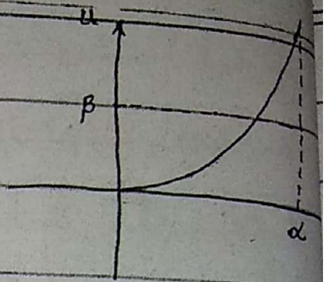
**PROOF:—** The inequality is trivially true if either $\alpha = 0$ or $\beta = 0$.




We have $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow p+q = pq$ so that

$1 + \frac{q}{p} = q$ & $\frac{p}{q} + 1 = p$ then

$(p-1)(q-1) = 1$ ——— (a)

Consider a fn $u = t^{p-1}$ ; $0 \le t \le \alpha$

then, $t = u^{1/p-1}$ ; $0 \le u \le \beta$ (inverse

$\Rightarrow$ $t = u^{q-1}$ (by a)

From fig we get

$$\alpha\beta \le \int_0^\alpha t^{p-1}\,dt + \int_0^\beta u^{q-1}\,du$$

$$\le \left|\frac{t^p}{p}\right|_0^\alpha + \left|\frac{u^q}{q}\right|_0^\beta$$

$$\le \left(\frac{\alpha^p}{p} - 0\right) + \left(\frac{\beta^q}{q} - 0\right)$$

$$\Rightarrow \alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

# HOLDER'S INEQUALITY:—

Let $x = (\xi_j)_1^\infty \in l^p$, $y = (\eta_j)_1^\infty \in l^q$ where $1/p + 1/$

Then, $\sum_{j=1}^{\infty} |\xi_j \eta_j| \le \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p} \left(\sum_{m=1}^{\infty} |\eta_m|^q\right)^{1/q}$.

**PROOF:—** Let $(\bar{\xi}_j)_1^\infty \in l^p$ & $(\bar{\eta}_j)_1^\infty \in l^q$

s.t $\sum_1^\infty |\bar{\xi}_j|^p = 1$ & $\sum_1^\infty |\bar{\eta}_j|^q = 1$

If we set $\alpha = |\bar{\xi}_j|$ & $\beta = |\bar{\eta}_j|$ so that by

Auxiliary inequality

$$|\bar{\xi}_j|\,|\bar{\eta}_j| \le \frac{|\bar{\xi}_j|^p}{p} + \frac{|\eta_j|^q}{q}$$

$$\sum_{j=1}^{\infty} |\bar{\xi}_j \bar{\eta}_j| \le \frac{1}{p}\sum_1^{\infty} |\bar{\xi}_j|^p + \frac{1}{q}\sum_1^{\infty} |\bar{\eta}_j|^q$$

$$\le \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \sum_{j=1}^{\infty} |\bar{\xi}_j \bar{\eta}_j| \le 1 \quad \text{---} \quad (1)$$

let $x = (\xi_j)_1^{\infty} \in l^p$ & $y = (\eta_j)_1^{\infty} \in l^q$ then define $(\hat{\xi}_j)_1^{\infty}$ & $(\hat{\eta}_j)_1^{\infty}$ by $\hat{\xi}_j = \dfrac{\xi_j}{\left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p}}$ &

$$\hat{\eta}_j = \frac{\eta_j}{\left(\sum_{m=1}^{\infty} |\eta_m|^q\right)^{1/q}}$$

It is easy to check that $\sum_1^{\infty} |\hat{\xi}_j|^p = 1$ & $\sum_1^{\infty} |\hat{\eta}_j|^q = 1$

so that (1) $\Rightarrow \sum_1^{\infty} |\hat{\xi}_j \hat{\eta}_j| \le 1$

$$\Rightarrow \sum_1^{\infty} \frac{|\xi_j|\,|\eta_j|}{\left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p} \left(\sum_1^{\infty} |\eta_j|^q\right)^{1/q}} \le 1$$

$$\Rightarrow \sum_{j=1}^{\infty} |\xi_j \eta_j| \le \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p} \left(\sum_{m=1}^{\infty} |\eta_m|^q\right)^{1/q}$$

## MINKOWSKI'S INEQUALITY :—

Let $x = (\xi_j)_1^{\infty}$, $y = (\eta_j)_1^{\infty} \in l^p$ & let $p \ge 1$. Then,

$$\left(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p\right)^{1/p} \le \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{1/p} + \left(\sum_{m=1}^{\infty} |\eta_m|^p\right)^{1/p} \quad (1)$$

**PROOF :—** if $p = 1$ then (1) holds trivially because $|\xi_j + \eta_j| \le |\xi_j| + |\eta_j|$ so that

$$\sum_1^{\infty} |\xi_j + \eta_j| \le \sum_1^{\infty} |\xi_k| + \sum_1^{\infty} |\eta_m|$$

Suppose that $p > 1$ and let $\xi_j + \eta_j = \omega_j$

Then $|\omega_j|^p = |\omega_j|\,|\omega_j|^{p-1} = |\xi_j + \eta_j|\,|\omega_j|^{p-1} \le |\xi_j|\,|\omega_j|^{p-1} + |\eta_j|\,|\omega_j|^{p-1}$

$$\sum_1^{\infty} |\omega_j|^p \le \sum_1^{\infty} |\xi_j|\,|\omega_j|^{p-1} + \sum_1^{\infty} |\eta_j|\,|\omega_j|^{p-1} \quad \text{---} \quad (2)$$

Note that $(|\omega_j|)_1^\infty \in l^p$ in order to apply Holder's inequality in eq ② we first need to show that $(|\omega_j|^{p-1})^q \in l^q$. For this consider $\sum_1^\infty (|\omega_j|^{p-1})^q = \sum_1^\infty |\omega_j|^{(p-1)q} = \sum_1^\infty |\omega_j|^p < \infty$

$\because (\omega_j)_1^\infty \in l^p$ so that $(|\omega_j|^{p-1})_1^\infty \in l^q$.

So, $\sum_1^\infty |\xi_j| |\omega_j|^{p-1} \leq \left(\sum_{K=1}^\infty |\xi_K|^p\right)^{1/p} \left(\sum_1^\infty (|\omega_j|^{p-1})^q\right)^{1/q}$ — (

and $\sum_1^\infty |\xi_j||\omega_j|^{p-1} \leq \left(\sum_{m=1}^\infty |\eta_m|^p\right)^{1/p} \left(\sum_1^\infty (|\omega_j|^{p-1})^q\right)^{1/q}$ —

② $\Rightarrow \sum_1^\infty |\omega_j|^p \leq \left[\left(\sum_{K=1}^\infty |\xi_K|^p\right)^{1/p} + \left(\sum_{m=1}^\infty |\eta_m|^p\right)^{1/p}\right]\left(\sum_1^\infty (|\omega_j|\right.$

$\sum_{j=1}^\infty |\omega_j|^p \leq \left[\left(\sum_{K=1}^\infty |\xi_K|^p\right)^{1/p} + \left(\sum_{m=1}^\infty |\eta_m|^p\right)^{1/p}\right]\left(\sum_1^\infty |\omega_j|^p\right)^{1/q}$

$\left(\sum_{j=1}^\infty (|\omega_j|)^p\right)^{1-1/q} \leq \left(\sum_{K=1}^\infty |\xi_K|^p\right)^{1/p} + \left(\sum_{m=1}^\infty |\eta_m|^p\right)^{1/p}$

$\Rightarrow \left(\sum_{j=1}^\infty |\xi_j + \eta_j|^p\right)^{1/p} \leq \left(\sum_{K=1}^\infty |\xi_K|^p\right)^{1/p} + \left(\sum_{m=1}^\infty |\eta_m|^p\right)^{1/p}$

**EXP#** Consider $X = l^p$ with $d$ defined by $d(x,y) = \left(\sum_1^\infty |\xi_i - \eta_i|^p\right)^{1/p}$. show that $d$ is metric on $X$.

**SOL:—** $M_1$: Since, $|\xi_i - \eta_i|^p \geq 0 \quad \forall i$

$d(x,y) = \left(\sum_1^\infty |\xi_i - \eta_i|^p\right)^{1/p} \geq 0$

$M_2$: $d(x,y) = \left(\sum_1^\infty |\xi_i - \eta_i|^p\right)^{1/p} = 0 \Leftrightarrow \xi_i = \eta_i \;\; \forall i$

$\Leftrightarrow x = y$

$M_3$: $d(x,y) = \left(\sum_1^\infty |\xi_i - \eta_i|^p\right)^{1/p} = d(y,x)$

$M_4$: $d(x,y) = \left(\sum_1^\infty |(\xi_i - e_i) + (e_i - \eta_i)|^p\right)^{1/p}$ ; $(e_i)_1^\infty = z$

$\leq \left(\sum_1^\infty |\xi_i - e_i|^p\right)^{1/p} + \left(\sum_1^\infty |e_i - \eta_i|^p\right)^{1/p}$

$= d(x,z) + d(z,y)$

i.e $d(x,y) \leq d(x,z) + d(z,y)$

**EXP #** let $X = \mathbb{R}^n$ define $d$ on $X$ by

$$d(x,y) = \left(\sum_1^\infty |\xi_i - \eta_i|^2\right)^{1/2} = \sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2+\cdots+(\xi_n-\eta_n)^2}$$

show that $d$ is metric on $X$.

**SOL:—** $M_1$: Since, $|\xi_i - \eta_i|^2 \geq 0 \quad \forall i$

$$\Rightarrow \quad d(x,y) = \left(\sum_1^\infty |\xi_i-\eta_i|^2\right)^{1/2} \geq 0$$

$M_2$: $d(x,y) = \left(\sum_1^\infty |\xi_i-\eta_i|^2\right)^{1/2} = 0 \iff \xi_i = \eta_i \ \forall i$

$\iff x = y$

$M_3$: $d(x,y) = \left(\sum_1^\infty |\xi_i-\eta_i|^2\right)^{1/2} = \left(\sum_1^\infty |\eta_i-\xi_i|^2\right)^{1/2} = d(y,x)$

$M_4$: let $z = (e_i, \ldots, e_n) \in \mathbb{R}^n$ then

$$d(x,y) = \left(\sum_1^\infty |\xi_i-\eta_i|^2\right)^{1/2} = \left(\sum_1^\infty |(\xi_i - e_i) + (e_i - \eta_i)|^2\right)^{1/2}$$

$$\leq \left(\sum_1^\infty |\xi_i - e_i|^2\right)^{1/2} + \left(\sum_1^\infty |e_i - \eta_i|^2\right)^{1/2}$$

$$= d(x,z) + d(y,z)$$

i.e $d(x,y) \leq d(x,z) + d(z,y)$

Hence, $d$ is metric on $X$.

# QUESTIONS:—

**1:** If $(X,d)$ is metric space then show that $d_1(x,y) = \dfrac{2d(x,y)}{1+2d(x,y)}$. show that $d$ is metric.

**SOL:—** $M_1$: $d_1(x,y) \geq 0$

$M_2$: $d_1(x,y) = \dfrac{2d(x,y)}{1+2d(x,y)} = 0$

$\iff 2d(x,y) = 0 \iff x = y$

$M_3$: $d_1(x,y) = \frac{2d(x,y)}{1+2d(x,y)} = \frac{2d(y,x)}{1+2d(y,x)} = d_1(y,x)$

$M_4$: $d(x,y) \le d(x,z) + d(z,y)$

xply by 2

$2d(x,y) \le 2d(x,z) + 2d(z,y)$

Add ①

$1+2d(x,y) \le 1+2d(x,z) + 1+2d(z,y)$

$$\frac{1}{1+2d(x,y)} \ge \frac{1}{1+2d(x,z)} + \frac{1}{1+2d(z,y)}$$

xply by (-1)

$$\frac{-1}{1+2d(x,y)} \le \frac{-1}{1+2d(x,z)} + \frac{-1}{1+2d(z,y)}$$

Adding (1)

$$1+\frac{-1}{1+2d(x,y)} \le 1+\frac{-1}{1+2d(x,z)} + 1-\frac{1}{1+2d(z,y)}$$

$$\frac{\cancel{1}+2d(x,y)-\cancel{1}}{1+2d(x,y)} \le \frac{\cancel{1}+2d(x,z)-\cancel{1}}{1+2d(x,z)+2d(z,y)}$$

$$\frac{2d(x,y)}{1+2d(x,y)} \le \frac{2d(x,z)}{1+2d(x,z)} + \frac{2d(z,y)}{1+2d(z,y)}$$

$d_1(x,y) \le d_1(x,z) + d_1(z,y)$

Hence, d is metric.

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**2:** show that $d(x,y) = \dfrac{|x-y|}{1+|x-y|}$ $\forall\ x,y \in \mathbb{R}$

is a metric on $\mathbb{R}$.

**SOL:—** $M_1$: $d(x,y) \geq 0$

$M_2$: $d(x,y) = \dfrac{|x-y|}{1+|x-y|} = 0 \iff |x-y| = 0$

$\iff x = y$

$M_3$: $d(x,y) = \dfrac{|x-y|}{1+|x-y|} = \dfrac{|y-x|}{1+|y-x|} = d(y,x)$

$M_4$: Define $f(t) = \dfrac{t}{1+t}$ then

$f'(t) = \dfrac{1+\cancel{t}-\cancel{t}}{(1+t)^2} = \dfrac{1}{(1+t)^2} > 0$

$f$ is monotonically increasing i.e

$f(t_1) \leq f(t_2)$ $\forall\ t_1, t_2$. Then $|a+b| \leq |a|+|b|$

$\Rightarrow \dfrac{|a+b|}{1+|a+b|} \leq \dfrac{|a|+|b|}{1+|a|+|b|} = \dfrac{|a|}{1+|a|+|b|} + \dfrac{|b|}{1+|a|+|b|}$

$\dfrac{|a+b|}{1+|a+b|} \leq \dfrac{|a|}{1+|a|} + \dfrac{|b|}{1+|b|}$

choose $a = x-z$ & $b = z-y$ then

$\dfrac{|x-y|}{1+|x-y|} \leq \dfrac{|x-z|}{1+|x-z|} + \dfrac{|z-y|}{1+|z-y|}$

$d(x,y) \leq d(x,z) + d(z,y)$

**3:** show that $d(x,y) = \sum_1^\infty |\xi_i - \eta_i|$, $x = (\xi_1, \dots, \xi_n)$, $y = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$ is metric on $\mathbb{R}^n$.

**SOL:—** $M_1$: $d(x,y) \geq 0$

$M_2$: $d(x,y) = \sum_1^\infty |\xi_i - \eta_i| = 0$

$\Leftrightarrow \xi_i = \eta_i \Leftrightarrow x = y$

$M_3$: $d(x,y) = \sum_1^\infty |\xi_i - \eta_i| = \sum_1^\infty |\eta_i - \xi_i|$

$= d(y,x)$

$M_4$: $d(x,y) = \sum_1^\infty |\xi_i - \eta_i|$

$= \sum_1^\infty |(\xi_i - e_i) + (e_i - \eta_i)|$

$\leq \sum_1^\infty |\xi_i - e_i| + \sum_1^\infty |e_i - \eta_i| = d(x,z) + d$

by Minkowski's inequality

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$

Hence, $d$ is metric on $\mathbb{R}^n$.

## SOME TOPOLOGICAL CONCEPTS:—

**DEFINITION:—** Let $(X,d)$ be a metric spa $x_0 \in X$ & $r > 0$ define

(i) $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ (open ball)

$\bar{B}(x_0, r) = \{x \in X : d(x, x_0) \leq r\}$ (closed ball)

$S(x_0, r) = \{x \in X : d(x, x_0) = r\}$ (sphere)

Note that $S(x_0, r) = \bar{B}(x_0, r) - B(x_0, r)$

**EXP #** Consider discrete metric space (X,

where $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if} \end{cases}$

**SOL:—** $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$

$B(x_0, 1) = \{x \in X : d(x, x_0) < 1\}$

$\Rightarrow \quad d(x, x_0) = 0 \quad , \quad x = x_0$

$\{x_0\}$

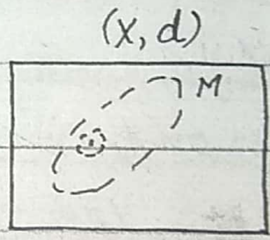
$S(x_0, 1) = \{x \in X : d(x, x_0) = 1\} \quad , \quad x \neq x_0$

$= X / \{x_0\}$

$S(x_0, r) = \{x \in X : d(x, x_0) = r\}$

$= \phi \quad \text{if} \quad r \neq 1.$

**DEF:—** Let $(X, d)$ be a metric space & $M \subseteq X$.
we say that M is an open set if it contains
an open ball about each of its pts.
A subset K of X is called closed
if its compliment is open.
i.e $K^c = X / K$ is open.

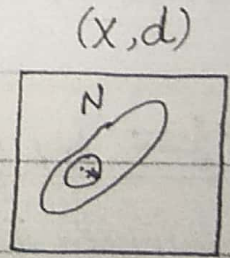



$x \in M$

$B(x, r) \subseteq M$

**DEF:—** An open ball $B(x_0, \varepsilon)$ is called an
$\varepsilon$-neighborhood of $x_0$.

**DEF:—** A subset $N \subseteq X$ is called
n.hood of a pt $x_0 \in X$ if it contains
an $\varepsilon$-n.hood of $B(x_0, \varepsilon)$ of $x_0$.




**DEF:—** A pt $x_0 \in X$ is called an interior pts
of $M \subseteq X$ if M is n.hood of $x_0$. The set of all interior
pts of M is denoted by $Int(M)$ or $M^\circ$.

**RESULT:—** Every open ball is an open set.