THEOREM:— Every Convergent seq in M.S is cauchy seq.

PROOF:— let $(x_n)_1^\infty$ be convergent seq in X.

Then $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ s.t

$d(x_n, x) < \frac{\varepsilon}{2} \quad \forall\ n > N$

Now, for $n, m > N$ we have

$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow d(x_n, x_m) < \varepsilon$

Hence, $(x_n)_1^\infty$ is cauchy seq.

THEOREM:— Let M be the non-empty subset of a M.S (X,d) & $\bar{M}$ be its closure. Then,

2) $x \in \bar{M}$ iff $\exists$ a seq $(x_n)_1^\infty$ in M s.t $x_n \to x$.

PROOF:— Note that $\bar{M} = M \cup \{$limit pts of M$\}$.

let $x \in \bar{M}$

CASE I: if $x \in M$ then there is seq $x, x, \ldots$ in M and converges to x.

CASE II: if $x \notin M$ then x is lim pt of M.

Hence for each $n = 1, 2, \ldots$ the ball $B(x, 1/n)$ contains pt $x_n \in M$ s.t $x_n \to x$. Since, $\frac{1}{n} \to 0$ as $n \to \infty$.

CONVERSELY:— Suppose that $(x_n)_1^\infty$ be seq in M s.t $x_n \to x$ then x is lim pt of M. So, that $x \in \bar{M}$.

(b) M is closed iff statement $x_n \in M$ s.t x_n implies $x \in M$.

PROOF:— suppose M is closed then $\bar{M} = M$. Let $x_n \in M$ s.t $x_n \to x$ then by (a) converse $x \in \bar{M} = M$ i.e $x \in M$.

CONVERSELY:— suppose that $x_n \in M$ s.t $x_n \to x$ implies $x \in M$. We show that $M = \bar{M}$ i.e $M \subseteq \bar{M}$ and $\bar{M} \subseteq M$. Obviously, $M \subseteq \bar{M}$. Let $x \in \bar{M}$ then by (a) $\exists$ $(x_n)_1^\infty$ seq in M s.t $x_n \to x$. Then $x \in M$ (by assumption). So that $\bar{M} \subseteq M$. Hence, M is closed i.e $M = \bar{M}$.

THEOREM:— A subspace M of a CMS (X,d) is complete iff M is closed.

PROOF:— suppose that M is complete. we show that M is closed i.e $M = \bar{M}$. obviously $M \subseteq \bar{M}$. let $x \in \bar{M}$ then by (a) of previous theorem $\exists$ (x_n) in M s.t $x_n \to x$. $(x_n)_1^\infty$ being Cnvrgnt seq is a Cauchy seq. Since, M is complete, $(x_n)_1^\infty$ converges to pt of M. Since, Limit is unique, $x \in M$ so that $\bar{M} \subseteq M$ and we have $M = \bar{M}$ i.e M is closed.

CONVERSELY:— suppose M is closed i.e $M = \bar{M}$. Let $(x_n)_1^\infty$ be a cauchy seq in M. Then, $(x_n)_1^\infty$ is a cauchy seq in X. Since, X is complete $\exists$

$x \in X$ s.t $x_n \to x$. Then by (a) of previous theorem $x \in \bar{M} = M$. So that M is complete.

THEOREM:— A mapping $T : (X, d) \to (Y, \bar{d})$ is continuous at a pt $x_0 \in X$ iff $x_n \to x_0 \Rightarrow Tx_n \to Tx_0$.

PROOF:— suppose that $T : (X,d) \to (Y, \bar{d})$ is continuous at $x_0 \in X$. Then $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ s.t

$$d(x, x_0) < \delta \Rightarrow \bar{d}(Tx, Tx_0) < \varepsilon \quad \text{(a)}$$

suppose that $x_n \to x$ $\exists\ n \in N$ s.t

$$d(x_0, x) < \delta \qquad \forall\ n > N$$

by (a) $\quad \bar{d}(Tx_n, Tx_0) < \varepsilon \qquad \forall\ n > N$

i.e $\quad Tx_n \to Tx_0$.

CONVERSELY:— suppose that $x_n \to x_0$ $Tx_n \to Tx_0$. we show that T is continuous at x_0.
we suppose on contrary that T is not continuous.
Then $\exists\ \varepsilon > 0$ s.t for $\delta > 0$ we have

$$d(x, x_0) < \delta \quad \text{but} \quad \bar{d}(Tx, Tx_0) \geq \varepsilon$$

In particular for $\delta = 1/n$ $\exists\ x_n$ s.t

$$d(x_n, x_0) < \delta = 1/n \quad \text{but} \quad \bar{d}(Tx_n, Tx_0) \geq \varepsilon$$

$\Rightarrow x_n \to x_0$ but $Tx_n \not\to Tx_0$, which is contradiction to our assumption. Hence, T is continuous.

COMPLETENESS PROOFS:—

To prove that a M.S. (X, d) is complete