Volterra Integral Equation

The Volterra Integral equation is special type of integral equations.

$$g(x)y(x) = f(x) + \lambda \int_0^x K(x,t)\, y(t)\, dt \longrightarrow i)$$

In this integral equation Lower Limit is constant and upper Limit is variable. $f(x)$, $g(x)$ and $K(x,t)$ are known functions. While λ is non-zero parameter. $K(x,t)$ is kernel.

Volterra Integral equation of first kind

A Linear integral equation of the form by using $g(x)=0$ in (i)

$$f(x) + \lambda \int_0^x K(x,t)y(t)dt = 0$$

Volterra Integral equation of Second kind

A Linear integral equation of the form by using $g(x)=1$ in (i)

$$y(x) = f(x) + \lambda \int_0^x K(x,t)y(t)dt$$

Volterra Integral equation of third kind

$$g(x)y(x) = f(x) + \lambda \int_0^x K(x,t)y(t)dt \quad (\because g(x) \neq 0)$$

Convolution Volterra Integral equation

$$y(x) = f(x) + \lambda \int_0^x K(x-t)y(t)dt$$

The function k in this integral is kernel.

Laplace Transformation

Laplace transform is a function of $f(x)$ in $(0, \infty)$ is defined as $F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\,dx$

Properties of Laplace transform

→ $f(x)$ be a real function which is continuous for $x \geq 0$ and of exponential order e^{ax} and Let $\frac{\partial f}{\partial x}$ be sectionally (piecewise) continuous in finite closed interval $0 \leq x \leq A$

→ $$\mathcal{L}\left\{\frac{df}{dx}\right\} = \int_0^\infty e^{-sx} \frac{df}{dx}\,dx$$

$$= \left| e^{-sx} f(x) \right|_0^\infty + s\int_0^\infty e^{-sx} f(x)\,dx$$

$$= 0 - f(0) + sF(s)$$

$$\mathcal{L}\left\{\frac{df}{dx}\right\} = sF(s) - f(0)$$

Convolution Theorem

$$u(x) = f(x) + \lambda\int_0^x K(x,t)\,u(t)\,dt$$

In the convolution theorem for the

Laplace transform: if the kernel $K(x,t)$ of the integral equation depends on the difference $x-t$, then it is called a difference kernel.

Examples of the difference kernel are e^{x-t}, $\cos(x-t)$ and $(x-t)$.

The integral equation can thus be expressed as

$$u(x) = f(x) + \lambda \int_0^x K(x-t)u(t)\,dt$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions needed for the existence of Laplace transform for each.

Let the Laplace transforms for the function $f_1(x)$ and $f_2(x)$ be given by

$$\mathcal{L}\{f_1(x)\} = F_1(x)$$
$$\mathcal{L}\{f_2(x)\} = F_2(x)$$

The Laplace convolution product of these two functions is defined by

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)\,dt$$

OR

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t)f_1(t)\,dt$$

$$(f_1 * f_2)(x) = (f_2 * f_1)(x)$$

We can easily show that the Laplace transform of the Convolution product $(f_1 * f_2)(x)$ is given by

$$\mathcal{L}\{(f_1 * f_2)(x)\} = \mathcal{L}\left\{\int_0^x f_1(x-t) f_2(t)\,dt\right\}$$

$$= F_1(s) F_2(s)$$

LAPLACE TRANSFORM OF VOLTERRA Integral Equation

In the Volterra integral equation where the Kernel is difference Kernel.

$$u(x) = f(x) + \lambda \int_0^x K(x-t)\,u(t)\,dt \quad \rightarrow i)$$

By taking Laplace transform of both sides of equation

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{f(x) + \lambda \int_0^x K(x-t)\,u(t)\,dt\right\}$$

$$= \mathcal{L}\{f(x)\} + \lambda \mathcal{L} \int_0^x K(x-t)\,u(t)\,dt$$

$$U(s) = F(s) + \lambda \mathcal{L}\left\{\int_0^x K(x-t)\,u(t)\,dt\right\} \rightarrow ii)$$

Using Convolution theorem

$$\mathcal{L}\left\{\int_0^x k(x-t)u(t)\,dt\right\} = K(s)U(s) \text{ put in (ii)}$$

Now $U(s) = F(s) + \lambda K(s)U(s) \longrightarrow \text{iii)}$

Solving eq (iii) for $U(s)$ gives

$$U(s) - \lambda K(s)U(s) = F(s)$$

$$U(s)(1 - \lambda K(s)) = F(s)$$

$$U(s) = \frac{F(s)}{1-\lambda K(s)} \longrightarrow \text{iv)} \quad \lambda K(s) \neq 1$$

The solution $u(x)$ is obtained by taking the inverse Laplace of both sides of eq (iv)

$$\mathcal{L}^{-1}[U(s)] = \mathcal{L}^{-1}\left\{\frac{F(s)}{1-\lambda K(s)}\right\}$$

$$u(x) = \mathcal{L}^{-1}\left\{\frac{F(s)}{1-\lambda K(s)}\right\}$$

Example 3.23 :-

Solve the votlera integral equation by using Laplace transform method

$$u(x) = 1 + \int_0^x u(t)\,dt$$

$$u(x) = 1 + \int_0^x u(t)\,dt \longrightarrow (1)$$

Taking Laplace transform of both sides of eq (1)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{1 + \int_0^x u(t)\,dt\right\}$$

$$U(s) = \mathcal{L}\{1\} + \mathcal{L}\left\{\int_0^x u(t)\,dt\right\}$$

$$U(s) = \frac{1}{s} + \frac{1}{s}U(s)$$

$$U(s) - \frac{1}{s}U(s) = \frac{1}{s}$$

$$U(s)\left(1 - \frac{1}{s}\right) = \frac{1}{s}$$

$$U(s)\left(\frac{s-1}{s}\right) = \frac{1}{s}$$

$$U(s) = \frac{1}{\cancel{s}} \times \frac{\cancel{s}}{s-1}$$

$$U(s) = \frac{1}{s-1}$$

Taking Inverse Laplace transform

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$u(x) = e^{x}$$

Example 3.24:-

Solve the Volterra integral equation by using Laplace transform method

$$u(x) = 1 - \int_0^x (x-t)u(t)\,dt$$

$$u(x) = 1 - \int_0^x (x-t)u(t)\,dt \longrightarrow (1)$$

$$K(x,t) = (x-t)$$

Taking Laplace on both sides of eq (1)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{1 - \int_0^x (x-t)u(t)\,dt\right\}$$

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{1\} - \mathcal{L}\left\{\int_0^x (x-t)u(t)\,dt\right\}$$

$$U(s) = \frac{1}{s} - \mathcal{L}\{x\}\cdot\mathcal{L}\{u(x)\} \quad \left(\because \text{by using convolution theorem}\right)$$

$$U(s) = \frac{1}{s} - \frac{1}{s^2}U(s)$$

$$U(s) + \frac{1}{s^2}U(s) = \frac{1}{s}$$

$$U(s)\left(1 + \frac{1}{s^2}\right) = \frac{1}{s}$$

$$U(s)\left(\frac{s^2+1}{s^2}\right) = \frac{1}{s}$$

$$U(s) = \frac{1}{s} \cdot \frac{s^2}{s^2+1}$$

$$U(s) = \frac{s}{s^2+1}$$

Taking Laplace inverse on both sides

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$\left(\because \mathcal{L}\{\cos x\} = \frac{s}{s^2+1}\right)$$

$$u(x) = \cos x$$

EXAMPLE 3.25:-

Solve the Volterra integral equation by using Laplace transform method

$$u(x) = \frac{x^3}{3!} - \int_0^x (x-t)u(t)dt$$

$$u(x) = \frac{x^3}{3!} - \int_0^x (x-t)u(t)dt \longrightarrow (1)$$

Taking Laplace transform on both sides of eq (1)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{\frac{x^3}{3!} - \int_0^x (x-t)u(t)dt\right\}$$

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{\frac{x^3}{3!}\right\} - \mathcal{L}\left\{\int_0^x (x-t)u(t)dt\right\}$$

$$U(s) = \frac{1}{3!} \times \frac{3!}{s^4} - \mathcal{L}\left\{\int_0^x (x-t)u(t)dt\right\} \longrightarrow ii)$$

using convolution theorem

$$\mathcal{L}\left\{\int_0^x (x-t)u(t)dt\right\} - \mathcal{L}\{x\}\mathcal{L}\{u(x)\}$$

$$= \frac{1}{s^2}U(s) \quad \text{put in (ii)}$$

$$(ii) \Longrightarrow \quad U(s) = \frac{1}{s^4} - \frac{1}{s^2}U(s)$$

$$U(s) + \frac{1}{s^2}U(s) = \frac{1}{s^4}$$

$$U(s)\left(1 + \frac{1}{s^2}\right) = \frac{1}{s^4}$$

$$U(s) = \frac{1}{s^4} \times \frac{s^2}{s^2+1} = \frac{1}{s^2(s^2+1)}$$

By Partial fraction

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} \longrightarrow iii)$$

$$1 = As(s^2+1) + B(s^2+1) + (Cs+D)(s^2)$$

$$1 = A(s^3+s) + Bs^2 + B + Cs^3 + Ds^2$$

$$1 = As^3 + As + Bs^2 + B + Cs^3 + Ds^2$$

Comparing coff

s^3; $\quad A + C = 0 \longrightarrow iv)$

s^2; $\quad B + D = 0 \longrightarrow v)$

s; $\quad A = 0 \longrightarrow vi)$

s^0; $\quad B = 1 \longrightarrow vii)$

$v) \Longrightarrow \quad D = -1$

$iv) \Longrightarrow \quad C = 0$

Use all values in eq ii)

$$= \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$U(s) = \frac{1}{s^2} - \frac{1}{s^2+1}$$

Taking Laplace inverse

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\}$$

$$U(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$U(x) = x - \sin x$$

EXAMPLE 3.26

Solve Voltera integral equation by using Laplace transform

$$U(x) = \sin x + \cos x + 2\int_0^x \sin(x-t)u(t)\,dt$$

$$U(x) = \sin x + \cos x + 2\int_0^x \sin(x-t)u(t)\,dt \longrightarrow i)$$

Taking Laplace transform on both sides of equation (i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{\sin x + \cos x + 2\int_0^x \sin(x-t)u(t)\,dt\right\}$$

$$= \mathcal{L}\{\sin x\} + \mathcal{L}\{\cos x\} + 2\mathcal{L}\left\{\int_0^x \sin(x-t)u(t)\,dt\right\}$$

$$= \frac{1}{s^2+1} + \frac{s}{s^2+1} + 2\mathcal{L}\left\{\int_0^x \sin(x-t)u(t)\,dt\right\} \longrightarrow \text{ii)}$$

Using convolution theorem

$$\mathcal{L}\left\{\int_0^x \sin(x-t)u(t)\,dt\right\} = \mathcal{L}\{\sin x\}\mathcal{L}\{u(t)\}$$

$$= \frac{1}{s^2+1}U(s) \quad \text{put in (ii)}$$

$$\text{ii)} \Longrightarrow \quad U(s) = \frac{1}{s^2+1} + \frac{s}{s^2+1} + \frac{2}{s^2+1}U(s)$$

$$U(s) - \frac{2}{s^2+1}U(s) = \frac{1+s}{s^2+1}$$

$$U(s)\left(1 - \frac{2}{s^2+1}\right) = \frac{1+s}{s^2+1}$$

$$U(s)\left(\frac{s^2+1-2}{s^2+1}\right) = \frac{s+1}{s^2+1}$$

$$U(s) = \frac{s+1}{\cancel{s^2+1}} \times \frac{\cancel{s^2+1}}{s^2-1}$$

$$= \frac{s+1}{(s^2-1)} = \frac{\cancel{s+1}}{(s-1)\cancel{(s+1)}}$$

$$U(s) = \frac{1}{s-1}$$

Taking Laplace inverse on both sides

$$\mathcal{L}^{-1}\{u(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$u(x) = e^{x}$$

EXERCISE

Use the Laplace transform method to solve the Volterra Integral Equations.

1- $u(x) = x + \int_0^x (x-t)u(t)dt$

Solution:-

$$u(x) = x + \int_0^x (x-t)u(t)dt \longrightarrow i)$$

Taking Laplace transform on both sides of equation (i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{x + \int_0^x (x-t)u(t)dt\right\}$$

$$= \mathcal{L}\{x\} + \mathcal{L}\left\{\int_0^x (x-t)u(t)dt\right\}$$

$$\mathcal{L}\{u(x)\} = \frac{1}{s^2} + \mathcal{L}\left\{\int_0^x (x-t)u(t)dt\right\} \longrightarrow ii)$$

$\because$ by Convolution theorem

$$\int_0^x f(x-t)\,f(t)\,dt = F_1(s)F_2(s)$$

$$\mathcal{L}\left\{\int_0^x (x-t)\,u(t)\,dt\right\} = \mathcal{L}\{x\}\cdot\mathcal{L}\{u(x)\}$$

$$= \frac{1}{s^2}\,u(s) \quad \text{put in (ii)}$$

Now

ii) $\Rightarrow$ $$u(s) = \frac{1}{s^2} + \frac{u(s)}{s^2}$$

$$u(s) - \frac{u(s)}{s^2} = \frac{1}{s^2}$$

$$u(s)\left(1 - \frac{1}{s^2}\right) = \frac{1}{s^2}$$

$$u(s)\left(\frac{s^2-1}{s^2}\right) = \frac{1}{s^2}$$

$$u(s) = \frac{1}{\cancel{s^2}} \times \frac{\cancel{s^2}}{s^2-1}$$

$$u(s) = \frac{1}{s^2-1}$$

Taking Laplace inverse on both sides

$$\mathcal{L}^{-1}\{u(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\}$$

$$u(x) = \sinh x$$

2- $u(x) = 1 - x - \int_0^x (x-t)u(t)dt$

Solution:-

$$u(x) = 1 - x - \int_0^x (x-t)u(t)dt \longrightarrow i)$$

Taking Laplace transform on both sides of eq (i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{1 - x - \int_0^x (x-t)u(t)dt\}$$

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{1\} - \mathcal{L}\{x\} - \mathcal{L}\{\int_0^x (x-t)u(t)dt\}$$

$$U(s) = \frac{1}{s} - \frac{1}{s^2} - \mathcal{L}\{\int_0^x (x-t)u(t)dt\} \longrightarrow ii)$$

$\because$ by convolution theorem

$$\mathcal{L}\{\int_0^x (x-t)u(t)dt\} = \mathcal{L}\{x\} \cdot \mathcal{L}\{u(x)\}$$

$$= \frac{1}{s^2} u(s) \quad \text{put in (ii)}$$

$$(ii) \Rightarrow U(s) = \frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^2} u(s)$$

$$U(s) + \frac{1}{s^2} u(s) = \frac{s-1}{s^2}$$

$$u(s)\left(1 + \frac{1}{s^2}\right) = \frac{s-1}{s^2}$$

$$\mathcal{L}\left\{\frac{s^2+1}{s^2}\right\}U(s) = \frac{s-1}{s^2}$$

$$U(s) = \frac{s-1}{s^2} \times \frac{s^2}{s^2+1}$$

$$= \frac{s-1}{s^2+1}$$

$$U(s) = \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

Taking Laplace inverse on B.S

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$u(x) = \cos x - \sin x$$

3. $u(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)\,dt$

Solution:-

$$u(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)\,dt \longrightarrow i)$$

Taking Laplace inverse on both sides

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)\,dt\right\}$$

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{1\} - \mathcal{L}\left\{\frac{x^2}{2}\right\} + \frac{1}{6}\mathcal{L}\left\{\int_0^x (x-t)^3 u(t)\,dt\right\}$$

$$U(s) = \frac{1}{s} - \frac{1}{2}\cdot\frac{2}{s^3} + \frac{1}{6}\mathcal{L}\left\{\int_0^x (x-t)^3 u(t)\,dt\right\} \longrightarrow \text{ii)}$$

∵ by convolution theorem

$$\mathcal{L}\left\{\int_0^x (x-t)^3 u(t)\,dt\right\} = \mathcal{L}\{x^3\}\cdot\mathcal{L}\{u(x)\}$$

$$= \frac{3!}{s^4}U(s) \quad \text{put in (ii)}$$

$$\text{(ii)} \Rightarrow U(s) = \frac{1}{s} - \frac{1}{s^3} + \frac{U(s)}{s^4}$$

$$U(s) - \frac{U(s)}{s^4} = \frac{s^2-1}{s^3}$$

$$U(s)\left(1 - \frac{1}{s^4}\right) = \frac{s^2-1}{s^3}$$

$$U(s) = \frac{s^2-1}{s^3}\cdot\frac{s^4}{s^4-1}$$

$$= \frac{s(s^2-1)}{(s^4-1)} = \frac{s\cancel{(s^2-1)}}{\cancel{(s^2-1)}(s^2+1)}$$

$$U(s) = \frac{s}{s^2+1}$$

Taking Laplace inverse

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$U(x) = \cos x$$

4- $u(x) = 1 + 3\int_0^x (x-t)u(t)\,dt$

Solution:-

$$u(x) = 1 + 3\int_0^x (x-t)u(t)\,dt$$

Taking Laplace on both sides

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{1 + 3\int_0^x (x-t)u(t)\,dt\}$$

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{1\} + 3\mathcal{L}\{\int_0^x (x-t)u(t)\,dt\}$$

$$U(s) = \frac{1}{s} + 3\mathcal{L}\{\int_0^x (x-t)u(t)\,dt\}$$

$\because$ using convolution theorem

$$\mathcal{L}\{\int_0^x (x-t)u(t)\,dt\} = \mathcal{L}\{x\}\cdot\mathcal{L}\{u(x)\}$$

$$= \frac{1}{s^2}U(s)$$

$$U(s) = \frac{1}{s} + \frac{3U(s)}{s^2}$$

$$U(s) - \frac{3}{s^2}U(s) = \frac{1}{s}$$

$$U(s)\left(1 - \frac{3}{s^2}\right) = \frac{1}{s}$$

$$U(s) = \frac{1}{s} * \frac{s^2}{s^2-3}$$

$$U(s) = \frac{s}{s^2-3}$$

Taking Laplace inverse

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-3}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2-(\sqrt{3})^2}\right\}$$

$$U(x) = \cosh\sqrt{3}x$$

5- $u(x) = x - 1 + \int_0^x (x-t)u(t)dt$

Solution:-

$$u(x) = x - 1 + \int_0^x (x-t)u(t)dt \longrightarrow i)$$

Taking Laplace on both sides of eq i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{x - 1 + \int_0^x (x-t)u(t)dt\right\}$$

$$= \mathcal{L}\{x\} - \mathcal{L}\{1\} + \mathcal{L}\left\{\int_0^x (x-t)u(t)dt\right\}$$

$$U(s) = \frac{1}{s^2} - \frac{1}{s} + \mathcal{L}\left\{\int_0^x (x-t)u(t)dt\right\} \longrightarrow ii)$$

$\because$ by convolution theorem

$$\mathcal{L}\left\{\int_0^x (x-t)u(t)dt\right\} = \mathcal{L}\{x\}\cdot\mathcal{L}\{u(t)\}$$

$$= \frac{1}{s^2}U(s) \quad \text{put in (ii)}$$

$$ii) \Rightarrow \quad U(s) = \frac{1}{s^2} - \frac{1}{s} + \frac{U(s)}{s^2}$$

$$U(s) - \frac{U(s)}{s^2} = \frac{1}{s^2} - \frac{1}{s}$$

$$U(s)\left(1 - \frac{1}{s^2}\right) = \frac{1-s}{s^2}$$

$$U(s) = \frac{1-s}{\cancel{s^2}} \times \frac{\cancel{s^2}}{s^2-1}$$

$$U(s) = \frac{1-s}{s^2-1}$$

$$U(s) = \frac{1}{s^2-1} - \frac{s}{s^2-1}$$

Taking Laplace inverse

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2-1} - \frac{s}{s^2-1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\}$$

$$u(x) = \sinh x - \cosh x$$

6- $u(x) = \cos x - \sin x + 2\int_0^x \cos(x-t)u(t)\,dt$

Solution:-

$$u(x) = \cos x - \sin x + 2\int_0^x \cos(x-t)u(t)\,dt \rightarrow i)$$

Taking Laplace on both sides of eq i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{\cos x - \sin x + 2\int_0^x \cos(x-t)u(t)\,dt\right\}$$

$$= \mathcal{L}\{\cos x\} - \mathcal{L}\{\sin x\} + 2\mathcal{L}\left\{\int_0^x \cos(x-t)u(t)\,dt\right\}$$

$$u(s) = \frac{s}{s^2+1} - \frac{1}{s^2+1} + 2\mathcal{L}\left\{\int_0^x \cos(x-t)u(t)\,dt\right\} \longrightarrow (ii)$$

$\because$ by Convolution theorem

$$\mathcal{L}\left\{\int_0^x \cos(x-t)u(t)\,dt\right\} = \mathcal{L}\{\cos x\}\cdot\mathcal{L}\{u(x)\}$$

$$= \frac{s}{s^2+1}u(s) \quad \text{put in (ii)}$$

$$(ii) \Longrightarrow u(s) = \frac{s}{s^2+1} - \frac{1}{s^2+1} + \frac{2s}{s^2+1}u(s)$$

$$u(s) - \frac{2s}{s^2+1}u(s) = \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

$$u(s)\left(1 - \frac{2s}{s^2+1}\right) = \frac{s-1}{s^2+1}$$

$$u(s) = \frac{s-1}{\cancel{s^2+1}} \times \frac{\cancel{s^2+1}}{s^2-2s+1}$$

$$u(s) = \frac{s-1}{(s-1)^2} = \frac{1}{s-1}$$

$$u(s) = \frac{1}{s-1}$$

Taking Laplace inverse

$$\mathcal{L}^{-1}\{u(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$u(x) = e^x$$

7- $u(x) = e^x - \cos x - 2\int_0^x e^{x-t}\, u(t)\,dt$

Solutions:-

$$u(x) = e^x - \cos x - 2\int_0^x e^{x-t}\, u(t)\,dt \longrightarrow i)$$

Taking Laplace on both sides on eq i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{e^x - \cos x - 2\int_0^x e^{x-t}\, u(t)\,dt\right\}$$

$$= \mathcal{L}\{e^x\} - \mathcal{L}\{\cos x\} - 2\mathcal{L}\left\{\int_0^x e^{x-t}\, u(t)\,dt\right\}$$

$$U(s) = \frac{1}{s-1} - \frac{s}{s^2+1} - 2\mathcal{L}\left\{\int_0^x e^{x-t}\, u(t)\,dt\right\}$$

$$= \frac{1}{s-1} - \frac{s}{s^2+1} - 2\left[\mathcal{L}\{e^x\}\cdot\mathcal{L}\{u(x)\}\right]$$

($\because$ by convolution)

$$= \frac{1}{s-1} - \frac{s}{s^2+1} - \frac{2}{s-1}U(s)$$

$$U(s) + \frac{2}{s-1}U(s) = \frac{s^2+1-s^2+s}{(s-1)(s^2+1)}$$

$$U(s)\left(1+\frac{2}{s-1}\right) = \frac{s+1}{(s-1)(s^2+1)}$$

$$U(s) = \frac{\cancel{s+1}}{\cancel{(s-1)}(s^2+1)} \times \frac{\cancel{s-1}}{\cancel{s+1}}$$

$$U(s) = \frac{1}{s^2+1}$$

Taking Laplace inverse

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$u(x) = \sin x$$

8- $u(x) = 1 - \int_0^x ((x-t)^2 - 1)\, u(t)\,dt$

Solution:-

$$u(x) = 1 - \int_0^x ((x-t)^2 - 1)\, u(t)\,dt \longrightarrow i)$$

Taking Laplace on both sides of eq i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{1 - \int_0^x ((x-t)^2-1)\, u(t)\,dt\right\}$$

$$= \mathcal{L}\{1\} - \mathcal{L}\left\{\int_0^x ((x-t)^2-1)\, u(t)\,dt\right\}$$

$$U(s) = \frac{1}{s} - \mathcal{L}\left\{\int_0^x ((x-t)^2-1)\, u(t)\,dt\right\} \longrightarrow ii)$$

$\because$ by convolution theorem

$$\mathcal{L}\left\{\int_0^x ((x-t)^2-1)\, u(t)\,dt\right\}$$

$$= \mathcal{L}\left\{\int_0^x (x^2-2xt+t^2-1)\, u(t)\,dt\right\}$$

$$= \mathcal{L}\{x^2-2x-1\} \cdot \mathcal{L}\{u(x)\}$$

$$= [\mathcal{L}\{x^2\} - \mathcal{L}\{2x\} - \mathcal{L}\{1\}]\, U(s)$$

$$= \left[\frac{2}{s^3} - \frac{2}{s^2} - \frac{1}{s}\right] U(s) \quad \text{put in (ii)}$$

$$U(s) = \frac{1}{s} - \left[\frac{2}{s^3} - \frac{2}{s^2} - \frac{1}{s}\right] U(s)$$

$$U(s) + \left[\frac{2}{s^3} - \frac{2}{s^2} - \frac{1}{s}\right] U(s) = \frac{1}{s}$$

$$U(s)\left[1 + \frac{2}{s^3} - \frac{2}{s^2} - \frac{1}{s}\right] = \frac{1}{s}$$

$$U(s)\left[\frac{s^3 + 2 - 2s - s^2}{s^3}\right] = \frac{1}{s}$$

$$U(s) = \frac{1}{s} \times \frac{s^3}{s^3 - 2s - s^2 + 2}$$

$$U(s) = \frac{s^2}{s^3 - s^2 - 2s + 2}$$

$$= \frac{s^2}{(s^2(s-1) - 2(s-1))}$$

$$= \frac{s^2}{(s^2-2)(s-1)}$$

$$\frac{s^2}{(s^2-2)(s-1)} = \frac{As+B}{s^2-2} + \frac{C}{s-1}$$

$$s^2 = (As+B)(s-1) + C(s^2-2) \longrightarrow ii)$$

$s - 1 = 0 \Rightarrow s = 1$ put in (ii)

$$(\cancel{1})^2 = (As+B)(0) + C(1-2)$$

$$s^2 1 = C(-1)$$

$$C = -1$$

$$s^2 = As^2 + Bs - As - B + Cs^2 - 2C$$

Comparing coff

s^2; $A + C = 1 \longrightarrow iii)$

s; $B - A = 0 \longrightarrow iv)$

$s^0;\quad -B-2C=0 \longrightarrow v)$

iii) $\Longrightarrow\quad A=2$

iv) $\Longrightarrow\quad B=2$

$$\frac{s^2}{(s^2-2)(s-1)} = \frac{2s+2}{s^2-2} + \frac{(-1)}{(s-1)}$$

$$U(s) = \frac{2(s+1)}{s^2-2} - \frac{1}{s-1}$$

$$= \frac{2s}{s^2-2} + \frac{2}{s^2-2} - \frac{1}{s-1}$$

$$U(s) = \frac{2(s)}{s^2-(\sqrt{2})^2} + \frac{(\sqrt{2})^2}{s^2-(\sqrt{2})^2} - \frac{1}{s-1}$$

Taking Laplace inverse

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{2s}{(s^2-(\sqrt{2})^2)} + \frac{(\sqrt{2})^2}{s^2-(\sqrt{2})^2} - \frac{1}{s-1}\right\}$$

$$= 2\mathcal{L}^{-1}\left\{\frac{s}{s^2-(\sqrt{2})^2}\right\} + \sqrt{2}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2-(\sqrt{2})^2}\right\}$$

$$- \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$U(x) = 2\cosh\sqrt{2}x + \sqrt{2}\sinh\sqrt{2}x - e^x$$

9- $U(x) = \sin x + \sinh x + \cosh x - 2\int_0^x \cos(x-t)U(t)\,dt$

Solution:-

$$U(x) = \sin x + \sinh x + \cosh x - 2\int_0^x \cos(x-t)U(t)\,dt \longrightarrow i)$$

Taking Laplace on both sides of equation (i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{\sin x + \sinh x + \cosh x - 2\int_0^x \cos(x-t)u(t)\,dt\right\}$$

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{\sin x\} + \mathcal{L}\{\sinh x\} + \mathcal{L}\{\cosh x\} - 2\mathcal{L}\left\{\int_0^x \cos(x-t)u(t)\,dt\right\}$$

$$U(s) = \frac{1}{s^2+1} + \frac{1}{s^2-1} + \frac{s}{s^2-1} - 2\mathcal{L}\left\{\int_0^x \cos(x-t)u(t)\,dt\right\} \longrightarrow \text{ii)}$$

Using Convolution theorem

$$\mathcal{L}\left\{\int_0^x \cos(x-t)u(t)\,dt\right\} = \mathcal{L}\{\cos x\}\cdot\mathcal{L}\{u(x)\}$$

$$= \frac{s}{s^2+1}U(s) \quad \text{put in (ii)}$$

$$\text{(ii)} \Rightarrow U(s) = \frac{1}{s^2+1} + \frac{1}{s^2-1} + \frac{s}{s^2-1} - \frac{2s}{s^2+1}U(s)$$

$$U(s) + \frac{2s}{s^2+1}U(s) = \frac{1}{s^2+1} + \frac{s+1}{s^2-1}$$

$$U(s)\left(1 + \frac{2s}{s^2+1}\right) = \frac{s^2 - \cancel{1} + s^2 + \cancel{1} + s(s^2+1)}{(s^2+1)(s^2-1)}$$

$$U(s)\left(\frac{s^2+1+2s}{s^2+1}\right) = \frac{2s^2+s^3+s}{(s^2+1)(s^2-1)}$$

$$U(s) = \frac{2s^2+s^3+s}{(\cancel{s^2+1})(s^2-1)} \times \frac{\cancel{s^2+1}}{s^2+1+2s}$$

$$U(s) = \frac{s(s^2+2s+1)}{(s^2-1)(s+1)^2}$$

$$= \frac{s\cancel{(s+1)^2}}{\cancel{(s+1)^2}(s^2-1)} = \frac{s}{s^2-1}$$

$$U(s) = \frac{s}{s^2-1}$$

Taking Laplace inverse on both sides

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\}$$

$$u(x) = \cosh x$$

10- $u(x) = \sinh x + \cosh x - \cos x - 2\int_0^x \cos(x-t)u(t)dt$

Solution :-

$$u(x) = \sinh x + \cosh x - \cos x - 2\int_0^x \cos(x-t)u(t)dt \longrightarrow i)$$

$K(x,t) = \cos(x-t) \quad \gamma = -2$

Taking Laplace on both sides of equation (i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{\sinh x + \cosh x - \cos x - 2\int_0^x \cos(x-t)u(t)dt\right\}$$

$$= \mathcal{L}\{\sinh x\} + \mathcal{L}\{\cosh x\} - \mathcal{L}\{\cos x\} - 2\,\mathcal{L}\left\{\int_0^x \cos(x-t)u(t)dt\right\}$$

$$U(s) = \frac{1}{s^2-1} + \frac{s}{s^2-1} - \frac{s}{s^2+1} - 2\mathcal{L}\left\{\int_0^x \cos(x-t)u(t)dt\right\} \longrightarrow ii)$$

Using Convolution theorem

$$\mathcal{L}\left\{\int_0^x \cos(x-t)u(t)dt\right\} = \mathcal{L}\{\cos x\}\mathcal{L}\{u(x)\}$$

$$= \frac{s}{s^2+1}U(s) \quad \text{put in (ii)}$$

$$(ii) \Longrightarrow U(s) = \frac{1}{s^2-1} + \frac{s}{s^2-1} - \frac{s}{s^2+1} - \frac{2s}{s^2+1}U(s)$$

$$U(s) + \frac{2s}{s^2+1}U(s) = \frac{1}{s^2-1} + \frac{s}{s^2-1} - \frac{s}{s^2+1}$$

$$U(s)\left(1 + \frac{2s}{s^2+1}\right) = \frac{s+1}{s^2-1} - \frac{s}{s^2+1}$$

$$U(s)\left(\frac{s^2+1+2s}{s^2+1}\right) = \frac{(s+1)(s^2+1) - s(s^2-1)}{(s^2-1)(s^2+1)}$$

$$U(s) = \frac{\cancel{s^3}+s+s^2+1-\cancel{s^3}+s}{(s^2-1)\cancel{(s^2+1)}} \times \frac{\cancel{s^2+1}}{s^2+1+2s}$$

$$= \frac{\cancel{s^2+2s+1}}{(s^2-1)(\cancel{s^2+1+2s})}$$

$$U(s) = \frac{1}{s^2-1}$$

Taking Laplace inverse on both sides

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\}$$

$$u(x) = \sinh x$$

11- $u(x) = \sin x - \cos x + \cosh x - 2\int_0^x \cosh(x-t)u(t)\,dt$

Solution :-

$$u(x) = \sin x - \cos x + \cosh x - 2\int_0^x \cosh(x-t)u(t)\,dt \longrightarrow i)$$

$K(x,t) = \cosh(x-t)$; $\gamma = -2$

Taking Laplace on both sides of eq (i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{\sin x - \cos x + \cosh x - 2\int_0^x \cosh(x-t)u(t)\,dt\right\}$$

$$= \mathcal{L}\{\sin x\} - \mathcal{L}\{\cos x\} + \mathcal{L}\{\cosh x\} - 2\mathcal{L}\left\{\int_0^x \cosh(x-t)u(t)\,dt\right\}$$

$$U(s) = \frac{1}{s^2+1} - \frac{s}{s^2+1} + \frac{s}{s^2-1} - 2\mathcal{L}\left\{\int_0^x \cosh(x-t)u(t)\,dt\right\} \longrightarrow ii)$$

using convolution theorem

$$\mathcal{L}\left\{\int_0^x \cosh(x-t)u(t)\,dt\right\} = \mathcal{L}\{\cosh x\}.\mathcal{L}\{u(x)\}$$

$$= \frac{s}{s^2-1}U(s) \quad \text{put in ii)}$$

$$(ii) \Rightarrow U(s) = \frac{1}{s^2+1} - \frac{s}{s^2+1} + \frac{s}{s^2-1} - \frac{2s}{s^2-1}U(s)$$

$$U(s) + \frac{2s}{s^2-1}U(s) = \frac{1}{s^2+1} - \frac{s}{s^2+1} + \frac{s}{s^2-1}$$

$$U(s)\left(1+\frac{2s}{s^2-1}\right) = \frac{1-s}{s^2+1} + \frac{s}{s^2-1}$$

$$U(s)\left(\frac{s^2-1+2s}{s^2-1}\right) = \frac{(1-s)(s^2-1)+s(s^2+1)}{(s^2+1)(s^2-1)}$$

$$U(s) = \frac{s^2-1-\cancel{s^3}+s+\cancel{s^3}+s}{(s^2+1)\cancel{(s^2-1)}} \times \frac{\cancel{s^2-1}}{s^2-1+2s}$$

$$= \frac{\cancel{(s^2+2s-1)}}{(s^2+1)\cancel{(s^2-1+2s)}}$$

$$U(s) = \frac{1}{s^2+1}$$

Taking Laplace inverse both sides

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$U(x) = \sin x$$

12- $u(x) = \sin x + \cos x + \sinh x - 2\int_0^x \cosh(x-t)u(t)dt$

Solution:-

$$u(x) = \sin x + \cos x + \sinh x - 2\int_0^x \cosh(x-t)u(t)dt \longrightarrow i)$$

Here $K(x,t) = \cosh(x-t)$, $\lambda = -2$

Taking Laplace inverse on both sides of eq (i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{\sin x + \cos x + \sinh x - 2\int_0^x \cosh(x-t)u(t)dt\}$$

$$= \mathcal{L}\{\sin x\} + \mathcal{L}\{\cos x\} + \mathcal{L}\{\sinh x\} - 2\mathcal{L}\left\{\int_0^x \cosh(x-t)u(t)\,dt\right\}$$

$$U(s) = \frac{1}{s^2+1} + \frac{s}{s^2+1} + \frac{1}{s^2-1} - 2\mathcal{L}\left\{\int_0^x \cosh(x-t)u(t)\,dt\right\} \longrightarrow \text{ii)}$$

using convolution theorem

$$\mathcal{L}\left\{\int_0^x \cosh(x-t)u(t)\,dt\right\} = \mathcal{L}\{\cosh x\}\mathcal{L}\{u(x)\}$$

$$= \frac{s}{s^2-1}U(s) \quad \text{put in (ii)}$$

$$\text{(ii)} \Rightarrow U(s) = \frac{1}{s^2+1} + \frac{s}{s^2+1} + \frac{1}{s^2-1} - \frac{2s}{s^2-1}U(s)$$

$$U(s) + \frac{2s}{s^2-1}U(s) = \frac{1+s}{s^2+1} + \frac{1}{s^2-1}$$

$$U(s)\left(1 + \frac{2s}{s^2-1}\right) = \frac{(1+s)(s^2-1) + s^2+1}{(s^2-1)(s^2+1)}$$

$$U(s)\left(\frac{s^2-1+2s}{s^2-1}\right) = \frac{s^2 - \cancel{1} + s^3 - s + s^2 + \cancel{1}}{(s^2-1)(s^2+1)}$$

$$U(s) = \frac{s^3+2s^2-s}{\cancel{(s^2-1)}(s^2+1)} \times \frac{\cancel{s^2-1}}{s^2-1+2s}$$

$$= \frac{s\cancel{(s^2+2s-1)}}{(s^2-1)\cancel{(s^2+2s-1)}}$$

$$U(s) = \frac{s}{s^2-1}$$

Taking Laplace inverse on both sides

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$u(x) = \cos x$$

13- $u(x) = 2e^x - 2 - x + \int_0^x (x-t)u(t)\,dt$

Solution:-

$$u(x) = 2e^x - 2 - x + \int_0^x (x-t)u(t)\,dt \longrightarrow i)$$

$K(x,t) = x-t \qquad \lambda = 1$

Taking Laplace on both sides of eq (i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{2e^x - 2 - x + \int_0^x (x-t)u(t)\,dt\right\}$$

$$= 2\mathcal{L}\{e^x\} - 2\mathcal{L}\{1\} - \mathcal{L}\{x\} + \mathcal{L}\left\{\int_0^x (x-t)u(t)\,dt\right\}$$

$$U(s) = \frac{2}{s-1} - \frac{2}{s} - \frac{1}{s^2} + \mathcal{L}\left\{\int_0^x (x-t)u(t)\,dt\right\} \longrightarrow ii)$$

Using convolution theorem

$$\mathcal{L}\left\{\int_0^x (x-t)u(t)\,dt\right\} = \mathcal{L}\{x\}.\mathcal{L}\{u(x)\}$$

$$= \frac{1}{s^2}U(s)$$ put in (ii)

ii) $\Rightarrow U(s) = \frac{2}{s-1} - \frac{2}{s} - \frac{1}{s^2} + \frac{U(s)}{s^2}$

$$U(s) - \frac{1}{s^2}U(s) = \frac{2s^2 - 2s(s-1) - (s-1)}{s^2(s-1)}$$

$$U(s)\left(1 - \frac{1}{s^2}\right) = \frac{\cancel{2s^2} - \cancel{2s^2} + 2s - s + 1}{s^2(s-1)}$$

$$U(s)\left(\frac{s^2-1}{s^2}\right) = \frac{s+1}{s^2(s-1)}$$

$$U(s) = \frac{s+1}{\cancel{s^2}(s-1)} \times \frac{\cancel{s^2}}{s^2-1}$$

$$= \frac{s+1}{(s-1)(s^2-1)}$$

$$= \frac{\cancel{s+1}}{(s-1)(s-1)\cancel{(s+1)}} = \frac{1}{(s-1)^2}$$

$$U(s) = \frac{1}{(s-1)^2}$$

Taking Laplace inverse on both sides

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$U(x) = xe^x$$

14- $U(x) = 2\cosh x - 2 + \int_0^x (x-t)u(t)\,dt$

Solution:-

Here $K(x,t) = x - t \qquad \gamma = 1$

$$U(x) = 2\cosh x - 2 + \int_0^x (x-t)u(t)\,dt \longrightarrow i)$$

Taking Laplace on both sides of equation (i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{2\cosh x - 2 + \int_0^x (x-t)u(t)\,dt\right\}$$

$$= 2\mathcal{L}\{\cosh x\} - 2\mathcal{L}\{1\} + \mathcal{L}\left\{\int_0^x (x-t)u(t)\,dt\right\}$$

$$U(s) = \frac{2s}{s^2-1} - \frac{2}{s} + \mathcal{L}\left\{\int_0^x (x-t)u(t)\,dt\right\} \longrightarrow \text{ii)}$$

using convolution theorem

$$\mathcal{L}\left\{\int_0^x (x-t)u(t)\,dt\right\} = \mathcal{L}\{x\}\cdot\mathcal{L}\{u(x)\}$$

$$= \frac{1}{s^2}U(s) \quad \text{put in (ii)}$$

$$\text{(ii)} \Rightarrow U(s) = \frac{2s}{s^2-1} - \frac{2}{s} + \frac{1}{s^2}U(s)$$

$$U(s) - \frac{1}{s^2}U(s) = \frac{2s}{s^2-1} - \frac{2}{s}$$

$$U(s)\left(1 - \frac{1}{s^2}\right) = \frac{2s^2 - 2(s^2-1)}{s(s^2-1)}$$

$$U(s)\left(\frac{s^2-1}{s^2}\right) = \frac{\cancel{2s^2} - \cancel{2s^2} + 2}{s(s^2-1)}$$

$$U(s) = \frac{2}{s(s^2-1)} \times \frac{s^2}{s^2-1}$$

$$U(s) = \frac{2s}{(s^2-1)^2}$$

$$U(s) = 2\left\{\frac{1}{(s^2-1)}\cdot\frac{s}{(s^2-1)}\right\}$$

Taking Laplace inverse on both sides

$$\mathcal{L}^{-1}\{U(s)\} = 2\mathcal{L}^{-1}\left[\frac{1}{(s^2-1)}\cdot\frac{s}{(s^2-1)}\right]$$

$$= 2\int_0^x \sinh(x-t)\cosh t\, dt$$

$\because \sinh(\alpha-\beta) = \sinh\alpha\cosh\beta - \cosh\alpha\sinh\beta$

$$= 2\int_0^x (\sinh x\cosh t - \cosh x\sinh t)\cosh t\, dt$$

$$= 2\int_0^x \sinh x\cosh^2 t\, dt - 2\cosh x\int_0^x \sinh t\cosh t\, dt$$

$$= 2\left[\sinh x\int_0^x\left(\frac{1+\cosh 2t}{2}\right)dt - \cosh x\int_0^x \sinh t\cosh t\, dt\right]$$

$$= 2\left[\sinh x\left(\frac{t}{2}+\frac{\sinh 2t}{4}\right)\Big|_0^x - \cosh x\left|\frac{(\sinh t)^2}{2}\right|_0^x\right]$$

$$= 2\left[\sinh x\left(\frac{x}{2}+\frac{\sinh 2x}{4} - 0 - \cosh x\left(\frac{(\sinh x)^2}{2} - 0\right)\right)\right]$$

$$= 2\left[\frac{x\sinh x}{2} + \frac{\sinh x\sinh 2x}{4} - \frac{\cosh x\sinh^2 x}{2}\right]$$

$$= 2\left[\frac{x\sinh x}{2} + \frac{\sinh x\sinh 2x}{4} - \frac{\cosh x}{2}\left(\frac{\cosh 2x-1}{2}\right)\right]$$

$$= 2\left[\frac{x\sinh x}{2} + \frac{1}{4}(\sinh x\sinh 2x) - \frac{\cosh x\cosh 2x}{4} + \frac{\cosh x}{4}\right]$$

$$= 2\left\{\frac{x\sinh x}{2} - \frac{1}{4}\left\{\cosh 2x\cosh x - \sinh 2x \sinh x\right\} + \frac{\cosh x}{4}\right\}$$

$$= 2\left[\frac{x\sinh x}{2} - \frac{\cosh(2x-x)}{4} + \frac{\cosh x}{4}\right]$$

$$= 2\left[\frac{x\sinh x}{2} - \cancel{\frac{\cosh x}{4}} + \cancel{\frac{\cosh x}{4}}\right]$$

$$= x\sinh x$$

$$u(x) = x\sinh x$$

15- $u(x) = 2 - 2\cos x - \int_0^x (x-t)u(t)\,dt$

Solution:-

$$u(x) = 2 - 2\cos x - \int_0^x (x-t)u(t)\,dt \longrightarrow i)$$

$K(x,t) = (x-t)$ $\quad \gamma = -1$

Taking Laplace on both sides of eq (i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{2 - 2\cos x - \int_0^x (x-t)u(t)\,dt\right\}$$

$$= 2\mathcal{L}\{1\} - 2\mathcal{L}\{\cos x\} - \mathcal{L}\left\{\int_0^x (x-t)u(t)\,dt\right\}$$

$$U(s) = \frac{2}{s} - \frac{2s}{s^2+1} - \mathcal{L}\left\{\int_0^x (x-t)u(t)\,dt\right\} \longrightarrow ii)$$

Using convolution theorem

$$\mathcal{L}\left\{\int_0^x (x-t)u(t)\,dt\right\} = \mathcal{L}\{x\}\cdot\mathcal{L}\{u(x)\}$$

$$= \frac{1}{s^2} U(s) \quad \text{put in (ii)}$$

$$\text{(ii)} \Rightarrow U(s) = \frac{2}{s} - \frac{2s}{s^2+1} - \frac{1}{s^2} U(s)$$

$$U(s) + \frac{1}{s^2} U(s) = \frac{2}{s} - \frac{2s}{s^2+1}$$

$$U(s)\left(1 + \frac{1}{s^2}\right) = \frac{\cancel{2s^2}+2-\cancel{2s^2}}{s(s^2+1)}$$

$$U(s) = \frac{2}{s(s^2+1)} \times \frac{s^2}{s^2+1}$$

$$U(s) = \frac{2s}{(s^2+1)^2}$$

Taking Laplace inverse on both sides

$$\mathcal{L}^{-1}\{U(s)\} = 2\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)} \cdot \frac{s}{(s^2+1)}\right\}$$

by convolution theorem

$$U(x) = 2\int_0^x \sin(x-t)\cos t \, dt$$

$$= 2\int_0^x (\sin x \cos t - \cos x \sin t)\cos t \, dt$$

$$= 2\left[\int_0^x (\sin x \cos^2 t - \cos x \sin t \cos t)\, dt\right]$$

$$= 2\left[\sin x \int_0^x \cos^2 t \, dt - \cos x \int_0^x \sin t \cos t \, dt\right]$$

$$= 2\left\{ \sin x \int_0^x \left(\frac{1+\cos 2t}{2}\right)dt - \frac{\cos x}{2}\int_0^x 2\sin t\cos t\, dt \right\}$$

$$= 2\left\{ \sin x \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^x - \frac{\cos x}{2}\int_0^x \sin 2t\, dt \right\}$$

$$= 2\left\{ \sin x \left[\frac{x}{2} + \frac{\sin 2x}{4} - 0\right] - \frac{\cos x}{2}\left[-\frac{\cos 2t}{2}\right]_0^x \right\}$$

$$= 2\left\{ \frac{x\sin x}{2} + \frac{\sin x \sin 2x}{4} - \frac{\cos x}{2}\left[-\frac{\cos 2x}{2} + \frac{\cos(0)}{2}\right] \right\}$$

$$= x\sin x + \frac{\sin x \sin 2x}{2} + \frac{\cos x \cos 2x}{2} - \frac{\cos x}{2}$$

$$= x\sin x + \frac{1}{2}(\cos 2x \cos x + \sin 2x \sin x) - \frac{\cos x}{2}$$

$$= x\sin x + \frac{\cos(2x - x)}{2} - \frac{\cos x}{2}$$

$$= x\sin x + \cancel{\frac{\cos x}{2}} - \cancel{\frac{\cos x}{2}}$$

$$u(x) = x\sin x$$

16- $u(x) = 1 + \int_0^x \sin(x-t)u(t)dt$

Solution:-

$$u(x) = 1 + \int_0^x \sin(x-t)u(t)dt \longrightarrow i)$$

$$K(x,t) = \sin(x-t)$$

Taking Laplace on both sides of eq (i)

$$\mathcal{L}\{u(x)\} = \mathcal{L}\left\{1 + \int_0^x \sin(x-t)u(t)\,dt\right\}$$

$$= \mathcal{L}\{1\} + \mathcal{L}\left\{\int_0^x \sin(x-t)u(t)\,dt\right\}$$

$$U(s) = \frac{1}{s} + \mathcal{L}\left\{\int_0^x \sin(x-t)u(t)\,dt\right\} \longrightarrow ii)$$

Using convolution theorem

$$\mathcal{L}\left\{\int_0^x \sin(x-t)u(t)\,dt\right\} = \mathcal{L}\{\sin x\}\cdot\mathcal{L}\{u(x)\}$$

$$= \frac{1}{s^2+1}U(s) \quad \text{put in (ii)}$$

$$U(s) = \frac{1}{s} + \frac{1}{s^2+1}U(s)$$

$$U(s) - \frac{1}{s^2+1}U(s) = \frac{1}{s}$$

$$U(s)\left(1 - \frac{1}{s^2+1}\right) = \frac{1}{s}$$

$$U(s)\left(\frac{s^2+1-1}{s^2+1}\right) = \frac{1}{s}$$

$$U(s) = \frac{1}{s} \times \frac{s^2+1}{s^2} = \frac{s^2+1}{s^3}$$

$$U(s) = \frac{1}{s} + \frac{1}{s^3}$$

Taking Laplace inverse on both sides

$$\mathcal{L}^{-1}\{U(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s^3}\right\}$$

$$\left(\text{Using } \mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}\right.$$

$$u(x) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$u(x) = 1 + \frac{x^2}{2}$$