Chapter 2
Introductory Concepts of Integral Equations

As stated in the previous chapter, an *integral equation* is the equation in which the unknown function $u(x)$ appears inside an integral sign [1–5]. The most standard type of integral equation in $u(x)$ is of the form

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt, \tag{2.1}$$

where $g(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, and $K(x,t)$ is a known function, of two variables x and t, called the *kernel* or the *nucleus* of the integral equation. The unknown function $u(x)$ that will be determined appears inside the integral sign. In many other cases, the unknown function $u(x)$ appears inside and outside the integral sign. The functions $f(x)$ and $K(x,t)$ are given in advance. It is to be noted that the limits of integration $g(x)$ and $h(x)$ may be both variables, constants, or mixed.

Integral equations appear in many forms. Two distinct ways that depend on the limits of integration are used to characterize integral equations, namely:

1. If the limits of integration are fixed, the integral equation is called a *Fredholm integral equation* given in the form:

$$u(x) = f(x) + \lambda \int_{a}^{b} K(x,t)u(t)dt, \tag{2.2}$$

where a and b are constants.

2. If at least one limit is a variable, the equation is called a *Volterra integral equation* given in the form:

$$u(x) = f(x) + \lambda \int_{a}^{x} K(x,t)u(t)dt. \tag{2.3}$$

Moreover, two other distinct kinds, that depend on the appearance of the unknown function $u(x)$, are defined as follows:

1. If the unknown function $u(x)$ appears only under the integral sign of Fredholm or Volterra equation, the integral equation is called *a first kind* Fredholm or Volterra integral equation respectively.

A-M. Wazwaz, *Linear and Nonlinear Integral Equations*

2. If the unknown function $u(x)$ appears both inside and outside the integral sign of Fredholm or Volterra equation, the integral equation is called *a second kind* Fredholm or Volterra equation integral equation respectively.

In all Fredholm or Volterra integral equations presented above, if $f(x)$ is identically zero, the resulting equation:

$$u(x) = \lambda \int_a^b K(x,t)u(t)dt \tag{2.4}$$

or

$$u(x) = \lambda \int_a^x K(x,t)u(t)dt \tag{2.5}$$

is called *homogeneous* Fredholm or *homogeneous* Volterra integral equation respectively.

It is interesting to point out that any equation that includes both integrals and derivatives of the unknown function $u(x)$ is called *integro-differential equation*. The Fredholm integro-differential equation is of the form:

$$u^{(k)}(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \quad u^{(k)} = \frac{d^k u}{dx^k}. \tag{2.6}$$

However, the Volterra integro-differential equation is of the form:

$$u^{(k)}(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt, \quad u^{(k)} = \frac{d^k u}{dx^k}. \tag{2.7}$$

The integro-differential equations [6] will be defined and classified in this text.

2.1 Classification of Integral Equations

Integral equations appear in many types. The types depend mainly on the limits of integration and the kernel of the equation. In this text we will be concerned on the following types of integral equations.

2.1.1 Fredholm Integral Equations

For Fredholm integral equations, the limits of integration are fixed. Moreover, the unknown function $u(x)$ may appear only inside integral equation in the form:

$$f(x) = \int_a^b K(x,t)u(t)dt. \tag{2.8}$$

This is called Fredholm integral equation of the *first kind*. However, for Fredholm integral equations of the *second kind*, the unknown function $u(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt. \tag{2.9}$$

Examples of the two kinds are given by

$$\frac{\sin x - x\cos x}{x^2} = \int_0^1 \sin(xt)u(t)dt, \tag{2.10}$$

and

$$u(x) = x + \frac{1}{2}\int_{-1}^{1}(x-t)u(t)dt, \tag{2.11}$$

respectively.

2.1.2 Volterra Integral Equations

In Volterra integral equations, at least one of the limits of integration is a variable. For the *first kind* Volterra integral equations, the unknown function $u(x)$ appears only inside integral sign in the form:

$$f(x) = \int_0^x K(x,t)u(t)dt. \tag{2.12}$$

However, Volterra integral equations of the *second kind*, the unknown function $u(x)$ appears inside and outside the integral sign. The second kind is represented by the form:

$$u(x) = f(x) + \lambda\int_0^x K(x,t)u(t)dt. \tag{2.13}$$

Examples of the Volterra integral equations of the first kind are

$$xe^{-x} = \int_0^x e^{t-x}u(t)dt, \tag{2.14}$$

and

$$5x^2 + x^3 = \int_0^x (5+3x-3t)u(t)dt. \tag{2.15}$$

However, examples of the Volterra integral equations of the second kind are

$$u(x) = 1 - \int_0^x u(t)dt, \tag{2.16}$$

and

$$u(x) = x + \int_0^x (x-t)u(t)dt. \tag{2.17}$$

2.1.3 Volterra-Fredholm Integral Equations

The Volterra-Fredholm integral equations [6,7] arise from parabolic boundary value problems, from the mathematical modelling of the spatio-temporal development of an epidemic, and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature in two forms, namely

$$u(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)dt + \lambda_2 \int_a^b K_2(x,t)u(t)dt, \tag{2.18}$$

and

$$u(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(x,t,\xi,\tau,u(\xi,\tau))d\xi d\tau, (x,t) \in \Omega \times [0,T], \tag{2.19}$$

where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on $D = \Omega \times [0,T]$, and Ω is a closed subset of $\mathbb{R}^n, n = 1,2,3$. It is interesting to note that (2.18) contains disjoint Volterra and Fredholm integral equations, whereas (2.19) contains mixed Volterra and Fredholm integral equations. Moreover, the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of first kind, but will not be examined in this text. Examples of the two types are given by

$$u(x) = 6x + 3x^2 + 2 - \int_0^x xu(t)dt - \int_0^1 tu(t)dt, \tag{2.20}$$

and

$$u(x,t) = x + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \xi)d\xi d\tau. \tag{2.21}$$

2.1.4 Singular Integral Equations

Volterra integral equations of the first kind [4,7]

$$f(x) = \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt \tag{2.22}$$

or of the second kind

$$u(x) = f(x) + \int_{g(x)}^{h(x)} K(x,t)u(t)dt \tag{2.23}$$

are called *singular* if one of the limits of integration $g(x)$, $h(x)$ or both are infinite. Moreover, the previous two equations are called singular if the kernel $K(x,t)$ becomes unbounded at one or more points in the interval of integration. In this text we will focus our concern on equations of the form:

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, \quad 0 < \alpha < 1, \tag{2.24}$$

or of the second kind:

$$u(x) = f(x) + \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, \quad 0 < \alpha < 1. \tag{2.25}$$

The last two standard forms are called *generalized Abel's integral equation* and *weakly singular integral equations* respectively. For $\alpha = \frac{1}{2}$, the equation:

$$f(x) = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt \tag{2.26}$$

is called the Abel's singular integral equation. It is to be noted that the kernel in each equation becomes infinity at the upper limit $t = x$. Examples of Abel's integral equation, generalized Abel's integral equation, and the weakly singular integral equation are given by

$$\sqrt{x} = \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt, \tag{2.27}$$

$$x^3 = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt, \tag{2.28}$$

and

$$u(x) = 1 + \sqrt{x} + \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt, \tag{2.29}$$

respectively.

Exercises 2.1

For each of the following integral equations, classify as Fredholm, Volterra, or Volterra-Fredholm integral equation and find its kind. Classify the equation as singular or not.

1. $u(x) = 1 + \int_0^x u(t)dt$
2. $x = \int_0^x (1 + x - t)u(t)dt$
3. $u(x) = e^x + e - 1 - \int_0^1 u(t)dt$
4. $x + 1 - \frac{\pi}{2} = \int_0^{\frac{\pi}{2}} (x - t)u(t)dt$
5. $u(x) = \frac{3}{2}x - \frac{1}{3} - \int_0^1 (x - t)u(t)dt$
6. $u(x) = x + \frac{1}{6}x^3 - \int_0^x (x - t)u(t)dt$
7. $\frac{1}{6}x^3 = \int_0^x (x - t)u(t)dt$
8. $\frac{1}{2}x^2 - \frac{2}{3}x + \frac{1}{4} = \int_0^1 (x - t)u(t)dt$
9. $u(x) = \frac{3}{2}x + \frac{1}{6}x^3 - \int_0^x (x - t)u(t)dt - \int_0^1 xu(t)dt$
10. $u(x,t) = x + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \xi)d\xi d\tau$
11. $x^3 + \sqrt{x} = \int_0^x \frac{1}{(x-t)^{\frac{1}{3}}} u(t)dt$
12. $u(x) = 1 + x^2 + \int_0^x \frac{1}{\sqrt{x-t}} u(t)dt$

2.2 Classification of Integro-Differential Equations

Integro-differential equations appear in many scientific applications, especially when we convert initial value problems or boundary value problems to integral equations. The integro-differential equations contain both integral and differential operators. The derivatives of the unknown functions may appear to any order. In classifying integro-differential equations, we will follow the same category used before.

2.2.1 Fredholm Integro-Differential Equations

Fredholm integro-differential equations appear when we convert differential equations to integral equations. The Fredholm integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x), n \geqslant 1$ inside and outside the integral sign respectively. The limits of integration in this case are fixed as in the Fredholm integral equations. The equation is labeled as integro-differential because it contains differential and integral operators in the same equation. It is important to note that initial conditions should be given for Fredholm integro-differential equations to obtain the particular solutions. The Fredholm integro-differential equation appears in the form:

$$u^{(n)}(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \tag{2.30}$$

where $u^{(n)}$ indicates the nth derivative of $u(x)$. Other derivatives of less order may appear with $u^{(n)}$ at the left side. Examples of the Fredholm integro-differential equations are given by

$$u'(x) = 1 - \frac{1}{3}x + \int_0^1 xu(t)dt, \quad u(0) = 0, \tag{2.31}$$

and

$$u''(x) + u'(x) = x - \sin x - \int_0^{\frac{\pi}{2}} xtu(t)dt, \quad u(0) = 0, \quad u'(0) = 1. \tag{2.32}$$

2.2.2 Volterra Integro-Differential Equations

Volterra integro-differential equations appear when we convert initial value problems to integral equations. The Volterra integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x), n \geqslant 1$ inside and outside the integral sign. At least one of the limits of integration in this case is a variable as in the Volterra integral equations. The equation is called integro-differential because differential and integral operators are involved in the same equation. It is important to note that initial conditions should be given for Volterra integro-differential equations to determine the particular solutions. The Volterra integro-differential equation appears in the form:

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \tag{2.33}$$

where $u^{(n)}$ indicates the nth derivative of $u(x)$. Other derivatives of less order may appear with $u^{(n)}$ at the left side. Examples of the Volterra integro-differential equations are given by

$$u'(x) = -1 + \frac{1}{2}x^2 - xe^x - \int_0^x tu(t)dt, \ u(0) = 0, \tag{2.34}$$

and

$$u''(x)+u'(x) = 1-x(\sin x+\cos x)-\int_0^x tu(t)dt, \ u(0) = -1, \ u'(0) = 1. \tag{2.35}$$

2.2.3 Volterra-Fredholm Integro-Differential Equations

The Volterra-Fredholm integro-differential equations arise in the same manner as Volterra-Fredholm integral equations with one or more of ordinary derivatives in addition to the integral operators. The Volterra-Fredholm integro-differential equations appear in the literature in two forms, namely

$$u^{(n)}(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)u(t)dt + \lambda_2 \int_a^b K_2(x,t)u(t)dt, \tag{2.36}$$

and

$$u^{(n)}(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(x,t,\xi,\tau,u(\xi,\tau))d\xi d\tau, (x,t) \in \Omega \times [0,T], \tag{2.37}$$

where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on $D = \Omega\times[0,T]$, and Ω is a closed subset of $\mathbb{R}^n, n = 1,2,3$. It is interesting to note that (2.36) contains disjoint Volterra and Fredholm integral equations, whereas (2.37) contains mixed integrals. Other derivatives of less order may appear as well. Moreover, the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of first kind. Initial conditions should be given to determine the particular solution. Examples of the two types are given by

$$u'(x) = 24x + x^4 + 3 - \int_0^x (x-t)u(t)dt - \int_0^1 tu(t)dt, \quad u(0) = 0, \tag{2.38}$$

and

$$u'(x,t) = 1 + t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - \int_0^t \int_0^1 (\tau - \xi)d\xi d\tau, \quad u(0,t) = t^3. \tag{2.39}$$

Exercises 2.2

For each of the following integro-differential equations, classify as Fredholm, Volterra, or Volterra-Fredholm integro-equation

1. $u'(x) = 1 + \int_0^x xu(t)dt,\ u(0) = 0$

2. $u''(x) = x + \int_0^1 (1 + x - t)u(t)dt,\ u(0) = 1,\ u'(0) = 0$

3. $u''(x) + u(x) = x + \int_0^x tu(t)dt + \int_0^1 u(t)dt,\ u(0) = 0,\ u'(0) = 1$

4. $u'''(x) + u'(x) = x + \int_0^x tu(t)dt + \int_0^1 u(t)dt,\ u(0) = 0, u'(0) = 1,\ u''(0) = 1$

5. $u'(x) + u(x) = x + \int_0^1 (x - t)u(t)dt,\ u(0) = 1$

6. $u''(x) = 1 + \int_0^x tu(t)dt,\ u(0) = 0,\ u'(0) = 1$

2.3 Linearity and Homogeneity

Integral equations and integro-differential equations fall into two other types of classifications according to *linearity* and *homogeneity* concepts. These two concepts play a major role in the structure of the solutions. In what follows we highlight the definitions of these concepts.

2.3.1 Linearity Concept

If the exponent of the unknown function $u(x)$ inside the integral sign is one, the integral equation or the integro-differential equation is called *linear* [6]. If the unknown function $u(x)$ has exponent other than one, or if the equation contains nonlinear functions of $u(x)$, such as $e^u, \sinh u, \cos u, \ln(1 + u)$, the integral equation or the integro-differential equation is called *nonlinear*. To explain this concept, we consider the equations:

$$u(x) = 1 - \int_0^x (x - t)u(t)dt, \tag{2.40}$$

$$u(x) = 1 - \int_0^1 (x - t)u(t)dt, \tag{2.41}$$

$$u(x) = 1 + \int_0^x (1 + x - t)u^4(t)dt, \tag{2.42}$$

and

$$u'(x) = 1 + \int_0^1 xte^{u(t)}dt, \quad u(0) = 1. \tag{2.43}$$

The first two examples are linear Volterra and Fredholm integral equations respectively, whereas the last two are nonlinear Volterra integral equation and nonlinear Fredholm integro-differential equation respectively.

It is important to point out that linear equations, except Fredholm integral equations of the first kind, give a unique solution if such a solution exists. However, solution of nonlinear equation may not be unique. Nonlinear equations usually give more than one solution and it is not usually easy to handle. Both linear and nonlinear integral equations of any kind will be investigated in this text by using traditional and new methods.

2.3.2 Homogeneity Concept

Integral equations and integro-differential equations of the second kind are classified as *homogeneous* or *inhomogeneous*, if the function $f(x)$ in the second kind of Volterra or Fredholm integral equations or integro-differential equations is identically zero, the equation is called homogeneous. Otherwise it is called inhomogeneous. Notice that this property holds for equations of the second kind only. To clarify this concept we consider the following equations:

$$u(x) = \sin x + \int_0^x xtu(t)dt, \tag{2.44}$$

$$u(x) = x + \int_0^1 (x-t)^2 u(t)dt, \tag{2.45}$$

$$u(x) = \int_0^x (1+x-t)u^4(t)dt, \tag{2.46}$$

and

$$u''(x) = \int_0^x xtu(t)dt, \quad u(0) = 1, \quad u'(0) = 0. \tag{2.47}$$

The first two equations are inhomogeneous because $f(x) = \sin x$ and $f(x) = x$, whereas the last two equations are homogeneous because $f(x) = 0$ for each equation. We usually use specific approaches for homogeneous equations, and other methods are used for inhomogeneous equations.

Exercises 2.3

Classify the following equations as Fredholm, or Volterra, linear or nonlinear, and homogeneous or inhomogeneous

1. $u(x) = 1 + \int_0^x (x-t)^2 u(t)dt$

2. $u(x) = \cosh x + \int_0^1 (x-t)u(t)dt$

3. $u(x) = \int_0^x (2+x-t)u(t)dt$

4. $u(x) = \lambda \int_{-1}^1 t^2 u(t)dt$

5. $u(x) = 1 + x + \int_0^x (x-t)\frac{1}{1+u^2}dt$ 6. $u(x) = 1 + \int_0^1 u^3(t)dt$

7. $u'(x) = 1 + \int_0^1 (x-t)u(t)dt,\ u(0) = 1$ 8. $u'(x) = \int_0^x (x-t)u(t)dt,\ u(0) = 0$

2.4 Origins of Integral Equations

Integral and integro-differential equations arise in many scientific and engineering applications. Volterra integral equations and Volterra integro-differential equations can be obtained from converting initial value problems with prescribed initial values. However, Fredholm integral equations and Fredholm integro-differential equations can be derived from boundary value problems with given boundary conditions.

It is important to point out that converting initial value problems to Volterra integral equations, and converting Volterra integral equations to initial value problems are commonly used in the literature. This will be explained in detail in the coming section. However, converting boundary value problems to Fredholm integral equations, and converting Fredholm integral equations to equivalent boundary value problems are rarely used. The conversion techniques will be examined and illustrated examples will be presented.

In what follows we will examine the steps that we will use to obtain these integral and integro-differential equations.

2.5 Converting IVP to Volterra Integral Equation

In this section, we will study the technique that will convert an initial value problem (IVP) to an equivalent Volterra integral equation and Volterra integro-differential equation as well [3]. For simplicity reasons, we will apply this process to a second order initial value problem given by

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x) \tag{2.48}$$

subject to the initial conditions:

$$y(0) = \alpha, \quad y'(0) = \beta, \tag{2.49}$$

where α and β are constants. The functions $p(x)$ and $q(x)$ are analytic functions, and $g(x)$ is continuous through the interval of discussion. To achieve our goal we first set

$$y''(x) = u(x), \tag{2.50}$$

where $u(x)$ is a continuous function. Integrating both sides of (2.50) from 0 to x yields

$$y'(x) - y'(0) = \int_0^x u(t)dt, \tag{2.51}$$

or equivalently

$$y'(x) = \beta + \int_0^x u(t)dt. \tag{2.52}$$

Integrating both sides of (2.52) from 0 to x yields

$$y(x) - y(0) = \beta x + \int_0^x \int_0^x u(t)dtdt, \tag{2.53}$$

or equivalently

$$y(x) = \alpha + \beta x + \int_0^x (x-t)u(t)dt, \tag{2.54}$$

obtained upon using the formula that reduce double integral to a single integral that was discussed in the previous chapter. Substituting (2.50), (2.52), and (2.54) into the initial value problem (2.48) yields the Volterra integral equation:

$$u(x) + p(x)\left[\beta + \int_0^x u(t)dt\right] + q(x)\left[\alpha + \beta x + \int_0^x (x-t)u(t)dt\right] = g(x). \tag{2.55}$$

The last equation can be written in the standard Volterra integral equation form:

$$u(x) = f(x) - \int_0^x K(x,t)u(t)dt, \tag{2.56}$$

where

$$K(x,t) = p(x) + q(x)(x-t), \tag{2.57}$$

and

$$f(x) = g(x) - \left[\beta p(x) + \alpha q(x) + \beta x q(x)\right]. \tag{2.58}$$

It is interesting to point out that by differentiating Volterra equation (2.56) with respect to x, using Leibnitz rule, we obtain an equivalent Volterra integro-differential equation in the form:

$$u'(x) + K(x,x)u(x) = f'(x) - \int_0^x \frac{\partial K(x,t)}{\partial x}u(t)dt, \quad u(0) = f(0). \tag{2.59}$$

The technique presented above to convert initial value problems to equivalent Volterra integral equations can be generalized by considering the general initial value problem:

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = g(x), \tag{2.60}$$

subject to the initial conditions

$$y(0) = c_0, y'(0) = c_1, y''(0) = c_2, \ldots, y^{(n-1)}(0) = c_{n-1}. \tag{2.61}$$

We assume that the functions $a_i(x), 1 \leqslant i \leqslant n$ are analytic at the origin, and the function $g(x)$ is continuous through the interval of discussion. Let $u(x)$ be a continuous function on the interval of discussion, and we consider the transformation:

$$y^{(n)}(x) = u(x). \tag{2.62}$$

Integrating both sides with respect to x gives

$$y^{(n-1)}(x) = c_{n-1} + \int_0^x u(t)dt. \tag{2.63}$$

Integrating again both sides with respect to x yields

$$\begin{aligned} y^{(n-2)}(x) &= c_{n-2} + c_{n-1}x + \int_0^x \int_0^x u(t)dtdt \\ &= c_{n-2} + c_{n-1}x + \int_0^x (x-t)u(t)dt, \end{aligned} \tag{2.64}$$

obtained by reducing the double integral to a single integral. Proceeding as before we find

$$\begin{aligned} y^{(n-3)}(x) &= c_{n-3} + c_{n-2}x + \frac{1}{2}c_{n-1}x^2 + \int_0^x \int_0^x \int_0^x u(t)dtdtdt \\ &= c_{n-3} + c_{n-2}x + \frac{1}{2}c_{n-1}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt. \end{aligned} \tag{2.65}$$

Continuing the integration process leads to

$$y(x) = \sum_{k=0}^{n-1} \frac{c_k}{k!}x^k + \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}u(t)dt. \tag{2.66}$$

Substituting (2.62)–(2.66) into (2.60) gives

$$u(x) = f(x) - \int_0^x K(x,t)u(t)dt, \tag{2.67}$$

where

$$K(x,t) = \sum_{k=1}^{n} \frac{a_n}{(k-1)!}(x-t)^{k-1}, \tag{2.68}$$

and

$$f(x) = g(x) - \sum_{j=1}^{n} a_j \left(\sum_{k=1}^{j} \frac{c_{n-k}}{(j-k)!}x^{j-k} \right). \tag{2.69}$$

Notice that the Volterra integro-differential equation can be obtained by differentiating (2.67) as many times as we like, and by obtaining the initial conditions of each resulting equation. The following examples will highlight the process to convert initial value problem to an equivalent Volterra integral equation.

Example 2.1

Convert the following initial value problem to an equivalent Volterra integral equation:

$$y'(x) - 2xy(x) = e^{x^2}, \quad y(0) = 1. \tag{2.70}$$

We first set

$$y'(x) = u(x). \tag{2.71}$$

Integrating both sides of (2.71), using the initial condition $y(0) = 1$ gives

$$y(x) - y(0) = \int_0^x u(t)dt, \tag{2.72}$$

or equivalently

$$y(x) = 1 + \int_0^x u(t)dt, \tag{2.73}$$

Substituting (2.71) and (2.73) into (2.70) gives the equivalent Volterra integral equation:

$$u(x) = 2x + e^{x^2} + 2x\int_0^x u(t)dt. \tag{2.74}$$

Example 2.2

Convert the following initial value problem to an equivalent Volterra integral equation:

$$y''(x) - y(x) = \sin x, \quad y(0) = 0, \quad y'(0) = 0. \tag{2.75}$$

Proceeding as before, we set

$$y''(x) = u(x). \tag{2.76}$$

Integrating both sides of (2.76), using the initial condition $y'(0) = 0$ gives

$$y'(x) = \int_0^x u(t)dt. \tag{2.77}$$

Integrating (2.77) again, using the initial condition $y(0) = 0$ yields

$$y(x) = \int_0^x\int_0^x u(t)dtdt = \int_0^x (x-t)u(t)dt, \tag{2.78}$$

obtained upon using the rule to convert double integral to a single integral. Inserting (2.76)–(2.78) into (2.70) leads to the following Volterra integral equation:

$$u(x) = \sin x + \int_0^x (x-t)u(t)dt. \tag{2.79}$$

Example 2.3

Convert the following initial value problem to an equivalent Volterra integral equation:

$$y''' - y'' - y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 3. \tag{2.80}$$

We first set

$$y'''(x) = u(x), \tag{2.81}$$

where by integrating both sides of (2.81) and using the initial condition $y''(0) = 3$ we obtain

$$y'' = 3 + \int_0^x u(t)dt. \tag{2.82}$$

Integrating again and using the initial condition $y'(0) = 2$ we find

$$y'(x) = 2 + 3x + \int_0^x\int_0^x u(t)dtdt = 2 + 3x + \int_0^x (x-t)u(t)dt. \tag{2.83}$$

Integrating again and using $y(0) = 1$ we obtain

$$
\begin{aligned}
y(x) &= 1+2x+\frac{3}{2}x^2+\int_0^x\int_0^x\int_0^x u(t)dtdtdt \\
&= 1+2x+\frac{3}{2}x^2+\frac{1}{2}\int_0^x (x-t)^2u(t)dt. \qquad (2.84)
\end{aligned}
$$

Notice that in (2.83) and (2.84) the multiple integrals were reduced to single integrals as used before. Substituting (2.81) – (2.84) into (2.80) leads to the Volterra integral equation:

$$
u(x) = 4+x+\frac{3}{2}x^2+\int_0^x [1+(x-t)-\frac{1}{2}(x-t)^2]u(t)dt. \qquad (2.85)
$$

Remark

We can also show that if $y^{(iv)}(x) = u(x)$, then

$$
\begin{aligned}
y'''(x) &= y'''(0)+\int_0^x u(t)dt \\
y''(x) &= y''(0)+xy'''(0)+\int_0^x (x-t)u(t)dt \\
y'(x) &= y'(0)+xy''(0)+\frac{1}{2}x^2y'''(0)+\frac{1}{2}\int_0^x (x-t)^2u(t)dt \\
y(x) &= y(0)+xy'(0)+\frac{1}{2}x^2y''(0)+\frac{1}{6}x^3y'''(0)+\frac{1}{6}\int_0^x (x-t)^3u(t)dt. \qquad (2.86)
\end{aligned}
$$

This process can be generalized to any derivative of a higher order.

In what follows we summarize the relation between derivatives of $y(x)$ and $u(x)$:

Table 2.1 The relation between derivatives of $y(x)$ and $u(x)$

$y^{(n)}(x)$	Integral Equations
$y'(x)=u(x)$	$y(x)=y(0)+\int_0^x u(t)dt$
$y''(x)=u(x)$	$y'(x)=y'(0)+\int_0^x u(t)dt$
	$y(x)=y(0)+xy'(0)+\int_0^x (x-t)u(t)dt$
	$y''(x)=y''(0)+\int_0^x u(t)dt$
$y'''(x)=u(x)$	$y'(x)=y'(0)+xy''(0)+\int_0^x (x-t)u(t)dt$
	$y(x)=y(0)+xy'(0)+\frac{1}{2}x^2y''(0)+\frac{1}{2}\int_0^x (x-t)^2u(t)dt$

2.5.1 Converting Volterra Integral Equation to IVP

A well-known method for solving Volterra integral and Volterra integro-differential equation, that we will use in the forthcoming chapters, converts these equations to equivalent initial value problems. The method is achieved simply by differentiating both sides of Volterra equations [6] with respect to x as many times as we need to get rid of the integral sign and come out with a differential equation. The conversion of Volterra equations requires the use of Leibnitz rule for differentiating the integral at the right hand side. The initial conditions can be obtained by substituting $x = 0$ into $u(x)$ and its derivatives. The resulting initial value problems can be solved easily by using ODEs methods that were summarized in Chapter 1. The conversion process will be illustrated by discussing the following examples.

Example 2.4

Find the initial value problem equivalent to the Volterra integral equation:

$$u(x) = e^x + \int_0^x u(t)dt. \tag{2.87}$$

Differentiating both sides of (2.87) and using Leibnitz rule we find

$$u'(x) = e^x + u(x). \tag{2.88}$$

It is clear that there is no need for differentiating again because we got rid of the integral sign. To determine the initial condition, we substitute $x = 0$ into both sides of (2.87) to find $u(0) = 1$. This in turn gives the initial value problem:

$$u'(x) - u(x) = e^x, \quad u(0) = 1. \tag{2.89}$$

Notice that the resulting ODE is a linear inhomogeneous equation of first order.

Example 2.5

Find the initial value problem equivalent to the Volterra integral equation:

$$u(x) = x^2 + \int_0^x (x-t)u(t)dt. \tag{2.90}$$

Differentiating both sides of (2.90) and using Leibnitz rule we find

$$u'(x) = 2x + \int_0^x u(t)dt. \tag{2.91}$$

To get rid of the integral sign we should differentiate (2.91) and by using Leibnitz rule we obtain the second order ODE:

$$u''(x) = 2 + u(x). \tag{2.92}$$

To determine the initial conditions, we substitute $x = 0$ into both sides of (2.90) and (2.91) to find $u(0) = 0$ and $u'(0) = 0$ respectively. This in turn gives the initial value problem:

$$u''(x) - u(x) = 2, \quad u(0) = 0, \quad u'(0) = 0. \tag{2.93}$$

Notice that the resulting ODE is a second order inhomogeneous equation.

Example 2.6

Find the initial value problem equivalent to the Volterra integral equation:

$$u(x) = \sin x - \frac{1}{2}\int_0^x (x-t)^2 u(t)dt. \tag{2.94}$$

Differentiating both sides of the integral equation three times to get rid of the integral sign to find

$$\begin{aligned} u'(x) &= \cos x - \int_0^x (x-t)u(t)dt, \\ u''(x) &= -\sin x - \int_0^x u(t)dt, \\ u'''(x) &= -\cos x - u(x). \end{aligned} \tag{2.95}$$

Substituting $x = 0$ into (2.94) and into the first two integro-differential equations in (2.95) gives the initial conditions:

$$u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 0. \tag{2.96}$$

In view of the last results, the initial value problem equivalent to the Volterra integral equation (2.94) is a third order inhomogeneous ODE given by

$$u'''(x) + u(x) = -\cos x, \quad u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 0. \tag{2.97}$$

Exercises 2.5

Convert each of the following IVPs in 1–8 to an equivalent Volterra integral equation:

1. $y' - 4y = 0,\ y(0) = 1$
2. $y' + 4xy = e^{-2x^2},\ y(0) = 0$
3. $y'' + 4y = 0,\ y(0) = 0,\ y'(0) = 1$
4. $y'' - 6y' + 8y = 1,\ y(0) = 1,\ y'(0) = 1$
5. $y''' - y' = 0,\ y(0) = 2,\ y'(0) = y''(0) = 1$
6. $y''' - 2y'' + y = x,\ y(0) = 1, y'(0) = 0,\ y''(0) = 1$
7. $y^{(iv)} - y'' = 1,\ y(0) = y'(0) = 0,\ y''(0) = y'''(0) = 1$
8. $y^{(iv)} + y'' + y = x,\ y(0) = y'(0) = 1,\ y''(0) = y'''(0) = 0$

Convert each of the following Volterra integral equation in 9–16 to an equivalent IVP:

9. $u(x) = x + 2\int_0^x u(t)dt$
10. $u(x) = 1 + e^x - \int_0^x u(t)dt$
11. $u(x) = 1 + x^2 + \int_0^x (x-t)u(t)dt$
12. $u(x) = \sin x - \int_0^x (x-t)u(t)dt$
13. $u(x) = 1 - \cos x + 2\int_0^x (x-t)^2 u(t)dt$
14. $u(x) = 2 + \sinh x + \int_0^x (x-t)^2 u(t)dt$
15. $u(x) = 1 + 2\int_0^x (x-t)^3 u(t)dt$
16. $u(x) = 1 + e^x + \int_0^x (1+x-t)^3 u(t)dt$

2.6 Converting BVP to Fredholm Integral Equation

In this section, we will present a method that will convert a boundary value problem to an equivalent Fredholm integral equation. The method is similar to the method that was presented in the previous section for converting Volterra equation to IVP, with the exception that boundary conditions will be used instead of initial values. In this case we will determine another initial condition that is not given in the problem. The technique requires more work if compared with the initial value problems when converted to Volterra integral equations. For this reason, the technique that will be presented is rarely used. Without loss of generality, we will present two specific distinct boundary value problems (BVPs) to derive two distinct formulas that can be used for converting BVP to an equivalent Fredholm integral equation.

Type I

We first consider the following boundary value problem:

$$y''(x) + g(x)y(x) = h(x), \quad 0 < x < 1, \tag{2.98}$$

with the boundary conditions:

$$y(0) = \alpha, \quad y(1) = \beta. \tag{2.99}$$

We start as in the previous section and set

$$y''(x) = u(x). \tag{2.100}$$

Integrating both sides of (2.100) from 0 to x we obtain

$$\int_0^x y''(t)dt = \int_0^x u(t)dt, \tag{2.101}$$

that gives

$$y'(x) = y'(0) + \int_0^x u(t)dt, \tag{2.102}$$

where the initial condition $y'(0)$ is not given in a boundary value problem. The condition $y'(0)$ will be determined later by using the boundary condition at $x = 1$. Integrating both sides of (2.102) from 0 to x gives

$$y(x) = y(0) + xy'(0) + \int_0^x \int_0^x u(t)dt, \tag{2.103}$$

or equivalently

$$y(x) = \alpha + xy'(0) + \int_0^x (x-t)u(t)dt, \tag{2.104}$$

obtained upon using the condition $y(0) = \alpha$ and by reducing double integral to a single integral. To determine $y'(0)$, we substitute $x = 1$ into both sides of (2.104) and using the boundary condition at $y(1) = \beta$ we find

$$y(1) = \alpha + y'(0) + \int_0^1 (1-t)u(t)dt, \tag{2.105}$$

that gives

$$\beta = \alpha + y'(0) + \int_0^1 (1-t)u(t)dt. \tag{2.106}$$

This in turn gives

$$y'(0) = (\beta - \alpha) - \int_0^1 (1-t)u(t)dt. \tag{2.107}$$

Substituting (2.107) into (2.104) gives

$$y(x) = \alpha + (\beta - \alpha)x - \int_0^1 x(1-t)u(t)dt + \int_0^x (x-t)u(t)dt. \tag{2.108}$$

Substituting (2.100) and (2.108) into (2.98) yields

$$\begin{aligned} u(x) + \alpha g(x) + (\beta - \alpha)xg(x) - \int_0^1 xg(x)(1-t)u(t)dt \\ + \int_0^x g(x)(x-t)u(t)dt = h(x). \end{aligned} \tag{2.109}$$

From calculus we can use the formula:

$$\int_0^1 (\cdot) = \int_0^x (\cdot) + \int_x^1 (\cdot), \tag{2.110}$$

to carry Eq. (2.109) to

$$\begin{aligned} u(x) = h(x) - \alpha g(x) - (\beta - \alpha)xg(x) - g(x)\int_0^x (x-t)u(t)dt \\ + xg(x)\left[\int_0^x (1-t)u(t)dt + \int_x^1 (1-t)u(t)dt\right], \end{aligned} \tag{2.111}$$

that gives

$$u(x) = f(x) + \int_0^x t(1-x)g(x)u(t)dt + \int_x^1 x(1-t)g(x)u(t)dt, \tag{2.112}$$

that leads to the Fredholm integral equation:

$$u(x) = f(x) + \int_0^1 K(x,t)u(t)dt, \tag{2.113}$$

where

$$f(x) = h(x) - \alpha g(x) - (\beta - \alpha)xg(x), \tag{2.114}$$

and the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} t(1-x)g(x), & \text{for } 0 \leqslant t \leqslant x, \\ x(1-t)g(x), & \text{for } x \leqslant t \leqslant 1. \end{cases} \tag{2.115}$$

An important conclusion can be made here. For the specific case where $y(0) = y(1) = 0$ which means that $\alpha = \beta = 0$, i.e. the two boundaries of a moving string are fixed, it is clear that $f(x) = h(x)$ in this case. This means that the resulting Fredholm equation in (2.113) is homogeneous or inhomogeneous if the boundary value problem in (2.98) is homogeneous or inhomogeneous respectively when $\alpha = \beta = 0$. The techniques presented above will be illustrated by the following two examples.

Example 2.7

Convert the following BVP to an equivalent Fredholm integral equation:

$$y''(x) + 9y(x) = \cos x, \quad y(0) = y(1) = 0. \tag{2.116}$$

We can easily observe that $\alpha = \beta = 0, g(x) = 9$ and $h(x) = \cos x$. This in turn gives

$$f(x) = \cos x. \tag{2.117}$$

Substituting this into (2.113) gives the Fredholm integral equation:

$$u(x) = \cos x + \int_0^1 K(x,t)u(t)dt, \tag{2.118}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 9t(1-x), & \text{for } 0 \leqslant t \leqslant x, \\ 9x(1-t), & \text{for } x \leqslant t \leqslant 1. \end{cases} \tag{2.119}$$

Example 2.8

Convert the following BVP to an equivalent Fredholm integral equation:

$$y''(x) + xy(x) = 0, \quad y(0) = 0, \quad y(1) = 2. \tag{2.120}$$

Recall that this is a boundary value problem because the conditions are given at the boundaries $x = 0$ and $x = 1$. Moreover, the coefficient of $y(x)$ is a variable and not a constant.

We can easily observe that $\alpha = 0, \beta = 2, g(x) = x$ and $h(x) = 0$. This in turn gives

$$f(x) = -2x^2. \tag{2.121}$$

Substituting this into (2.113) gives the Fredholm integral equation:

$$u(x) = -2x^2 + \int_0^1 K(x,t)u(t)dt, \tag{2.122}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} tx(1-x), & \text{for } 0 \leqslant t \leqslant x, \\ x^2(1-t), & \text{for } x \leqslant t \leqslant 1. \end{cases} \tag{2.123}$$

Type II

We next consider the following boundary value problem:

$$y''(x) + g(x)y(x) = h(x), \quad 0 < x < 1, \tag{2.124}$$

with the boundary conditions:

$$y(0) = \alpha_1, \quad y'(1) = \beta_1. \tag{2.125}$$

We again set

$$y''(x) = u(x). \tag{2.126}$$

Integrating both sides of (2.126) from 0 to x we obtain

$$\int_0^x y''(t)dt = \int_0^x u(t)dt, \tag{2.127}$$

that gives

$$y'(x) = y'(0) + \int_0^x u(t)dt, \tag{2.128}$$

where the initial condition $y'(0)$ is not given. The condition $y'(0)$ will be derived later by using the boundary condition at $y'(1) = \beta_1$. Integrating both sides of (2.128) from 0 to x gives

$$y(x) = y(0) + xy'(0) + \int_0^x \int_0^x u(t)dtdt, \tag{2.129}$$

or equivalently

$$y(x) = \alpha_1 + xy'(0) + \int_0^x (x-t)u(t)dt, \tag{2.130}$$

obtained upon using the condition $y(0) = \alpha_1$ and by reducing double integral to a single integral. To determine $y'(0)$, we first differentiate (2.130) with respect to x to get

$$y'(x) = y'(0) + \int_0^x u(t)dt, \tag{2.131}$$

where by substituting $x = 1$ into both sides of (2.131) and using the boundary condition at $y'(1) = \beta_1$ we find

$$y'(1) = y'(0) + \int_0^1 u(t)dt, \tag{2.132}$$

that gives

$$y'(0) = \beta_1 - \int_0^1 u(t)dt. \tag{2.133}$$

Using (2.133) into (2.130) gives

$$y(x) = \alpha_1 + x\left[\beta_1 - \int_0^1 u(t)dt\right] + \int_0^x (x-t)u(t)dt. \tag{2.134}$$

Substituting (2.126) and (2.134) into (2.124) yields

$$u(x) + \alpha_1 g(x) + \beta_1 x g(x) - \int_0^1 xg(x)u(t)dt + \int_0^x g(x)(x-t)u(t)dt = h(x). \tag{2.135}$$

From calculus we can use the formula:

$$\int_0^1 (\cdot) = \int_0^x (\cdot) + \int_x^1 (\cdot), \tag{2.136}$$

to carry Eq. (2.135) to

$$\begin{aligned} u(x) &= h(x) - (\alpha_1 + \beta_1 x)g(x) \\ &\quad + xg(x)\left[\int_0^x u(t)dt + \int_x^1 u(t)dt\right] - g(x)\int_0^x (x-t)u(t)dt. \end{aligned} \tag{2.137}$$

The last equation can be written as

$$u(x) = f(x) + \int_0^x tg(x)u(t)dt + \int_x^1 xg(x)u(t)dt, \tag{2.138}$$

that leads to the Fredholm integral equation:

$$u(x) = f(x) + \int_0^1 K(x,t)u(t)dt, \tag{2.139}$$

where

$$f(x) = h(x) - (\alpha_1 + \beta_1 x)g(x), \tag{2.140}$$

and the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} tg(x), & \text{for } 0 \leqslant t \leqslant x, \\ xg(x), & \text{for } x \leqslant t \leqslant 1. \end{cases} \tag{2.141}$$

An important conclusion can be made here. For the specific case where $y(0) = y'(1) = 0$ which means that $\alpha_1 = \beta_1 = 0$, it is clear that $f(x) = h(x)$ in this case. This means that the resulting Fredholm equation in (2.139) is homogeneous or inhomogeneous if the boundary value problem in (2.124) is homogeneous or inhomogeneous respectively.

The second type of conversion that was presented above will be illustrated by the following two examples.

Example 2.9

Convert the following BVP to an equivalent Fredholm integral equation:

$$y''(x) + y(x) = 0, \quad y(0) = y'(1) = 0. \tag{2.142}$$

We can easily observe that $\alpha_1 = \beta_1 = 0, g(x) = 1$ and $h(x) = 0$. This in turn gives

$$f(x) = 0. \tag{2.143}$$

Substituting this into (2.139) gives the homogeneous Fredholm integral equation:

$$u(x) = \int_0^1 K(x,t)u(t)dt, \tag{2.144}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} t, & \text{for } 0 \leqslant t \leqslant x, \\ x, & \text{for } x \leqslant t \leqslant 1. \end{cases} \tag{2.145}$$

Example 2.10

Convert the following BVP to an equivalent Fredholm integral equation:

$$y''(x) + 2y(x) = 4, \quad y(0) = 0, \quad y'(1) = 1. \tag{2.146}$$

We can easily observe that $\alpha_1 = 0, \beta_1 = 1, g(x) = 2$ and $h(x) = 4$. This in turn gives

$$f(x) = 4 - 2x. \tag{2.147}$$

Substituting this into (2.139) gives the inhomogeneous Fredholm integral equation:

$$u(x) = 4 - 2x + \int_0^1 K(x,t)u(t)dt, \tag{2.148}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 2t, & \text{for } 0 \leqslant t \leqslant x, \\ 2x, & \text{for } x \leqslant t \leqslant 1. \end{cases} \tag{2.149}$$

2.6.1 Converting Fredholm Integral Equation to BVP

In a previous section, we presented a technique to convert Volterra integral equation to an equivalent initial value problem. In a similar manner, we will present another technique that will convert Fredholm integral equation to an equivalent boundary value problem (BVP). In what follows we will examine two types of problems:

Type I

We first consider the Fredholm integral equation given by

$$u(x) = f(x) + \int_0^1 K(x,t)u(t)dt, \tag{2.150}$$

where $f(x)$ is a given function, and the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} t(1-x)g(x), & \text{for } 0 \leqslant t \leqslant x, \\ x(1-t)g(x), & \text{for } x \leqslant t \leqslant 1. \end{cases} \tag{2.151}$$

For simplicity reasons, we may consider $g(x) = \lambda$ where λ is constant. Equation (2.150) can be written as

$$u(x) = f(x) + \lambda\int_0^x t(1-x)u(t)dt + \lambda\int_x^1 x(1-t)u(t)dt, \tag{2.152}$$

or equivalently

$$u(x) = f(x) + \lambda(1-x)\int_0^x tu(t)dt + \lambda x\int_x^1 (1-t)u(t)dt. \tag{2.153}$$

Each term of the last two terms at the right side of (2.153) is a product of two functions of x. Differentiating both sides of (2.153), using the product rule of differentiation and using Leibnitz rule we obtain

$$\begin{aligned} u'(x) &= f'(x) + \lambda x(1-x)u(x) - \lambda\int_0^x tu(t) \\ &\quad -\lambda x(1-x)u(x) + \lambda\int_x^1 (1-t)u(t)dt \\ &= f'(x) - \lambda\int_0^x tu(t) + \lambda\int_x^1 (1-t)u(t)dt. \end{aligned} \tag{2.154}$$

To get rid of integral signs, we differentiate both sides of (2.154) again with respect to x to find that

$$u''(x) = f''(x) - \lambda x u(x) - \lambda(1-x)u(x), \tag{2.155}$$

that gives the ordinary differential equations:

$$u''(x) + \lambda u(x) = f''(x). \tag{2.156}$$

The related boundary conditions can be obtained by substituting $x = 0$ and $x = 1$ in (2.153) to find that

$$u(0) = f(0), \quad u(1) = f(1). \tag{2.157}$$

Combining (2.156) and (2.157) gives the boundary value problem equivalent to the Fredholm equation (2.150).

Recall that $y''(x) = u(x)$. Moreover, if $g(x)$ is not a constant, we can proceed in a manner similar to the discussion presented above to obtain the boundary value problem. The technique above for type I will be explained by studying the following examples.

Example 2.11

Convert the Fredholm integral equation

$$u(x) = e^x + \int_0^1 K(x,t)u(t)dt, \tag{2.158}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 9t(1-x), & \text{for } 0 \leqslant t \leqslant x, \\ 9x(1-t), & \text{for } x \leqslant t \leqslant 1, \end{cases} \tag{2.159}$$

to an equivalent boundary value problem.

The Fredholm integral equation can be written as

$$u(x) = e^x + 9(1-x)\int_0^x tu(t)dt + 9x\int_x^1 (1-t)u(t)dt. \tag{2.160}$$

Differentiating (2.160) twice with respect to x gives

$$u'(x) = e^x - 9\int_0^x tu(t)dt + 9\int_x^1 (1-t)u(t)dt, \tag{2.161}$$

and

$$u''(x) = e^x - 9u(x). \tag{2.162}$$

This in turn gives the ODE:

$$u''(x) + 9u(x) = e^x. \tag{2.163}$$

The related boundary conditions are given by

$$u(0) = f(0) = 1, \quad u(1) = f(1) = e, \tag{2.164}$$

obtained upon substituting $x = 0$ and $x = 1$ into (2.160).

Example 2.12

Convert the Fredholm integral equation

$$u(x) = x^3 + \int_0^1 K(x,t)u(t)dt, \tag{2.165}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 4t(1-x), & \text{for } 0 \leqslant t \leqslant x, \\ 4x(1-t), & \text{for } x \leqslant t \leqslant 1, \end{cases} \tag{2.166}$$

to an equivalent boundary value problem.

The Fredholm integral equation can be written as

$$u(x) = x^3 + 4(1-x)\int_0^x tu(t)dt + 4x\int_x^1 (1-t)u(t)dt. \tag{2.167}$$

Proceeding as before we find

$$u''(x) = 6x - 4u(x). \tag{2.168}$$

This in turn gives the ODE:

$$u''(x) + 4u(x) = 6x, \tag{2.169}$$

with the related boundary conditions:

$$u(0) = f(0) = 0, \quad u(1) = f(1) = 1. \tag{2.170}$$

Type II

We next consider the Fredholm integral equation given by

$$u(x) = f(x) + \int_0^1 K(x,t)u(t)dt, \tag{2.171}$$

where $f(x)$ is a given function, and the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} tg(x), & \text{for } 0 \leqslant t \leqslant x, \\ xg(x), & \text{for } x \leqslant t \leqslant 1. \end{cases} \tag{2.172}$$

For simplicity reasons, we again consider $g(x) = \lambda$ where λ is constant. Equation (2.171) can be written as

$$u(x) = f(x) + \lambda\int_0^x tu(t)dt + \lambda x\int_x^1 u(t)dt. \tag{2.173}$$

Each integral at the right side of (2.173) is a product of two functions of x. Differentiating both sides of (2.173), using the product rule of differentiation and using Leibnitz rule we obtain

$$u'(x) = f'(x) + \lambda\int_x^1 u(t)dt. \tag{2.174}$$

To get rid of integral signs, we differentiate again with respect to x to find that

$$u''(x) = f''(x) - \lambda u(x), \tag{2.175}$$

that gives the ordinary differential equations. Also change equations to equation

$$u''(x) + \lambda u(x) = f''(x). \tag{2.176}$$

Notice that the boundary condition $u(1)$ in this case cannot be obtained from (2.173). Therefore, the related boundary conditions can be obtained by substituting $x = 0$ and $x = 1$ in (2.173) and (2.174) respectively to find that

$$u(0) = f(0), \quad u'(1) = f'(1). \tag{2.177}$$

Combining (2.176) and (2.177) gives the boundary value problem equivalent to the Fredholm equation (2.171).

Recall that $y''(x) = u(x)$. Moreover, if $g(x)$ is not a constant, we can proceed in a manner similar to the discussion presented above to obtain the boundary value problem. The approach presented above for type II will be illustrated by studying the following examples.

Example 2.13

Convert the Fredholm integral equation:

$$u(x) = e^x + \int_0^1 K(x,t)u(t)dt, \tag{2.178}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 4t, & \text{for } 0 \leqslant t \leqslant x, \\ 4x, & \text{for } x \leqslant t \leqslant 1, \end{cases} \tag{2.179}$$

to an equivalent boundary value problem.

The Fredholm integral equation can be written as

$$u(x) = e^x + 4\int_0^x tu(t)dt + 4x\int_x^1 u(t)dt. \tag{2.180}$$

Differentiating (2.180) twice with respect to x gives

$$u'(x) = e^x + 4\int_x^1 u(t)dt, \tag{2.181}$$

and

$$u''(x) = e^x - 4u(x). \tag{2.182}$$

This in turn gives the ODE:

$$u''(x) + 4u(x) = e^x. \tag{2.183}$$

The related boundary conditions are given by

$$u(0) = f(0) = 1, \quad u'(1) = f'(1) = e, \tag{2.184}$$

obtained upon substituting $x = 0$ and $x = 1$ into (2.180) and (2.181) respectively. Recall that the boundary condition $u(1)$ cannot obtained in this case.

Example 2.14

Convert the Fredholm integral equation

$$u(x) = x^2 + \int_0^1 K(x,t)u(t)dt, \tag{2.185}$$

where the kernel $K(x,t)$ is given by

$$K(x,t) = \begin{cases} 6t, & \text{for } 0 \leqslant t \leqslant x, \\ 6x, & \text{for } x \leqslant t \leqslant 1 \end{cases} \tag{2.186}$$

to an equivalent boundary value problem.

The Fredholm integral equation can be written as

$$u(x) = x^2 + 6\int_0^x tu(t)dt + 6x\int_x^1 u(t)dt. \tag{2.187}$$

Proceeding as before we find

$$u'(x) = 2x + 6\int_x^1 u(t)dt. \tag{2.188}$$

and

$$u''(x) + 6u(x) = 2, \tag{2.189}$$

with the related boundary conditions

$$u(0) = f(0) = 0, \quad u'(1) = f'(1) = 2. \tag{2.190}$$

Exercises 2.6

Convert each of the following BVPs in 1–8 to an equivalent Fredholm integral equation:

1. $y'' + 4y = 0,\ 0 < x < 1,\ y(0) = y(1) = 0$
2. $y'' + xy = 0,\ y(0) = y(1) = 0$
3. $y'' + 2y = x,\ 0 < x < 1,\ y(0) = 1, y(1) = 0$
4. $y'' + 3xy = 4,\ 0 < x < 1,\ y(0) = 0, y(1) = 0$
5. $y'' + 4y = 0,\ 0 < x < 1,\ y(0) = 0, y'(1) = 0$
6. $y'' + xy = 0,\ y(0) = 0,\ y'(1) = 0$
7. $y'' + 4y = x,\ 0 < x < 1,\ y(0) = 1, y'(1) = 0$
8. $y'' + 4xy = 2,\ 0 < x < 1,\ y(0) = 0, y'(1) = 1$

Convert each of the following Fredholm integral equation in 9–16 to an equivalent BVP:

9. $u(x) = e^{2x} + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 3t(1-x), & \text{for } 0 \leqslant t \leqslant x \\ 3x(1-t), & \text{for } x \leqslant t \leqslant 1 \end{cases}$
10. $u(x) = 3x^2 + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} t(1-x), & \text{for } 0 \leqslant t \leqslant x \\ x(1-t), & \text{for } x \leqslant t \leqslant 1 \end{cases}$
11. $u(x) = \cos x + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 6t(1-x), & \text{for } 0 \leqslant t \leqslant x \\ 6x(1-t), & \text{for } x \leqslant t \leqslant 1 \end{cases}$
12. $u(x) = \sinh x + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 4t(1-x), & \text{for } 0 \leqslant t \leqslant x \\ 4x(1-t), & \text{for } x \leqslant t \leqslant 1 \end{cases}$

13. $u(x) = e^{3x} + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} t, & \text{for } 0 \leqslant t \leqslant x \\ x, & \text{for } x \leqslant t \leqslant 1 \end{cases}$

14. $u(x) = x^4 + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 6t, & \text{for } 0 \leqslant t \leqslant x \\ 6x, & \text{for } x \leqslant t \leqslant 1 \end{cases}$

15. $u(x) = 2x^2 + 3 + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 4t, & \text{for } 0 \leqslant t \leqslant x \\ 4x, & \text{for } x \leqslant t \leqslant 1 \end{cases}$

16. $u(x) = e^x + 1 + \int_0^1 K(x,t)u(t)dt,\ K(x,t) = \begin{cases} 2t, & \text{for } 0 \leqslant t \leqslant x \\ 2x, & \text{for } x \leqslant t \leqslant 1 \end{cases}$

2.7 Solution of an Integral Equation

A solution of a differential or an integral equation arises in any of the following two types:

1). **Exact solution**:
The solution is called exact if it can be expressed in a closed form, such as a polynomial, exponential function, trigonometric function or the combination of two or more of these elementary functions. Examples of exact solutions are as follows:

$$\begin{aligned} u(x) &= x + e^x, \\ u(x) &= \sin x + e^{2x}, \\ u(x) &= 1 + \cosh x + \tan x, \end{aligned} \tag{2.191}$$

and many others.

2). **Series solution**:
For concrete problems, sometimes we cannot obtain exact solutions. In this case we determine the solution in a series form that may converge to exact soliton if such a solution exists. Other series may not give exact solution, and in this case the obtained series can be used for numerical purposes. The more terms that we determine the higher accuracy level that we can achieve.

A solution of an integral equation or integro-differential equation is a function $u(x)$ that satisfies the given equation. In other words, the obtained solution $u(x)$ must satisfy both sides of the examined equation. The following examples will be examined to explain the meaning of a solution.

Example 2.15

Show that $u(x) = \sinh x$ is a solution of the Volterra integral equation:

$$u(x) = x + \int_0^x (x-t)u(t)dt. \tag{2.192}$$

Substituting $u(x) = \sinh x$ in the right hand side (RHS) of (2.192) yields

$$\begin{aligned}\text{RHS} &= x + \int_0^x (x-t)\sinh t\,dt \\ &= x + (\sinh t - t)|_0^x \\ &= \sinh x = u(x) = \text{LHS}.\end{aligned} \tag{2.193}$$

Example 2.16

Show that $u(x) = \sec^2 x$ is a solution of the Fredholm integral equation

$$u(x) = -\frac{1}{2} + \sec^2 x + \frac{1}{2}\int_0^{\frac{\pi}{4}} u(t)dt. \tag{2.194}$$

Substituting $u(x) = \sec^2 x$ in the right hand side of (2.194) gives

$$\begin{aligned}\text{RHS} &= -\frac{1}{2} + \sec^2 x + \frac{1}{2}\int_0^{\frac{\pi}{4}} u(t)dt \\ &= -\frac{1}{2} + \sec^2 x + \frac{1}{2}(\tan t)\,|_0^{\frac{\pi}{4}} \\ &= \sec^2 x = u(x) = \text{LHS}.\end{aligned} \tag{2.195}$$

Example 2.17

Show that $u(x) = \sin x$ is a solution of the Volterra integro-differential equation:

$$u'(x) = 1 - \int_0^x u(t)dt. \tag{2.196}$$

Proceeding as before, and using $u(x) = \sin x$ into both sides of (2.196) we find

$$\begin{aligned}\text{LHS} &= u'(x) = \cos x, \\ \text{RHS} &= 1 - \int_0^x \sin t\,dt = 1 - (-\cos t)\big|_0^x = \cos x.\end{aligned} \tag{2.197}$$

Example 2.18

Show that $u(x) = x + e^x$ is a solution of the Fredholm integro-differential equation:

$$u''(x) = e^x - \frac{4}{3}x + \int_0^1 xtu(t)dt. \tag{2.198}$$

Substituting $u(x) = x + e^x$ into both sides of (2.198) we find

$$\begin{aligned}\text{LHS} &= u''(x) = e^x, \\ \text{RHS} &= e^x - \frac{4}{3}x + x\int_0^1 t(t+e^t)dt \\ &= e^x - \frac{4}{3}x + x\left(\frac{1}{3}t^3 + te^t - e^t\right)\Big|_0^1 = e^x.\end{aligned} \tag{2.199}$$

Example 2.19

Show that $u(x) = \cos x$ is a solution of the Volterra-Fredholm integral equation:

$$u(x) = \cos x - x + \int_0^x \int_0^{\frac{\pi}{2}} u(t)dtdt. \quad (2.200)$$

Proceeding as before, and using $u(x) = \cos x$ into both sides of (2.200) we find

$$\text{LHS} = \cos x,$$
$$\text{RHS} = \cos x - x + \int_0^x \int_0^{\frac{\pi}{2}} \cos t dt = \cos x. \quad (2.201)$$

Example 2.20

Show that $u(x) = e^x$ is a solution of the Fredholm integral equation of the first kind:

$$\frac{e^{x^2+1} - 1}{x^2+1} = \int_0^1 e^{x^2 t} u(t)dt. \quad (2.202)$$

Proceeding as before, and using $u(x) = e^x$ into the right side of (2.202) we find

$$\text{RHS} = \int_0^1 e^{(x^2+1)t} dt = \left. \frac{e^{(x^2+1)t}}{x^2+1} \right|_{t=0}^{t=1} = \frac{e^{x^2+1} - 1}{x^2+1} = \text{LHS}. \quad (2.203)$$

Example 2.21

Show that $u(x) = x$ is a solution of the nonlinear Fredholm integral equation

$$u(x) = x - \frac{\pi}{12} + \frac{1}{3} \int_0^1 \frac{1}{1+u^2(t)} dt. \quad (2.204)$$

Using $u(x) = x$ into the right side of (2.204) we find

$$\text{RHS} = x - \frac{\pi}{12} + \frac{1}{3} \int_0^1 \frac{1}{1+t^2} dt = x - \frac{\pi}{12} + \frac{1}{3} \tan^{-1} t \Big|_{t=0}^{t=1}$$
$$= x - \frac{\pi}{12} + \frac{1}{3} \left(\frac{\pi}{4} - 0 \right) = x = \text{LHS}. \quad (2.205)$$

Example 2.22

Find $f(x)$ if $u(x) = x^2 + x^3$ is a solution of the Fredholm integral equation

$$u(x) = f(x) + \frac{5}{2} \int_{-1}^1 (xt^2 + x^2 t) u(t)dt. \quad (2.206)$$

Using $u(x) = x^2 + x^3$ into both sides of (2.206) we find

$$\text{LHS} = x^2 + x^3,$$
$$\text{RHS} = f(x) + \frac{5}{2} \int_{-1}^1 (xt^2 + x^2 t) u(t)dt = f(x) + x^2 + x. \quad (2.207)$$

Equating the left and right sides gives

$$f(x) = x^3 - x. \tag{2.208}$$

Exercises 2.7

In Exercises 1–4, show that the given function $u(x)$ is a solution of the corresponding Fredholm integral equation:

1. $u(x) = \cos x + \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x u(t)dt,\ \ u(x) = \sin x + \cos x$

2. $u(x) = e^{2x+\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{2x-\frac{5}{3}t} u(t)dt,\ \ u(x) = e^{2x}$

3. $u(x) = x + \int_{-1}^1 (x^4 - t^4)u(t)dt,\ \ -1 \leqslant x \leqslant 1,\ u(x) = x$

4. $u(x) = x + (1-x)e^x + \int_0^1 x^2 e^{t(x-1)} u(t)dt,\ \ u(x) = e^x$

In Exercises 5–8, show that the given function $u(x)$ is a solution of the corresponding Volterra integral equation:

5. $u(x) = 1 + \frac{1}{2}\int_0^x u(t)dt,\ \ u(x) = e^{2x}$

6. $u(x) = 4x + \sin x + 2x^2 - \cos x + 1 - \int_0^x u(t)dt,\ \ u(x) = 4x + \sin x$

7. $u(x) = 1 - \frac{1}{2}x^2 - \int_0^x (x-t)u(t)dt,\ \ u(x) = 2\cos x - 1$

8. $u(x) = 1 + 2x + \sin x + x^2 - \cos x - \int_0^x u(t)dt,\ \ u(x) = 2x + \sin x$

In Exercises 9–12, show that the given function $u(x)$ is a solution of the corresponding Fredholm integro-differential equation:

9. $u'(x) = xe^x + e^x - x + \frac{1}{2}\int_0^1 x u(t)dt, u(0) = 0,\ \ u(x) = xe^x$

10. $u'(x) = e^x + (e-1) - \int_0^1 u(t)dt,\ \ u(0) = 1,\ \ u(x) = e^x$

11. $u''(x) = 1 - \sin x - \int_0^{\frac{\pi}{2}} t u(t)dt,\ \ u(0) = 0,\ \ u'(0) = 1,\ \ u(x) = \sin x$

12. $u'''(x) = 1 + \sin x - \int_0^{\frac{\pi}{2}} (x-t)u(t)dt,$

    $u(0) = 1,\ \ u'(0) = 0,\ \ u''(0) = -1,\ \ u(x) = \cos x$

In Exercises 13–16, show that the given function $u(x)$ is a solution of the corresponding Volterra integro-differential equation:

13. $u'(x) = 2 + x + x^2 - \int_0^x u(t)dt,\ \ u(0) = 1,\ \ u(x) = 1 + 2x$

14. $u''(x) = x\cos x - 2\sin x + \int_0^x t u(t)dt,\ \ u(0) = 0,\ \ u'(0) = 1,\ \ u(x) = \sin x$

15. $u''(x) = 1 + \int_0^x (x-t)u(t)dt,\ \ u(0) = 1,\ \ u'(0) = 0,\ \ u(x) = \cosh x$

16. $u''(x) = 1 - xe^{-x} - \int_0^x tu(t)dt,\ u(0) = 1,\ u'(0) = -1,\ u(x) = e^{-x}$

In Exercises 17–24, find the unknown if the solution of each equation is given:

17. If $u(x) = e^{4x}$ is a solution of $u(x) = f(x) + 16\int_0^x (x-t)u(t)dt$, find $f(x)$

18. If $u(x) = e^{2x}$ is a solution of $u(x) = e^{2x} - \alpha(e^2+1)x + \int_0^1 xtu(t)dt$, find α

19. If $u(x) = \sin x$ is a solution of $u(x) = f(x) + \sin x - \int_0^{\frac{\pi}{2}} xu(t)dt$, find $f(x)$

20. If $u(x) = e^{-x^2}$ is a solution of $u(x) = 1 - \alpha\int_0^x tu(t)dt$, find α

21. If $u(x) = e^x$ is a solution of $u(x) = f(x) + \int_0^x (2u^2(t) + u(t))dt$, find $f(x)$

22. If $u(x) = \sin x$ is a solution of $u(x) = f(x) + \frac{4}{\pi}\int_0^x \int_0^{\frac{\pi}{2}} u^2(t)dtdt$, find $f(x)$

23. If $u(x) = 2 + 12x^2$ is a solution of $u'(x) = f(x) + 20x - \int_0^x \int_0^1 (x-t)u(t)dtdt$, find $f(x)$

24. If $u(x) = 6x$ is a solution of $u(x) = f(x) + \int_0^x (1-t)u(t)dt - x\int_0^1 (x-t)u(t)dtdt$, find $f(x)$

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