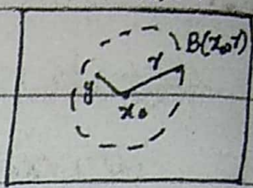
PROOF:— Let $B(x_0, r)$ be an open ball with centre x_0 and radius r.

(X, d)



Let $y \in B(x_0, r)$ define

$$r_1 = r - d(x_0, y)$$

we claim that $B(y, r_1) \subseteq B(x_0, r)$

Let $z \in B(y, r_1)$ then $d(y, z) < r_1$

then $d(x_0, z) \leq d(x_0, y) + d(y, z)$

$$= r - r_1 + r_1 = r$$

$\Rightarrow d(x_0, z) < r$

$\Rightarrow z \in B(x_0, r)$

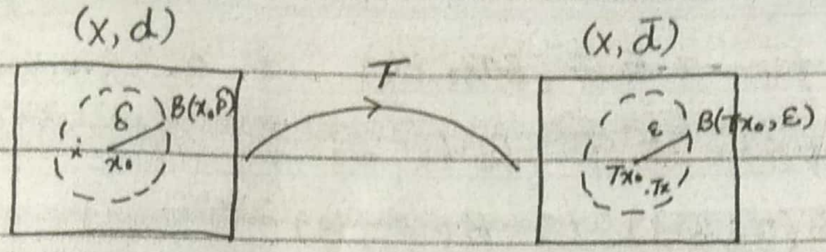
$\Rightarrow B(y, r_1) \subseteq B(x_0, r)$

Hence, $B(x_0, r)$ is an open set.

EXP # Let $\{G_i\}_1^\infty$ be a collection of open interval in R defined by $G_i = \left(\frac{-1}{i}, \frac{1}{i}\right)$ $\forall\ (i = 1, 2, \ldots)$ then $\bigcap_1^\infty G_i = \{0\}$ which is not open.

DEF:— Let (X, d) & $(X, \bar{d})$ be two metric spaces. A mapping $T: (X, d) \to (X, \bar{d})$ is said to be continuous at pt $x_0 \in X$ if $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ [illegible] $\bar{d}(Tx, Tx_0) < \varepsilon$ whenever $d(x, x_0) < \delta$ or equivalently

$$d(x, x_0) < \delta \Rightarrow T(Tx, Tx_0) < \varepsilon \quad \text{—— (a)}$$



Consider (a) is equivalent to

$T(B(x_0, \delta)) \subseteq B(x_0, \varepsilon)$

If T is continuous at each $x \in X$ we say that T = continuous.

THEOREM:— A mapping $T: (X, d) \to (Y, \bar{d})$ is continuous iff inverse image of every open subset of Y is an open subset of X.

PROOF:— suppose T is continuous and O is the open subset of Y. We show that $T^{-1}(O)$ is open subset of X.

Let $x \in T^{-1}(O)$ then $Tx \in O$. Since, O is open then $\exists$ $\varepsilon > 0$ s.t $B(Tx, \varepsilon) \subseteq O$. Since, T is Continuous $\exists$ $\delta > 0$ s.t $T(B(x_0, \delta)) \subseteq B(Tx, \varepsilon) \subseteq O$

$$B(x_0, \delta) \subseteq T^{-1}(O)$$

i.e $T^{-1}(O)$ is open.

CONVERSELY:— Suppose $T^{-1}(O)$ is open where O is open subset of Y. we show that T is continuous. Let $x \in X$ & $\varepsilon > 0$. Then, $B(Tx, \varepsilon)$ is open in Y ($\because$ every open ball is an open set).

Then, by assumption $T^{-1}(B(Tx, \varepsilon))$ is open in X.

Then for $x \in T^{-1}(B(Tx_0,\varepsilon))$ $\exists$ $\delta > 0$ s.t

$B(x_0,\delta) \subset T^{-1}(B(Tx,\varepsilon))$

$\Rightarrow$ $T(B(x_0,\delta)) \subseteq B(Tx,\varepsilon)$

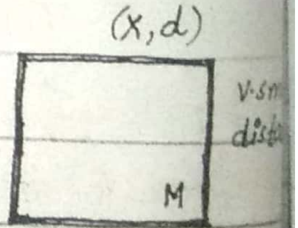
Hence, T is continuous

DEF:— Let (X,d) be a metric space and $M \subset$
A pt $x \in X$ (may or may not be pt of M) is said to be **limit pt** of M if every n-hood of x_0 contains atleast one element of M other than x_0 i.e $N_{x_0} / \{x_0\} \cap M \neq \phi$ where N_{x_0} is a n-hood of x_0.

DEF:— A subset M together with all its Lim pt is called **closure** of M and is denoted by $\bar{M}$
i.e $\bar{M} = M \cup \{\text{Lim pts of } M\}$.

e.g $M = (0,1]$, $(\frac{1}{n})_1^\infty$ in M, $\frac{1}{n} \to 0 \notin M$

DEF:— A subset M of metric space (X,d) is said to be **dense** if $\bar{M} = X$.



EXP# $\mathbb{Q} \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ because $\bar{\mathbb{Q}} = \mathbb{R}$.

REMARK: If M is dense in X then every open ball in X no matter how small must contains pt of M. In other words there is no pt $x \in X$ which has n-hood that does not contain pts of M.

DEF:— A metric space (X,d) is said to be **separable** if it contains **countable subset** which is **dense** in X.

EXP: Real line $\mathbb{R}$ is **separable** because $\mathbb{Q} \subseteq \mathbb{R}$ is Countable and dense.

EXP: Consider $X = \mathbb{C}$ i.e $X = \{x+iy \mid x, y \in \mathbb{R}\}$.

Define $M = \{p+iq \mid p, q \in \mathbb{Q}\} \subset \mathbb{C}$.

obviously M is Countable. since, $\mathbb{Q}$ is **dense** in $\mathbb{R}$

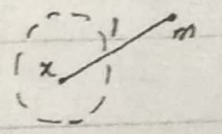
therefore $\bar{M} = \mathbb{C}$. So that $\mathbb{C}$ is **separable**.

RESULT:— A **discrete metric space** (X,d) is separable iff X is countable. $\bar{X} = X$

PROOF:— let (X,d) be discrete metric space

then $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

suppose that (X,d) is separable. let $M \subset X$ and let $x \in X/M$ then for any $m \in M$, $d(x,m) = 1$.



If we draw an open ball with centre x and radius $1/3$ (say). It will not intersect with M. So, M is not dense in X.

It means that no proper subset of X is dense in X. So, only dense subset of X is X itself.

Therefore, X is countable ($\because$ X is separable).

CONVERSELY:— suppose X is countable then since $\bar{X} = X$. Therefore, X is separable.

EXP# The space l^∞ is not separable.

SOL:— we have $l^\infty = \{x = (\xi_i)_1^\infty \mid \sup_i |\xi_i| < \infty\}$ with $d(x,y) = \sup_i \{\xi_i - \eta_i\}$ where $x = (\xi_i)_1^\infty$, $y = (\eta_i)_1^\infty \in l^\infty$.

let $\bar{y} = \{\eta_1, \eta_2, \dots\}$ s.t $\eta_i = 0$ or 1 $\forall i$.

then $\bar{y} \in l^\infty$ ($\because \sup_i |\eta_i| = 1 < \infty$).

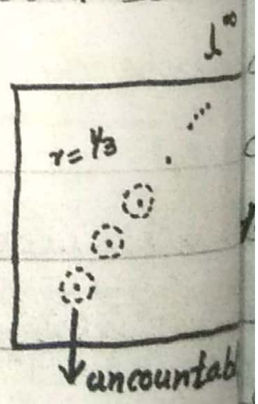
with this $\bar{y}$ we construct an element $\hat{y}$ s.t $\hat{y} = \{\frac{\eta_1}{2^1} + \frac{\eta_2}{2^2} + \dots\} \in [0,1]$

Note that $\hat{y}$ is binary representation of real no in $[0,1]$.

Then, there is one-to-one correspondence b/w $[0,1]$ and sequences of 0's & 1's in l^∞.

since, $[0,1]$ is **uncountable** then there are **uncountably many** sequences of 0's & 1's in l^∞

If we let each of these seq as centre of small balls of radius $1/3$ **(say)** then these balls are non-intersecting and uncountably many.



If **M** is any dense subset of l^∞ then each of these balls must contain one element of M

So, M cannot be **countable**.

It shows that l^∞ is not separable.

EXP# Show that l^p with $1 \le P < \infty$ is separable.

SOL:— we have $l^p = \{x = (\xi_i)_1^\infty \mid \sum_1^\infty |\xi_i|^p < \infty\}$ with $d(x,y) = (\sum_1^\infty |\xi_i - \eta_i|^p)^{1/p}$ where $x = (\xi_i)_1^\infty$, $y = (\eta_i)_1^\infty \in l^p$. we show that l^p is separable.

To show this first we show existance of set which is countable & dense.

let $x = (\xi_i)_1^\infty \in l^p$ then $\sum_1^\infty |\xi_i|^p < \infty$

then $\forall \; \varepsilon > 0 \;\; \exists \;\; n \in N$ s.t

$$\sum_{i=n+1}^{\infty} |\xi_i|^p < \varepsilon^p/2 \qquad \text{①}$$

construct $M = \{y = (\eta_1, \eta_2, \dots, \eta_n) \mid \eta_i \in Q\}$ then $M \subseteq l^p$ ($\because \sum_1^\infty |\eta_i|^p < \infty$) is countable since Q is countable.

Since, Q is dense in $\mathbb{R}$ therefore, for each $\xi_i \in \mathbb{R}$ we can find a rational η_i as close to ξ_i as we like.

Hence, we can find a ~~rational~~ $y \in M$ s.t

$$\sum_1^\infty |\xi_i - \eta_i|^p < \varepsilon^p/2 \qquad \text{②}$$

let ① & ② then consider

$$d^p(x,y) = \sum_{n=1}^{\infty} |\xi_i - \eta_i|^p$$

$$= \sum_1^n |\xi_i - \eta_i|^p + \sum_{i=n+1}^{\infty} |\xi_i - \eta_i|^p$$

$$= \sum_1^n |\xi_i - \eta_i|^p + \sum_{i=n+1}^{\infty} |\xi_i|^p < \varepsilon^p/2 + \varepsilon^p/2 < \varepsilon^p$$