i.e $x_m \longrightarrow x$

Hence, (l^p, d) is complete.

DEF:— Let X be a vector space. A real valued fn $\|\cdot\| : X \longrightarrow \mathbb{R}$ is said to be norm on X if

N_1: $\|x\| \geq 0 \quad \forall \; x \in X$.

N_2: $\|x\| = 0 \Leftrightarrow x = 0$

N_3: $\|\alpha x\| = |\alpha|\|x\|$ $\forall$ $x \in X$ & $\forall$ scalar α

N_4: $\|x+y\| \leq \|x\| + \|y\|$ $\forall$ $x, y \in X$

$(X, \|\cdot\|)$ the pair is called normed space.

A Banach space is complete normed space (complete under metric induced by norm).

REMARK: The norm generalizes the concept of length of a vector in $\mathbb{R}^3$ to a general vector space X. In this case we write $|x| = \|x\|$.

DEF:— The metric d on a vector space X can be defined by using the norm on X as follow:

$d(x,y) = \|x-y\|$

This metric is called metric induced by norm.

M_1: $d(x,y) = \|x-y\| \geq 0$ (by N_1) i.e $d(x,y) \geq 0$

M_2: $d(x,y) = \|x-y\| = 0$ (by N_2) $\Leftrightarrow x-y=0 \Leftrightarrow x=y$

M_3: $d(x,y) = \|x-y\| = \|-(y-x)\| = |-1|\|y-x\| = d(y,x)$

M_4: Let $x, y, z \in X$ then

$$d(x,y) = \|x-y\|$$
$$= \|(x-z)+(z-y)\| \leq \|x-z\| + \|z-y\|$$
$$= d(x,z) + d(z,y)$$

So, $d(x,y) = \|x-y\|$ is metric on X. So that (X, d) is a metric space. Consequently, we have.

REMARK: Every normed space is metric space but converse is not true in general.

EXP# Consider $S = \{x / x$ is bdd or unbdd se
with $d(x,y) = \sum_1^\infty \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|}$. we prove that
(S,d) is M.S.

SOL:— Suppose that d is induced by norm
$\|\cdot\|$ on S i.e $d(x,y) = \|x-y\|$

Then $\|x\| = d(x,0) = \sum_1^\infty \frac{1}{2^i} \frac{|\xi_i|}{1+|\xi_i|}$

Then $\|\alpha x\| = \sum_1^\infty \frac{1}{2^i} \frac{|\alpha \xi_i|}{1+|\alpha \xi_i|} \neq |\alpha| \sum_1^\infty \frac{1}{2^i} \frac{|\xi_i|}{1+|\xi_i|} = |\alpha|$

$\Rightarrow \|\alpha x\| \neq |\alpha| \|x\|$

So, S is not MS.

EXP# Consider $X = \mathbb{R}^n$ with $\|x\| = \left(\sum_1^\infty |\xi_i|^2\right)^{1/2}$ wher
$x = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. show that $(X, \|\cdot\|)$ is a N.S.

SOL:— N_1: since, $|\xi_i|^2 \geq 0 \;\; \forall\, i = 1, 2, \dots, n$ therefore $\|x\| = \left(\sum_1^\infty |\xi_i|^2\right)^{1/2} \geq$

N_2: $\|x\| = \left(\sum_1^\infty |\xi_i|^2\right)^{1/2} = 0 \iff \xi_i = 0 \iff x = 0 \quad \forall\, i = 1,$

N_3: $\|\alpha x\| = \left(\sum_1^\infty |\alpha \xi_i|^2\right)^{1/2} = |\alpha^2|^{1/2} \left(\sum_1^\infty |\xi_i|^2\right)^{1/2} = |\alpha| \|x\|$

N_4: $\|x+y\| = \left(\sum_1^\infty |\xi_i + \eta_i|^2\right)^{1/2} \underset{\substack{M.I \\ \text{For } p=2}}{\leq} \left(\sum_1^\infty |\xi_i|^2\right)^{1/2} + \left(\sum_1^\infty |\eta_i\right.$

i.e $\|x+y\| \leq \|x\| + \|y\|$

$\Rightarrow (X, \|\cdot\|)$ is N.S.

Metric induced by $\|\cdot\|$ on $\mathbb{R}^n$

$$d(x,y) = \|x-y\| = \left(\sum_1^\infty |\xi_i - \eta_i|^2\right)^{1/2}$$

Since, $\mathbb{R}^n$ is complete w.r.t this induced met

So, $(\mathbb{R}^n, \|\cdot\|)$ is Banach space.

REMARK: By similar argument we can show that $\mathbb{C}^n$ with $\|x\| = \left(\sum_{i}^{\infty} |\xi_i|^2\right)^{1/2}$ where $x = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ is a Banach space.

LEMMA:— A metric d induced by norm $\|\cdot\|$ on a normed space $(X, \|\cdot\|)$ satisfies properties:

(i) $d(x+z, y+z) = d(x,y)$

(ii) $d(\alpha x, \alpha y) = |\alpha| d(x,y)$

PROOF:— Since d is induced by norm $\|\cdot\|$ therefore $d(x,y) = \|x-y\|$

(i) $d(x+z, y+z) = \|x+z-y-z\|$

$= \|x-y\|$

$= d(x,y)$

(ii) $d(\alpha x, \alpha y) = \|\alpha x - \alpha y\|$

$= \|\alpha(x-y)\|$

$= |\alpha| \, \|x-y\|$

$= |\alpha| d(x,y)$

REMARK: The above lemma gives a criteria to judge that whether the given metric can be induced by norm or not.

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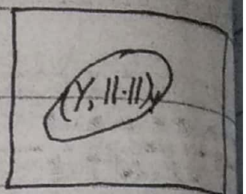
THEOREM:— A subspace Y of a Banach space $(X, \|\cdot\|)$ is complete iff Y is closed in X.

$(X, \|\cdot\|)$



PROOF:— suppose Y is complete. We prove that Y is closed i.e $Y = \bar{Y}$. obviously $Y \subseteq \bar{Y}$. let $y \in \bar{Y}$. Then, there is seq $(y_n)_1^\infty$ in Y s.t $y_n \to y$. Now, $(y_n)_1^\infty$ being convergent seq is cauchy seq. Since Y is complete So, $(y_n)_1^\infty$ converges in Y. so that $y \in Y$ ($\because$ limit is unique). $Y = \bar{Y}$.

$\because f: A \to$
$A \subset X$
$f/A: A \to$

CONVERSELY:— suppose that Y is closed. we prove that Y is complete. Let $(y_n)_1^\infty$ be a cauchy seq in Y. Then $(y_n)_1^\infty$ be cauchy seq in X. ($\because$ Y is subspace of X). Since X is complete $\exists$ $y \in X$ s.t $y_n \to y$. Then, y is a limit pt of Y. Since, Y is closed. Then $y \in \bar{Y} = Y$. So, Y is complete.

DEF:— Convergence in normed space:— A seq $(x_n)_1^\infty$ in a normed space X is convergent if $\exists$ $x \in X$ s.t $\lim_{n \to \infty} \|x_n - x\| = 0$. In this case we write $x_n \xrightarrow{\|\cdot\|} x$. and call x is the limit of

DEF:— A seq $(x_n)_1^\infty$ in normed space $(X, \|\cdot\|)$ cauchy if $\exists$ $N \in \mathbb{N}$ $\forall$ $\varepsilon > 0$ s.t

$\|x_n - x_m\| < \varepsilon \quad \forall \quad n, m \geq N$

DEF:— If $(x_K)_1^\infty$ is a seq in normed space X. Then $\sum_1^\infty x_K$ is a series in X. we associate $(x_K)_1^\infty$ a seq $(S_n)_1^\infty$ of partial sums where $S_n = x_1 + \cdots + x_n$; $n = 1, 2, 3, \ldots$

If $(S_n)_1^\infty$ converges say $S_n \to S$ i.e $\|S_n - S\| \to 0$ as $n \to \infty$ then $\sum_1^\infty x_K$ is said to converge and we write $S = \sum_1^\infty x_K = \lim_{n \to \infty} S_n$

If $\sum_1^\infty \|x_K\|$ converges then series $\sum_1^\infty x_K$ is said to be absolutely convergent in normed space.

NOTE:— In case of $\mathbb{R}$ or $\mathbb{C}$

Absolute convergence $\Rightarrow$ convergence

i.e $\sum_1^\infty |a_n| < \infty \Rightarrow \sum_1^\infty a_n < \infty$

But in case of normed space

Absolute convergence $\not\Rightarrow$ convergence

i.e $\sum_1^\infty \|x_n\| < \infty \not\Rightarrow \sum_1^\infty x_n < \infty$

EXP# Consider $l^\infty = \{x = (\xi_i)_1^\infty \mid \sup |\xi_i| < \infty\}$ with $\|x\| = \sup_i |\xi_i|$. $x = (\xi_i)_1^\infty \in l^\infty$ then $(l^\infty, \|\cdot\|)$ is Banach space.

SOL:— Let Y be the set of all seq of finitely many non-zeros terms ie

$Y = \{y = (\eta_i)_1^\infty \mid y_i = 0$ for $i > n$, $n \in N\}$. Then $Y \subset l^\infty$ ($\because \sup_i |\eta_i| < \infty$).

But Y is not closed. Since, seq $(y_n)_1^\infty$ defined by $y_n = (1, 1/2, 1/3, \dots, 1/n, 0, 0, \dots)$ lie in Y. Then $\lim_{n\to\infty} y_n = (1, 1/2, \dots, 1/n, 1/(n+1), \dots) \notin Y$. So, Y is not closed and therefore not complete.

Now, we show that absolute convergence $\not\Rightarrow$ convergence.

Let $(y_n)_1^\infty$ be a seq in Y.

where $y_n = (\eta_j^{(n)})_{j=1}^\infty$. Then $\eta_j^{(n)} = \begin{cases} 1/n^2 & ; j=n \\ 0 & ; j\neq n \end{cases}$

$$\Rightarrow y_1 = \eta_j^{(1)} = (1/1^2, 0, 0, \dots) \Rightarrow \|y_1\| = 1/1^2$$

$$y_2 = \eta_j^{(2)} = (0, 1/2^2, 0, 0, \dots) \Rightarrow \|y_2\| = 1/2^2$$

$$\vdots$$

$$y_n = \eta_j^{(n)} = (0, 0, \dots, 1/n^2) \Rightarrow \|y_n\| = 1/n^2$$

$$\Rightarrow \sum_1^\infty \|y_n\| = (1/1^2 + 1/2^2 + \dots + 1/n^2)$$

$$= \sum_1^\infty \frac{1}{n^2} < \infty \qquad \text{(by P-series test)}$$

Now, $\sum_1^\infty y_n = (1/1^2, 1/2^2, \dots) \notin Y$

$\Rightarrow \sum_1^\infty y_n$ does not converge

$\Rightarrow \sum_1^\infty \|y_n\| < \infty$ but $\sum_1^\infty y_n$ does not converge.