# METRIC SPACE:—

Let $X$ be a non-empty set then function $d : X \times X \to \mathbb{R}$ is called metric on $X$ if it satisfies following axioms:

$M_1$ : $d(x,y) \geq 0 \quad \forall\, x,y \in X$

$M_2$ : $d(x,y) = 0 \iff x = y$

$M_3$ : $d(x,y) = d(y,x)$

$M_4$ : $d(x,y) \leq d(x,z) + d(z,y) \quad \forall\, x,y,z \in X$

The pair $(X,d)$ is called metric space.

**NOTE:—** (i) If there pts $x,y,z \in X$ are colinear then

$$d(x,y) \leq d(x,z) + d(z,y)$$

(ii) If pts are not collinear then triangle is formed and length of one side of triangle is always less than sum of length of other two sides so that we have

$$d(x,y) < d(x,z) + d(z,y)$$

(iii) The triangle inequality can be generalised for n pts between x and y as follows:

$$d(x,y) \leq d(x,x_1) + d(x_1,x_2) + \cdots + d(x_{n-1},x_n) + d(x_n,y)$$

**EXP # 1** Let $X = \mathbb{R}$ and $d: \mathbb{R}\times\mathbb{R}$ be defined by $d(x,y) = |x-y| \ \forall \ x,y \in \mathbb{R}$. Show that $d$ is metric on $X$.

**SOL:—** $M_1$: $d(x,y) = |x-y| \geq 0$ ; $|x| \geq$

$M_2$: $d(x,y) = 0 \iff |x-y| = 0 \iff x = y$

$M_3$: $d(x,y) = |x-y| = |-(y-x)| = |y-x| = d($

$M_4$: $d(x,y) = |x-y| = |(x-z)+(z-y)|$

$\leq |x-z| + |z-y|$

$= d(x,z) + d(z,y)$

i.e $d(x,y) \leq d(x,z) + d(z,y)$

Hence, $d$ is metric on $X$

**EXP # 2** Let $X = \mathbb{R}^2$ with $d: \mathbb{R}^2\times\mathbb{R}^2 \to$ defined by $d(x,y) = \sqrt{(\xi_i - \eta_i)^2 + (\xi_i - \eta_i)^2}$ whe $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2) \in \mathbb{R}^2$ show th $d$ is metric on $X$.

**SOL:—** $M_1$: $d(x,y) = \sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2} \geq 0$

$M_2$: $d(x,y) = \sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2} = 0$

$\iff (\xi_1-\eta_1)^2 = 0$ & $(\xi_2-\eta_2)^2 = 0$

$\iff \xi_1-\eta_1 = 0$ & $\xi_2-\eta_2 = 0$

$\iff \xi_1 = \eta_1$ & $\xi_2 = \eta_2$

$\iff x = y$

$M_3$: $d(x,y) = \sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2}$

$= \sqrt{(\eta_1-\xi_1)^2+(\eta_2-\xi_2)^2} = d(y,x)$

Scanned with CamScanner

Numerical
computational methods

$M_4$: If we interpret $x, y \in \mathbb{R}^2$ as pts in complex plane then $x = \xi_1 + i\eta_1$ and $y = \xi_2 + i\eta_2$

then
$$x - y = (\xi_1 - \eta_1) + i(\xi_2 - \eta_2)$$
$$|x-y| = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}$$
$$|x-y| = d(x,y)$$

Let $x, y, z \in \mathbb{R}^2$ then
$$d(x,y) = d|x-y| = |(x-z) + (z-y)|$$
$$\leq |x-z| + |z-y|$$
$$= d(x,z) + d(z,y)$$

i.e $d(x,y) \leq d(x,z) + d(z,y)$

Hence, $d$ is metric on $X$.

**EXP # 3** Consider $\mathbb{R}^2$ with $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by $d(x,y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|$ ; $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2) \in \mathbb{R}^2$ show that $d$ is metric.

**SOL:—** $M_1$: $d(x,y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2| \geq 0$

$M_2$: $d(x,y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2| = 0$
$$\Leftrightarrow \quad |\xi_1 - \eta_1| = 0 \quad \& \quad |\xi_2 - \eta_2| = 0$$
$$\Leftrightarrow \quad \xi_1 - \eta_1 = 0 \quad \& \quad \xi_2 - \eta_2 = 0$$
$$\Leftrightarrow \quad \xi_1 = \eta_1 \quad \& \quad \xi_2 = \eta_2$$
$$\Leftrightarrow \quad x = y$$

$M_3$: $d(x,y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|$
$$= |\eta_1 - \xi_1| + |\eta_2 - \xi_2| = d(y,x)$$

$M_4$: let $z = (e_1, e_2) \in \mathbb{R}^2$

$$d(x,y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|$$
$$= |(\xi_1 - e_1) + (e_1 - \eta_1)| + |(\xi_2 - e_2) + (e_2 - \eta_2)|$$
$$\leq |(\xi_1 - e_1) + (\xi_2 - e_2)| + |(e_1 - \eta_1) + (e_2 - \eta_2)|$$
$$= d(x,z) + d(z,y)$$

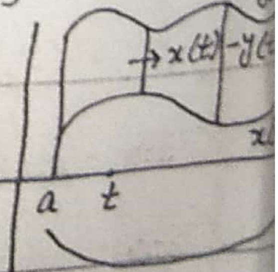
i.e $d(x,y) \leq d(x,z) + d(z,y)$

Hence, $d$ is metric on $\mathbb{R}^2$.

**NOTE:—** From exp ② & ③ it follows that more than one metric can be defined on same space.

**EXP # 4** consider space of all real valued continuous functions defined on closed interval $[a,b]$ this space is denoted by $c[a,b]$ i.e $c[a,b] = \{x \mid x : [a,b] \to \mathbb{R} \text{ is cont}\}$ defined $d$ on $c[a,b]$ by

$d(x,y) = \max |x(t) - y(t)|$ $t \in [a,b]$. show that $d$ is metric.




**SOL:—** $M_1$: $d(x,y) =$

$$\max_{t \in [a,b]} |x(t) - y(t)| \geq 0$$

$M_2$: $d(x,y) = \max\limits_{t \in [a,b]} |x(t) - y(t)| = 0$

$\iff x(t) - y(t) = 0 \quad \forall\, t \in [a,b]$

$\iff x = y$

$$= \max_{t\in[a,b]} |y(t) - x(t)| = d(y,x)$$

$M_4$ : let $z \in c[a,b]$ then

$$d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|$$

$$= \max_{t\in[a,b]} |(x(t) - z(t)) + (z(t) - y(t)|$$

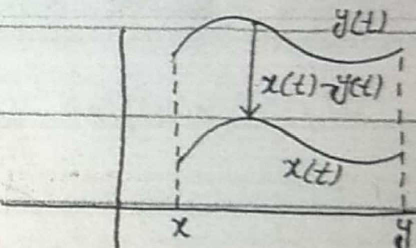
$$\leq \max_{t\in[a,b]} |x(t) - z(t)| + |z(t) - y(t)|$$

$$= d(x,z) + d(z,y)$$

i.e $d(x,y) \leq d(x,z) + d(z,y)$

Hence, d is metric.

**EXP # 5** consider $X = c[a,b]$ defined d on $c[a,b]$ by $d(x,y) = \int_a^b |x(t) - y(t)|\,dt$. show that d is metric.




**SOL:—** $M_1$ : $d(x,y) \geq 0$

$\Rightarrow |x(t) - y(t)| \geq 0 \quad \forall t$

$M_2$ : $d(x,y) = 0 \Leftrightarrow |x(t) - y(t)| = 0 \quad \forall\ t \in [a,b]$

$\Leftrightarrow x(t) = y(t) \Leftrightarrow x = y$

$M_3$ : $d(x,y) = \int_a^b |x(t) - y(t)|\,dt$

$$= \int_a^b |y(t) - x(t)|\,dt = d(y,x)$$

$M_4$ : $d(x,y) = \int_a^b |x(t) - y(t)|\,dt = \int_a^b |(x(t) - z(t)) + (z(t) - y(t))|\,dt$

$$\leq \int_a^b |x(t) - z(t)|\,dt + \int_a^b |z(t) - y(t)|\,dt$$

$$= d(x,z) + d(z,y)$$

e $d(x,y) \leq d(x,z) + d(z,y)$

ence, d is metric.

**XP # 6** consider the space of all bounded or unbounded sequence of complex numbers. This space is denoted by $s$ defined $d$ on by

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|} \; ; \; x = (\xi_i)_1^{\infty},\; y = (\eta_i) \in s.$$

show that $d$ is metric.

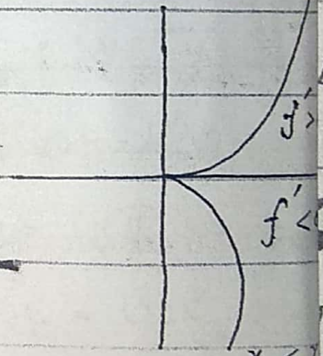
**SOL:—** $M_1$: $d(x,y) \geq 0$

$M_2$: $d(x,y) = 0 \iff \xi_i = \eta_i \;\forall i \iff x = y$

$M_3$: $$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|}$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\eta_i - \xi_i|}{1+|\eta_i - \xi_i|} = d(y,x)$$

$M_4$: Define $f(t) = \dfrac{t}{1+t}$ then




$$f'(t) = \frac{1+t-t}{(1+t)^2} = \frac{1}{(1+t)^2} > 0$$

then, $f$ is monotonically increasing

i.e $t_1 \leq t_2$ implies $f(t_1) \leq f(t_2)$.

then, $|a+b| \leq |a| + |b|$

$$f(|a+b|) \leq f(|a|+|b|)$$

$$\Rightarrow \frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|} = \frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|}$$

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

This choose $a = \xi_i - e_i$ and $b = e_i - \eta_i$ then,

$$\frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|} \leq \frac{|\xi_i - e_i|}{1+|\xi_i - e_i|} + \frac{|e_i - \eta_i|}{1+|e_i - \eta_i|}$$

$$\frac{1}{2^i}\frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|} \leq \frac{1}{2^i}\frac{|\xi_i - e_i|}{1+|\xi_i - e_i|} + \frac{1}{2^i}\frac{|e_i - \eta_i|}{1+|e_i - \eta_i|}$$

$$\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|\xi_i - \eta_i|}{1+|\xi_i - \eta_i|} \leq \sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|\xi_i - e_i|}{1+|\xi_i - e_i|} + \sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|e_i - \eta_i|}{1+|e_i - \eta_i|}$$

$$d(x,y) \leq d(x,z) + d(z,y)$$

Hence, $d$ is metric.

## EXP # 7

consider the space $B(A)$ of all functions which on defined and bounded on set $A$. Define $d$ on $B(A)$ by $d(x,y) = \sup_{t \in A} |x(t) - y(t)|$ $\forall$ $x, y \in B(A)$. show that $d$ is metric.

**SOL:—** $M_1$: $d(x,y) \geq 0$

$M_2$: $d(x,y) = \sup_{t\in A} |x(t)-y(t)| \iff x(t)-y(t) = 0 \ \forall t \in A$

$\iff x(t) = y(t) \iff x = y$

$M_3$: $d(x,y) = \sup_{t\in A} |x(t) - y(t)| = \sup_{t\in A} |y(t) - x(t)|$

$= d(y,x)$

$M_4$: $d(x,y) = \sup_{t\in A} |x(t)-y(t)| = \sup_{t\in A} |(x(t)-z(t))+(z(t)-y(t))|$

$\leq \sup_{t\in A} |x(t)-z(t)| + \sup_{t\in A} |z(t)-y(t)|$

$= d(x,z) + d(z,y)$

i.e $d(x,y) \leq d(x,z) + d(z,y)$

Hence $d$ is metric.

**EXP # 8** Let $X$ be any set define $d$ on $X$ by $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

show that $d$ is metric.

**SOL:—** $M_1$: since $d(x,y) = 0$ or $1$

$\Rightarrow \quad d(x,y) \geq 0$

$M_2$: $d(x,y) = 0 \iff x = y$ (by def of $d$)

$M_3$: $d(x,y) = d(y,x)$ (by def of $d$)

$M_4$: let $x, y, z \in X$ then

**CASE I** $x = y = z$ then

$d(x,y) = 0$ and

$d(x,z) + d(z,y) = 0$ i.e

$d(x,y) \leq d(x,z) + d(z,y)$

**CASE II** $x = y \neq z$ then

$d(x,y) = 0$ and

$d(x,z) + d(z,y) = 2$ i.e

$d(x,y) \leq d(x,z) + d(z,y)$

**CASE III** $x \neq y = z$ then

$d(x,y) = 1$ and

$d(x,z) + d(z,y) = 1$ i.e

$d(x,y) \leq d(x,z) + d(z,y)$

**CASE IV** $x \neq y \neq z$ then

$d(x,y) = 1$ and

$d(x,z) + d(y,z) = 1$ i.e

$d(x,y) \leq d(x,z) + d(z,y)$

Hence, d is called metric and pair (X,d) is called metric space. Here, d is called discrete metric and space is called discrete metric space.

**DEFINITION:—** let $P \geq 1$ be a fixed real number then space of all sequences of type $x = (\xi_i)_1^\infty$ s.t $\sum_{i=1}^{\infty} |\xi_i|^p$ converges is called $l^p$ space. i.e $l^p = \{ x = (\xi_i)_1^\infty : \sum_{i=1}^{\infty} |\xi_i|^p < \infty \}$

i) If $\xi_i$'s are real number then $l^p$ space is called real $l^p$ space.

ii) If $\xi_i$'s are complex number then $l^p$ space is called complex $l^p$ space.

we prove some inequalities which are helpful in proving $M_4$ (triangular inequality) for different spaces.

**AUXILIARY INEQUALITY:—**

et $\alpha$ & $\beta$ be any non-negative real no & let >1. Define q s.t $1/p + 1/q = 1$ (i.e p & q are gebraic conjugate) then prove that

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

Scanned with CamScanner