$\Rightarrow d(x,y) < \varepsilon$

$\Rightarrow M$ is dense in ℓ^p.

$\Rightarrow \ell^p$ is separable.

DEF:— A sequence $(x_n)_1^\infty$ in metric space (X,d) is said to be **convergent** if $\exists\, x \in X$ s.t $\lim\limits_{n\to\infty} (x_n, x) = 0$. $x_n \longrightarrow x$ if $d(x_n, x) \longrightarrow 0$ as $n \to$

this means that $\forall\, \varepsilon > 0\ \exists\, N \in \mathbb{N}$ s.t

$d(x_n, x) < \varepsilon \quad \forall\ n > N$.

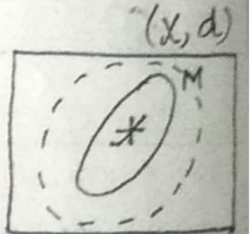
REMARK: The limit x of convergent sequence $(x_n)_1^\infty$ in a metric space (X,d) must be a pt of X.

EXP # consider $X = (0,1]$ with $d(x,y) = |x-y|$. the $(x_n)_1^\infty$ is a seq in X defined by $x_n = \frac{1}{n}$ $\forall\ n \in \mathbb{N}$

then although $\lim\limits_{n\to\infty} d(x_n, 0) = \lim\limits_{n\to\infty} |x_n - 0| = \lim\limits_{n\to\infty} x_n$

but $0 \notin X$. So, $x_n \not\to 0$ because $0 \notin X$.

DEF:— A non-empty subset of metric space (X,d) is said to be **bounded** if its diameter



$\delta(M) = \sup\limits_{x,y\in M} d(x,y)$ is finite. So, if M is bounded then $M \subset B(x_0, r)$ where x_0 is any pt of M & r is sufficiently large real number & vice versa.

DEF:— A seq $(x_n)_1^\infty$ in metric space is bounded if corresponding set $M = \{x_1, x_2, \ldots\}$ is bounded

THEOREM:— Let (X,d) be a metric space then (2) A convergent seq in X is bounded & its limit is unique.

PROOF:— Let $(x_n)_1^\infty$ be a convergent seq in (X,d).

Let $x_n \to x$ where $x \in X$ then $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ s.t

$d(x_n, x) < \varepsilon \quad \forall\ n > N$

In particular if $\varepsilon = 1\ \exists\ N_1 \in \mathbb{N}$ s.t

$d(x_n, x) < 1 \quad \forall\ n > N_1$

Define $r = \max\{1, d(x_1, x), \dots, d(x_{N_1}, x)\}$

then obviously $d(x_n, x) < r \quad \forall\ n \in \mathbb{N}$.

Hence, $(x_n)_1^\infty$ is bounded.

Now, we show that limit of $(x_n)_1^\infty$ is unique.

Consider $x_n \to x$ & $x_n \to z$ as $n \to \infty$

then $d(x, z) \leq d(x_n, x) + d(x_n, z) \to 0$ as $n \to \infty$

$(\because x_n \to x$ & $x_n \to z)$

i.e $d(x, z) \leq 0$ but $d(x, z) \geq 0$

that $d(x, z) = 0 \Rightarrow x = z$.

Hence, Limit is unique.

(3) If $x_n \to x$ & $y_n \to y$ in X then $d(x_n, y_n) \to d(x, y)$.

PROOF:— consider

$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y_n, y)$

$d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y)$ —— ①

Interchange x_n with x & y_n with y in ①

$d(x,y) - d(x_n,y_n) \le d(x_n,x) + d(y_n,y)$

$-(d(x_n,y_n) - d(x,y)) \le d(x_n,x) + d(y_n,y)$ —— ②

① & ② $\Rightarrow$

$|d(x_n,y_n) - d(x,y)| \le d(x_n,x) + d(y_n,y) \longrightarrow 0$ as $n \to \infty$

$\Rightarrow$ $d(x_n,y_n) \longrightarrow d(x,y)$ as $n \to \infty$.

REMARK: The converse of (a) in above theorem is not true in general i.e a bounded seq need not be convergent. **e.g** the seq $(-1)^n$ is bounded but not convergent.

DEF:— A seq $(x_n)_1^\infty$ in metric space (X,d) is said to be (cauchy or fundamental) if $\forall$ $\varepsilon > 0$ $\exists$ $n \in N$ s.t $d(x_n,x_m) < \varepsilon$ $\forall$ $n,m \ge N$. OR equivalently $d(x,x_n) \to 0$ as $n,m \to \infty$.

DEF:— A metric space (X,d) is said to be complete if every cauchy seq in X converges in X i.e has Limit pt which is an element of X.

EXP # $\mathbb{R}$ & $\mathbb{C}$ are complete by cauchy criterion.

EXP # let $X = (0,1]$ with $d(x,y) = |x-y|$ & let $(x_n)_1^\infty$ be seq given by $x_n = \frac{1}{n}$ $\forall$ $n \in N$. Then $(x_n)_1^\infty$ is cauchy seq because $d(x_n,x_m) = |x_n - x_m|$

$= \left|\frac{1}{n} - \frac{1}{m}\right| \le \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m} \longrightarrow 0$ as $n \to \infty$.

Now, $x_n \to 0 \notin X$. So, that X is not complete M.S