## 6.10 INNER PRODUCT REAL AND COMPLEX SPACES

In this section, we explore some extra structure on a vector space V. We have experienced the real spaces to the abstract setting. The concept of scalar product (dot product) of vectors, which attaches to the (i) length (magnitude) of a vector (ii) angle between two vectors and perpendicularity of a vector to given vector.

A more generalized concept of inner product on V is a more useful tool in linear algebra. It is important to note that inner products for the real spaces are some what different from the complex vector spaces. Thus we define inner product notion on real and complex spaces separately as follows:

### 6.10.1 DEFINITION

Let V($\Re$) be a real vector space. A function f : V×V $\longrightarrow \Re$, from V×V into $\Re$ is called inner product on V($\Re$) when defined by $f(\underline{u},\underline{v}) = (f(\underline{u}), f(\underline{v})) = \langle \underline{u},\underline{v} \rangle$ observing the following properties;

(a) $\langle \alpha\underline{u}+\beta\underline{v},\underline{w}\rangle = \alpha\langle \underline{u},\underline{w}\rangle + \beta\langle \underline{v},\underline{w}\rangle$

(b) $\langle \underline{u},\alpha\underline{v}+\beta\underline{w}\rangle = \alpha\langle \underline{u},\underline{v}\rangle + \beta\langle \underline{u},\underline{w}\rangle$

(c) $\langle \underline{u},\underline{v}\rangle = \langle \underline{v},\underline{u}\rangle$

(d) $\langle \underline{u},\underline{v}\rangle \geq 0$

(e) $\langle \underline{u},\underline{v}\rangle = 0$, if and only if $\underline{u} = 0_v$

### 6.10.2 DEFINITION

A vector space V($\Re$) together with inner product < , > on V($\Re$) is called real **inner product space.**

### 6.10.3 IMMEDIATE OBSERVATIONS FROM THE DEFINITION

(1) An inner product < , > on V is a special type of multilinear function from V x V into $\Re$.

(2) Inner product < , > is a bilinear form, which is symmetric from axiom (c) of definition 6.10.1.

(3) It is customary to adopt a special notation < , > for inner product. It observes the properties of dot product on V(F), when defined by $\langle \underline{u},\underline{v}\rangle = \sum_{i=1}^{n} \alpha_i u_i v_i$, if $\underline{u} = (u_1, u_2, \ldots, u_n)'$ and $\underline{v} = [v_1, v_2, \ldots, v_n]'$, $\alpha_i \in \Re$.

(4) A space V(F) can have infinitely many different inner products on it.

### 6.10.4 DEFINITION

Let $V(\mathbb{C})$ be a complex vector space. A complex inner product on $V(\mathbb{C})$ is a function $\langle\ \rangle$ similar from V×V into $\mathbb{C}$ which satisfies the following axioms;

(a) $\langle \alpha_1 \underline{u} + \beta_1 \underline{v}, \underline{w} \rangle = \alpha_1 (\underline{u}, \underline{w}) + \beta_1 (\underline{v}, \underline{w})$

(b) $\langle \underline{w}, \alpha_1 \underline{u} + \beta_1 \underline{v} \rangle = \overline{\alpha_1} (\underline{w}, \underline{u}) + \overline{\beta_1} (\underline{w}, \underline{v})$

(c) $\langle \underline{u}, \underline{v} \rangle = \overline{(\underline{v}, \underline{u})}$

d) $\langle \underline{u}, \underline{v} \rangle \geq 0$

e) $\langle \underline{u}, \underline{v} \rangle = 0$, if and only if $\underline{u} = 0_V$

for all $\underline{u}, \underline{v}, \underline{w} \in V(\mathbb{C})$ and $\alpha_1, \beta_1 \in \mathbb{C}$.

Of course, our principle example of complex inner product is defined by

$$\langle \underline{u}, \underline{v} \rangle = \underline{u}' \cdot \underline{v} \text{ on } \mathbb{C}^n - space$$

If $\underline{u} = (\alpha_1, \alpha_2, ...., \alpha_n)$ and $\underline{v} = (\beta_1, \beta_2, ...., \beta_n) \in \mathbb{C}^n$ then

$$\langle \underline{u}, \underline{v} \rangle = \alpha_1 \overline{\beta}_1 + \alpha_2 \overline{\beta}_2 + ... + \alpha_n \overline{\beta}_n = \sum_{i=1}^{n} \alpha_i \overline{\beta}_i$$

### 6.10.5 DEFINITION

A vector space V over F is said to be an **inner product space**, if there is defined, for any two vectors $\underline{u}$, $\underline{v} \in V$, an element ($\underline{u}$, $\underline{v}$) in F such that

(1) $(u, v) = \overline{(v, u)}$

(2) $(u, u) \geq 0$

(3) $(u, u) = 0$ if and only if $\underline{u} = ov$

(4) $(\alpha u + \beta v, w) = \alpha (u, w) + \beta (v, w)$

for any u, v, w ∈ V and α, β∈ F.

### 6.10.6 IMMEDIATE OBSERVATIONS FROM DEFINITION

(i) If F is the field of complex numbers, the property (1) of inner product implies that ($\underline{u}$, $\underline{u}$) is real and property (2) makes sense.

(ii) $(\underline{u}, \alpha v + \beta w) = \overline{(\alpha v + \beta w, u)} = (\overline{\alpha (v,u) + \beta (w,u)})$

$$= \overline{\alpha}(\overline{(v,u)}) + \overline{\beta}\overline{(w,u)}$$

$$= \overline{\alpha}\ (u, v) + \overline{\beta}\ (u, w), \text{ by (1)}$$

(iii) If $u = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $v = (\beta_1, \beta_2, \ldots, \beta_n) \in F^n$ and $(u, v) = \alpha_1 \overline{\beta_1} + \alpha_2 \overline{\beta_2} + \cdots + \alpha_n \overline{\beta_n}$ defines an inner product on $F^n$ and makes $F^n$ an inner product space.

(iv) If $u = (\alpha_1, \alpha_2)$ and $v = (\beta_1, \beta_2)$, then $(u, v) = 2\alpha_1\overline{\beta_1} + \alpha_1\overline{\beta_2} + \alpha_2\overline{\beta_1} + \alpha_2\overline{\beta_2}$ defines an inner product on $F^2$ and makes $F^2$ an inner product space.

## 6.10.7 DEFINITION

Let V be an inner product space. If $v \in V$, then the length of v is called the norm of v, written by $\| v \|$, and defined by $\| v \| = \sqrt{(v, v)}$

## 6.10.8 LEMMA

$\| \alpha \underline{u} \| = | \alpha | \, \| \underline{u} \|$

## PROOF

$\| \alpha u \|^2 = (\alpha u, \alpha u) = \alpha\bar{\alpha} \ (u, u)$

Since $\alpha \cdot \bar{\alpha} = | \alpha |^2$ and $(u, u) = \| u \|^2$, therefore,

$\| \alpha u \|^2 = | \alpha |^2 \cdot \| u \|^2$

$\Rightarrow$ $\| \alpha u \| = \alpha \cdot \| u \|$

## 6.10.9 DEFINITION

Let V be an inner product space. If $u, v \in V$ then u is said to be orthogonal to v if $(u, v) = 0v$.

It is important to note that if $\underline{u}$ is orthogonal to v then $\underline{v}$ is orthogonal to u, by

$(v, u) = \overline{(u, v)} = \bar{0} = 0$

## 6.10.10 DEFINITION

If W is a subspace of a normed space V, the orthogonal complement $W^\perp$ of W, is defined by $W^\perp = \{x \in V : (x, w) = 0v, \text{ for all } w \in W\}$.

### 6.10.11 LEMMA

$W^{\perp}$ is a subspace of V.

**PROOF**

Let $x, y \in W^{\perp}$. Then $(x, w) = 0_V = (y, w)$, for all $w \in W$, a subspace of V. If $\alpha, \beta \in F$, then,

$$(\alpha x + \beta y, w) = \alpha(x, w) + \beta(y, w)$$
$$= \alpha \cdot 0_V + \beta \cdot 0_V = 0_V$$

Thus $W^{\perp}$ is a subspace of V(F).

### 6.10.12 LEMMA

$W \cap W^{\perp} = \{0_V\}$:

**PROOF**

Let $u \in W \cap W^{\perp}$. Then $u \in W^{\perp}$ and hence $(u, u) = 0_V$, which is not true by definition of inner product unless $u = 0_V$.

**COR.**

Any normed vector space V is the direct sum of its spaces W and $W^{\perp}$, for each subspace W of V.

i.e., $V = W \oplus W^{\perp}$

For example, if $V = R^3 = \{(x, y, z) : x, y, z \in R\}$ is a vector space, then

$W = \{(x, y, 0) : x, y, \in R\}$ is a subspace of $R^3$ with $W^{\perp} = \{(0, 0, z) : z \in R\}$ as a subspace of $R^3$ with $W \cap W^{\perp} = \{0\}$ and $W + W^{\perp} = R^3$. Consequently $W \oplus W^{\perp} = R^3$.

### 6.10.13 THEOREM

Let V be a finite dimensional inner product space and W be a subspace of V. Then $(w^{\perp})^{\perp} = W$.

**PROOF**

Take $w \in W$, where W is a subspace of V. If $u \in W_1$, then $(w, u) = 0$, by definition, for all $u \in W^{\perp}$, which implies that $w \in W^{\perp}$ for all w and hence $w \subset W^{\perp}$.

Now $V = W_1 \oplus W = W_1 \oplus (W_1)^{\perp}$.

Since $\dim (W) = \dim (W^{\perp})^{\perp}$ and $W \subset W^{\perp} \subset (W^{\perp})^{\perp} \Rightarrow W \subset (W^{\perp})^{\perp}$, therefore $W = (W^{\perp})^{\perp}$, which proves the assertion of the theorem.

## 6.11 ORTHONORMAL BASIS OF FINITE DIMENSIONAL VECTOR SPACES

We have learnt already at the earlier part of this chapter that each finite-dimensional vector space V attains a basis of V. Basis of each vector space plays a central role in determining the local behaviour of V within itself. Since, each element of a vector space has also been declared a vector, therefore study of vectors shoulders the responsibility of the inner study of a vector space. With reference to the development of vector space structure, it is important to recall the content of chapter 2 on vectors, where orthogonality of a non-zero vector is defined to another non-zero vector. If u and v are two vectors of same space then vector $\left(v-\left(\frac{v \cdot u}{u \cdot u}\right)u\right)$ is orthogonal to the vector u, makes it possible to locate a vector orthogonal to any given vector. Thus basis vectors can this way be transformed to an orthogonal basis of the same vector space. The basis vectors can be located to be orthonormal, which are each of length unity.

### 6.11.1 DEFINITION

The set of vectors $\{v_i\}$ in V is an **orthonormal** set if

(1) each $v_i$ is of length 1 (i.e., $(v_i, v_i) = 1$)

(2) for $i \neq j$, $(v_i, v_j) = 0$

### 6.11.2 LEMMA

If $\{v_i\}$ is an orthonormal set of vectors of a normed space V, then the vectors of $\{v_i\}$ are linearly independent.