we take a cauchy seq in X and show that converges in X. For this, we generally use the following three steps:

1: construct an element x.

2: Prove that x is in space considered.

3: Prove convergence $x_n \to x$.

where $(x_n)_1^\infty$ is a cauchy seq in X.

EXP#1 show that the space $\mathbb{R}^n$ with $d(x,y) = \left(\sum_{i=1}^{n} |\xi_i - \eta_i|^2\right)^{1/2}$, where $x = (\xi_1, \dots, \xi_n)$, $y = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$ is Complete metric space.

SOL:— let $(x_n)_1^\infty$ be a cauchy seq in $\mathbb{R}^n$ where $x_m = (\xi_1^{(m)}, \dots, \xi_n^{(m)})$ then $\forall\ \varepsilon > 0\ \exists\ n \in$ s.t $d(x_m, x_r) < \varepsilon \quad \forall\ m, r > N$

$$\Rightarrow \left(\sum_{i=1}^{n} |\xi_i^{(m)} - \xi_i^{(r)}|^2\right)^{1/2} < \varepsilon \quad \forall\ m, r > N \quad \text{—}$$

$$\Rightarrow \sum_{i=1}^{n} |\xi_i^{(m)} - \xi_i^{(r)}|^2 < \varepsilon^2 \quad \forall\ m, r > N$$

For each fixed i $(1 \le i \le n)$ we have

$$\Rightarrow |\xi_i^{(m)} - \xi_i^{(r)}|^2 < \varepsilon^2 \quad \forall\ m, r > N$$

$$\Rightarrow |\xi_i^{(m)} - \xi_i^{(r)}| < \varepsilon \quad \forall\ m, r > N$$

$\Rightarrow (\xi_i^m)_{m=1}^\infty$ is cauchy seq in $\mathbb{R}/\mathbb{C}$ where $(i = 1, 2, \dots$

since, $\mathbb{R}$ is complete $\exists\ \xi_i \in \mathbb{R}$ s.t $\xi_i^m \to \xi_i$ as $m \to \infty$. using these n limits construct $x = (\xi_1, \dots$

obviously $x \in \mathbb{R}^n$.

letting $r \to \infty$ in (i) we get

$$\left(\sum_{i=1}^{n} |\xi_i^m - \xi_i|^2\right)^{1/2} < \varepsilon \quad \forall\ m > N$$

$\Rightarrow d(x_n, x) < \varepsilon \quad \forall\ m > N$

i.e $x_n \xrightarrow{d} x$

Hence, $(\mathbb{R}^n, d)$ is complete metric space

REMARK:— By similar argument show that $\mathbb{C}^n$ with $d(x,y) = \left(\sum_{i=1}^{n} |\xi_i - \eta_i|^2\right)^{1/2}$ is CMS where $x = (\xi_1, \ldots, \xi_n)$, $y = (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n$.

EXP # 2 show that the space $l^\infty = \{x = (\xi_i)_1^\infty \mid \sup_i |\xi_i| < \infty\}$ with $d(x,y) = \sup_i |\xi_i - \eta_i|$ where $x = (\xi_i)_1^\infty, y = (\eta_i)_1^\infty \in l^\infty$ is a CMS.

SOL:— let $(x_n)_1^\infty$ be a cauchy seq ie l^∞ where $x_n = (\xi_i^n)_{i=1}^\infty$ then $\forall\ \varepsilon > 0\ \exists\ n \in N$ s.t

$$d(x_m, x_r) < \varepsilon \quad \forall \quad m, r > N$$

$$\Rightarrow \sup_i |\xi_i^{(m)} - \xi_i^{(r)}| < \varepsilon \quad \forall \quad m, r > N \quad \text{--- (i)}$$

r fixed i $(1 \le i \le \infty)$ we have

$$|\xi_i^{(m)} - \xi_i^{(r)}| < \varepsilon \quad \forall \quad m, r > N \quad \text{--- (ii)}$$

For each fixed i $(\xi_i^m)_{m=1}^\infty$ be a cauchy seq in $\mathbb{R}/\mathbb{C}$.

$\mathbb{R}/\mathbb{C}$ are Complete $\exists\ \xi_i \in \mathbb{R}$ s.t $\xi_i^m \to \xi_i$ as $n \to \infty$.

using these infinitely many limits construct

$x = (\xi_i)_{i=1}^\infty$

letting $r \to \infty$ in (ii) we have

$$|\xi_i^m - \xi_i| \le \varepsilon \quad \forall \quad m > N \quad \text{(iii)}$$

$x_m = (\xi_i^m)_{i=1}^\infty \in l^\infty$ therefore $\exists\ K_m \in \mathbb{R}$

s.t $|\xi_i^m| \le K_m \quad \forall\ i.$

Hence, $|\xi_i| \le |\xi_i - \xi_i^m| + |\xi_i^m| \le \varepsilon + K_m$

$\Rightarrow \sup_i |\xi_i| < \infty$ so that $x \in l^\infty$.

letting $r \to \infty$ in (i) we get

$\Rightarrow \sup_i |\xi_i^m - \xi_i| < \varepsilon \quad \forall \quad m > N$

$\Rightarrow \quad d(x_m, x) < \varepsilon \quad \forall \quad m > N$

i.e $x_m \xrightarrow{d} x$

Hence, l^∞ is CMS.

REMARK: sometimes we use diff line of proof to show completeness as explained in next example.

EXP# show that $c = \{x = (\xi_i)_1^\infty \mid x \text{ is converg}$ is a complete metric space.

SOL:— since, every convergent seq is bound therefore $c \subset l^\infty$. since, c is subspace of CM Therefore, we show that c is complete it is sufficient to prove that c is closed. we c use same metric $d(x,y) = \sup|\xi_i - \eta_i|$ on c. obviously $c \subseteq \bar{c}$. let $x = (\xi_i)_1^\infty \in \bar{c}$ then $\exists$ a $x_n = (\xi_i^{(n)})_{i=1}^\infty$ in c. Then $\forall \; \varepsilon > 0 \; \exists \; N \in \mathbb{N}$ s

$|\xi_i^{(n)} - \xi_i| \le d(x_n, x) < \varepsilon/3 \quad \forall \; n \ge N, \forall i$

In particular for $n = N$ we have

$x_N = (\xi_i^{(N)})_{i=1}^\infty \in c$. Therefore, its terms $\xi_i^{(N)}$ f convergent seq. since, every convergent seq is cauchy seq. Therefore $\exists$ $N_1 \in \mathbb{N}$ s.t

$$|\xi_i^{(N)} - \xi_K^{(N)}| < \varepsilon/3 \quad \forall\ i, K > N_1 \qquad \text{(iii)}$$

Now, for $i, K > 1$ we have

$$|\xi_i - \xi_K| \le |\xi_i - \xi_i^{(N)}| + |\xi_i^{(N)} - \xi_K^{(N)}| + |\xi_K^{(N)} - \xi_K|$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \qquad \text{(by (i), (ii) \& (iii) holds)}$$

$$< \varepsilon$$

$\Rightarrow$ $x = (\xi_i)_1^{\infty}$ be a cauchy seq in $\mathbb{R}/\mathbb{C}$.

Since, $\mathbb{R}, \mathbb{C}$ are complete therefore x is convergent i.e $x \in C$. so that $\bar{C} \subseteq C$. Hence, $\bar{C} = C$ i.e C is closed. So, C is complete.

EXP# Consider $C[a,b] = \{x \,|\, x$ is convergent on $[a,b]\}$. Introduce two metrices on $C[a,b]$ as

$$d_1(x,y) = \max_{t \in [a,b]} |x(t) - y(t)|\, dt \quad \text{and}$$

$$d_2(x,y) = \int_a^b |x(t) - y(t)|\, dt$$

show that $(C[a,b], d_1)$ is complete but $(C[a,b], d_2)$ is not complete.

SOL:— (a) Let $(x_n)_1^{\infty}$ be a cauchy seq in $[a,b]$.

Then $\forall\ \varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t

$$d_1(x_m, x_r) < \varepsilon \quad \forall\ m, r > N$$

$$\Rightarrow \max_{t \in [a,b]} |x_m(t) - x_r(t)| < \varepsilon \quad \forall\ m, r > N \qquad \text{(i)}$$

$\Rightarrow$ $\exists$ $t_0 \in [a,b]$ at which max is attained

So that $|x_m(t_0) - x_r(t_0)| < \varepsilon \quad \forall\ m, r > N$

$\Rightarrow$ $(x_m(t_0))_{m=1}^{\infty}$ is cauchy seq in $\mathbb{R}$. Since, $\mathbb{R}$ is complete, $(x_m(t_0))_{m=1}^{\infty}$ converges in $\mathbb{R}$ i.e. $x_m(t_0) \to$ (say) as $m \to \infty$. In this way we can associate with each $t \in [a,b]$ a unique real no $x(t)$ (∵ limit converges). This defines a fn $x : [a,b] \to \mathbb{R}$. Letting $r \to \infty$ in (i)

$$\max_{t \in [a,b]} |x_m(t) - x(t)| < \varepsilon \quad \forall\ m > N \text{ --- (ii)}$$

$\Rightarrow \forall\ t \in [a,b],\ |x_m(t) - x(t)| < \varepsilon \quad \forall\ m > N$

$\Rightarrow x_m \longrightarrow x$ uniformally on $[a,b]$ (∵ it does depend upon)

Since, (x_m) is cauchy seq on $[a,b]$ and convergence is uniformally. Then, limit fn x is continuous i.e $x \in C[a,b]$. (∵ if seq (x_n) of continuous fn converges uniformally then limit fn is also continuous).

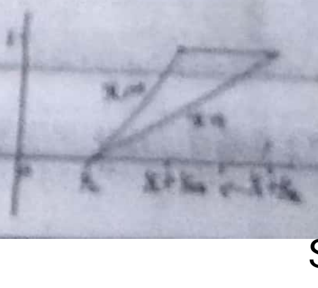
Now, (ii) implies

$$d_\infty(x_m, x) < \varepsilon \quad \forall\ m > N$$

i.e $x_m \xrightarrow{d_\infty} x$

$\Rightarrow (C[a,b], d_\infty)$ is CMS.

(b) Here, we need to identify a cauchy seq in $[a,b]$ which is not convergent under metric d_1. Let $[a,b] = [0,1]$. So, consider $(x_m)_{m=1}^{\infty}$ is a cauchy seq as shown below:



Let $m > n$ so $\frac{1}{m} < \frac{1}{n}$. consider

$$d_2(x_m, x_n) = \int_0^1 |x_m(t) - x_n(t)|\, dt$$

$$= \int_0^{1/2} |\overset{0}{x_m(t)} - \overset{0}{x_n(t)}|\, dt + \int_{1/2}^{1/2+1/m} |x_m(t) - x_n(t)|\, dt + \int_{\frac{1}{2}+\frac{1}{m}}^{\frac{1}{2}+\frac{1}{n}} |x_m(t) - x_n(t)|\, dt + \int_{\frac{1}{2}+\frac{1}{n}}^{1} |\overset{0}{x_m(t)} - \overset{0}{x_n(t)}|\, dt$$

$$\Rightarrow d_2(x_m, x_n) = \int_{1/2}^{\frac{1}{2}+\frac{1}{m}} |x_m(t) - x_n(t)|\, dt + \int_{\frac{1}{2}+\frac{1}{m}}^{\frac{1}{2}+\frac{1}{n}} |x_m(t) - x_n(t)|\, dt \longrightarrow 0 \text{ as } m, n \to \infty$$

$\left(\because \frac{1}{2} + \frac{1}{m} \to \frac{1}{2},\ \frac{1}{2} + \frac{1}{n} \to \frac{1}{2} \text{ as } m, n \to \infty \right)$

$\Rightarrow (x_m)_1^\infty$ is cauchy seq in $C[a,b]$.

suppose $\exists\ x \in C[a,b]$ s.t $x_m \xrightarrow{d_2} x$ then

$$d_2(x_m, x) = \int_0^1 |x_m(t) - x(t)|\, dt$$

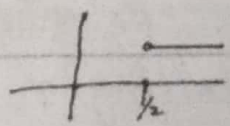
$$= \int_0^{1/2} |x_m(t) - x(t)|\, dt + \int_{1/2}^{\frac{1}{2}+\frac{1}{m}} |x_m(t) - x(t)|\, dt + \int_{\frac{1}{2}+\frac{1}{m}}^{1} |x_m(t) - x(t)|\, dt$$

$$= \int_0^{1/2} |x(t)|\, dt + \int_{1/2}^{\frac{1}{2}+\frac{1}{m}} |x_m(t) - x(t)|\, dt + \int_{\frac{1}{2}+\frac{1}{m}}^{1} |1 - x(t)|\, dt$$

i.e $d_2(x_m, x) \longrightarrow \int_0^{1/2} |x(t)|\, dt + 0 + \int_{1/2}^{1} |1 - x(t)|\, dt$ as $m \to \infty$

but $d_2(x_m, x) \to 0$ as $m \to \infty$ this is possible if

$$x(t) = \begin{cases} 0 & ; \quad 0 \le t \le 1/2 \\ 1 & ; \quad 1/2 < t \le 1 \end{cases}$$



$\Rightarrow x \notin C[a,b]$. This is contradiction to our supposition. Hence, $(C[a,b], d_2)$ is not CMS.

EXP# Show that l^p is complete where p is fixed and $1 \le p < \infty$.

SOL:- we have

$l^p = \{ x = (\xi_i)_1^\infty \mid \sum_1^\infty |\xi_i|^p \cancel{\eta_i} < \infty \}$ with $d(x,y) = \left(\sum_1^\infty |\xi_i - \eta_i|^p \right)^{1/p}$.

Let $(x_m)_1^\infty$ be a cauchy seq in l^p where $x_m = (\xi_i^{(m)})_{i=1}^\infty$.

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Then, $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ s.t

$d(x_m, x_r) < \varepsilon \quad \forall\ m, r > N$

i.e $\left(\sum_{1}^{\infty} |\xi_i^{(m)} - \xi_i^{(r)}|^p\right)^{1/p} < \varepsilon \quad \forall\ m, r > N$ —— (i)

$\Rightarrow \forall\ i = 1, 2, \ldots$ we have

$|\xi_i^{(m)} - \xi_i^{(r)}| < \varepsilon \quad \forall\ m, r > N$

$\Rightarrow (\xi_i^{(m)})_{m=1}^{\infty}$ be a cauchy seq in $\mathbb{R}/\mathbb{C}$ for each fixed i. Since, $\mathbb{R}/\mathbb{C}$ is complete $\exists\ \xi_i \in \mathbb{R}$ s.t $\xi_i^{(m)} \to \xi_i$ as $m \to \infty$. use these infinitely many limits, construct $x = (\xi_i)_1^{\infty}$

From (i) we have $\forall\ m, r > N$

$\sum_{1}^{\infty} |\xi_i^{(m)} - \xi_i^{(r)}|^p < \varepsilon^p \quad \forall\ m, r > N$ —— (ii)

$\Rightarrow$ letting $r \to \infty$ in (ii)

$\sum_{1}^{\infty} |\xi_i^{(m)} - \xi_i|^p < \varepsilon^p \quad \forall\ m > N$

$\Rightarrow x_n - x \in l^p$ so that $x = x_m - (x_m - x) \in l^p$

Letting $r \to \infty$ in (i)

$\left(\sum_{1}^{\infty} |\xi_i^{(m)} - \xi_i|^p\right)^{1/p} < \varepsilon \quad \forall\ m > N$

$\Rightarrow d(x_m, x) < \varepsilon \quad \forall\ m > N$

i.e $x_m \xrightarrow{d} x$

Hence, (l^p, d) is complete.

DEF:— Let X be a vector space. A real valued fn $\|\cdot\|: X \to \mathbb{R}$ is said to be norm on X if

N_1: $\|x\| \geq 0 \quad \forall\ x \in X$

N_2: $\|x\| = 0 \Leftrightarrow x = 0$