Functional Analysis	**Spring 2020**

Lecture 2

Lecturer: Muhammad Yaseen — *Topic: The Dual Space*

Definition 2.1 ***Let X be a normed space. Then the set of all bounded linear functionals on X forms a normed space with norm***

$$\|f\| = \sup_{\substack{x \in \mathcal{D}(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|}$$

and is called the dual space of X and is denoted by X'.

Remark 2.2 ***Note that $X' = B(X, \mathbb{R})$ or $X' = B(X, \mathbb{C})$ so that by previous theorem X' is complete because $\mathbb{R}$ and $\mathbb{C}$ are complete (whether or not X is). So we have the result***

Theorem 2.3 ***The dual space X' of a normed space X is a Banach space (whether or not X is).***

Recall that a matrix $A = (a_{ij})_{m \times n}$ can be used as an operator from $\mathbb{R}^n \to \mathbb{R}^m$ defined by $Ax = y$, where $x = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}$, $y = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{bmatrix}$ and $\eta_i = \sum_{j=1}^{n} a_{ij} \xi_j$.

Example 2.4 ***Show that every linear operator defined on a finite dimensional vector space can be represented by means of a matrix.***

Solution: Let X and Y be finite dimensional vector space over the same field and let $T : X \to Y$ be a linear operator. Let $\dim X = n$ and $\dim Y = r$ and

$E = \{e_1, e_2, \cdots, e_n\}$ and $B = \{b_1, b_2, \cdots, b_r\}$ be basis for X and Y respectively. Then every $x \in X$ can be uniquely expressed as

$$x = \xi_1 e_1 + \xi_2 e_2 + \cdots \xi_n e_n \tag{1}$$

$$\begin{aligned} \Rightarrow Tx &= T(\xi_1 e_1 + \xi_2 e_2 + \cdots \xi_n e_n) = T(\sum_{i=1}^{n} \xi_i e_i) \\ &= \sum_{i=1}^{n} \xi_i T(e_i) \\ \Rightarrow y = Tx &= \sum_{i=1}^{n} \xi_i T(e_i) \end{aligned} \tag{2}$$

Since representation 1 is unique, therefore T is uniquely determined if the images $y_i = Te_i$ of n basis vectors $e_1, \cdots, e_n$ are prescribed. Since y and y_i are in Y, they have unique representation of the form

$$\begin{aligned} y &= \sum_{j=1}^{r} \eta_j b_j \\ y_i = Te_i &= \sum_{j=1}^{r} \tau_{ji} b_j \end{aligned} \tag{3}$$

Now consider

$$\begin{aligned} y &= \sum_{j=1}^{r} \eta_j b_j \\ &= \sum_{i=1}^{n} \xi_i T(e_i) \qquad \text{(by (2))} \\ &= \sum_{i=1}^{n} \xi_i (\sum_{j=1}^{r} \tau_{ji} b_j) \qquad \text{(by (3))} \\ &= \sum_{j=1}^{r} (\sum_{i=1}^{n} \xi_i \tau_{ji}) b_j \end{aligned}$$

Comparing first and last sum, we obtain

$$\eta_j = \sum_{i=1}^{n} \xi_i \tau_{ji} = \sum_{i=1}^{n} \tau_{ji} \xi_i \tag{4}$$

Denoting $T_{EB} = (\tau_{ji})_{r\times n}$, (4) can be written in matrix form as

$$y = T_{EB}x$$

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Theorem 2.5 *Let X be an n-dimensional vector space and let $E = \{e_1, e_2, \cdots, e_n\}$ be the basis for X. Then $F = \{f_1, f_2, \cdots, f_n\}$ satisfying*

$$f_j(e_k) = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$$

is a basis for the algebraic dual space X^* of X and $\dim X^* = \dim X$.

Proof: To prove that F is a basis, we have to prove that F is linearly independent and spans X^*.
To prove that F is linearly independent:
Consider

$$\sum_{k=1}^{n} \beta_k f_k = O, \text{ where } O \text{ is zero functional}$$

$$\Rightarrow (\sum_{k=1}^{n} \beta_k f_k)(x) = O(x) \quad \forall\, x \in X$$

$$\Rightarrow \sum_{k=1}^{n} \beta_k f_k(x) = 0 \quad \forall\, x \in X \tag{5}$$

In particular for $x = e_j, \quad j = 1, 2, \cdots, n$, we get from (5)

$$\sum_{k=1}^{n} \beta_k f_k(e_j) = 0$$

$$\Rightarrow \sum_{k=1}^{n} \beta_k \delta_{kj} = 0$$

$$\Rightarrow \qquad \beta_j = 0, \quad \forall\, j = 1, 2, \cdots, n$$

$$\Rightarrow \qquad \beta_k = 0, \quad \forall\, k = 1, 2, \cdots, n$$

so that F is linearly independent.

To prove that F spans X^*:

Let $f \in X^*$ and $x \in X$. Then

$$\begin{aligned} x &= \sum_{k=1}^{n} \xi_k e_k \\ \Rightarrow f(x) &= f(\sum_{k=1}^{n} \xi_k e_k) \\ \Rightarrow f(x) &= \sum_{k=1}^{n} \xi_k f(e_k) \quad \text{because } f \text{ is linear} \\ \Rightarrow f(x) &= \sum_{k=1}^{n} \xi_k \alpha_k, \quad \text{where } \alpha_k = f(e_k) \end{aligned} \tag{6}$$

On the other hand, for $j = 1, 2, \cdots, n$, consider

$$\begin{aligned} f_j(x) &= f_j(\sum_{k=1}^{n} \xi_k e_k) \\ &= \sum_{k=1}^{n} \xi_k f_j(e_k) \\ &= \sum_{k=1}^{n} \xi_k \delta_{jk} \\ &= \xi_j \end{aligned} \tag{7}$$

So (6) implies

$$\begin{aligned} f(x) &= \sum_{k=1}^{n} \alpha_k \xi_k \\ &= \sum_{k=1}^{n} \alpha_k f_k(x) \\ &= (\sum_{k=1}^{n} \alpha_k f_k)(x) \\ \Rightarrow f &= \sum_{k=1}^{n} \alpha_k f_k \end{aligned} \tag{8}$$

$\Rightarrow f$ spans X^* ■