frac{d}{dx}(0)$$

$$\Rightarrow x\sin x = \int_0^x 2\cosh(x-t)u(t)dt$$

Which is still a Voltera Integral equation of first kind

Differentiating again w.r.t x and using Leibnitz Rule.

$$\sin x + x\cos x = \int_0^x 2\sinh(x-t)\,u(t)\,dt$$
$$+ 2\cosh(x-x)u(x)(1)$$
$$- 2\cosh(x-0)\,u(0)\,(0)$$

$$\Rightarrow \sin x + x\cos x = 2u(x) + 2\int_0^x \sinh(x-t)\, u(t)\, dt$$

or

$$u(x) = \frac{1}{2}(\sin x + x\cos x) - \int_0^x \sinh(x-t)\, u(t)\, dt \quad \text{—①}$$

which is volterra integral equation of second kind

Now we will solve this equation Using Laplace transform on ①

$$L[u(x)] = L\left[\frac{1}{2}(\sin x + x\cos x)\right] - L\left[\int_0^x \sinh(x-t)\, u(t)\, dt\right]$$

$$U(s) = \frac{1}{2}\left\{L[\sin x] + L[x\cos x]\right\} - L[\sinh x]\, L[u(x)]$$

$$= \frac{1}{2}\left[\frac{1}{s^2+1} - \frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right] - \frac{1}{s^2-1}\, U(s)$$

$$\Rightarrow U(s) + \frac{1}{s^2-1}U(s) = \frac{1}{2}\left[\frac{1}{s^2+1} - \frac{(s^2+1) - s(2s)}{(s^2+1)^2}\right]$$

$$U(s)\left[1 + \frac{1}{s^2-1}\right] = \frac{1}{2}\left[\frac{s^2+1 + s^2 - 1}{(s^2+1)^2}\right]$$

$$\Rightarrow U(s)\left[\frac{s^2-1+1}{s^2-1}\right] = \frac{1}{2}\left[\frac{2s^2}{(s^2+1)^2}\right]$$

$$\Rightarrow U(s) = \frac{s^2-1}{(s^2+1)^2}$$

$$= \frac{s^2+1}{(s^2+1)^2} - \frac{2}{(s^2+1)^2}$$

$$= \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}$$

Taking Inverse Laplace Transform

$$L^{-1}[U(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] - 2L^{-1}\left[\frac{1}{(s^2+1)^2}\right]$$

$$u(x) = \sin x - 2L^{-1}\left[\frac{1}{(s^2+1)^2}\right] \quad \text{—— (2)}$$

Consider

$$L^{-1}\left[\frac{1}{(s^2+1)^2}\right] = L^{-1}\left[\frac{1}{s^2+1} \cdot \frac{1}{s^2+1}\right]$$

$$= \int_0^x \sin(x-t)\sin t\,dt$$

$$= \int_0^x \sin x \cos t \sin t\,dt - \int_0^x \cos x \sin t \sin t\,dt$$

$$= \sin x \left|\frac{\sin^2 t}{2}\right|_0^x - \cos x \int_0^x \left(\frac{1-\cos 2t}{2}\right)dt$$

$$= \frac{\sin^3 x}{2} - \cos x\left(\frac{1}{2}\right)\int_0^x dt + \frac{\cos x}{2}\left|\frac{\sin 2t}{2}\right|_0^x$$

$$= \frac{\sin^3 x}{2} - \frac{x\cos x}{2} + \frac{2\sin x\cos^2 x}{4}$$

$$= \frac{\sin x}{2}(\sin^2 x + \cos^2 x) - \frac{x\cos x}{2}$$

$$= \frac{\sin x}{2} - \frac{x\cos x}{2}$$

from (2)

$$u(x) = \sin x - 2\left(\frac{\sin x}{2} - \frac{x\cos x}{2}\right)$$

$$= \sin x - \sin x + x\cos x$$

$$\Rightarrow \quad u(x) = x\cos x.$$