The dual space Hom (V, F) of V(F) is generally denoted by $\hat{V}$, each element of which is $f \in$ Hom(V,F).

REMARK

We have exhibited that for finite dimensional vector space V(F), a vector space $\hat{V}$ (F) is attached to V(F) under the same field having the same dimension i.e., dim V(F) = dim $\hat{V}$ (F).

QUESTION?

"What is the dual space $\hat{\hat{V}}$ (F) of $\hat{V}$ (F), (the dual space of V(F)), if V(F) is finite dimensional vector space over F?". As an answer of the above question, we prove the following theorem:

6.5.14 THEOREM

Let V be a vector space with dual space $\hat{V} = \{f : f \in \text{Hom}(V,F)\}$.

If $\hat{\hat{V}} = \{T : T \in \text{Hom}(\hat{V}, F)\}$ then $\hat{\hat{V}} \cong V$.

PROOF

Let $v_0 \in$ V(F) be a fixed vector. If $f \in \hat{V}$ (F) then $f(v_0)$ defines a functional f from $\hat{V}$ into F. Let us denote it a function T_{v_0} defined by,

$T_{v_0}(f) = f(v_0)$, for every $f \in \hat{V}$, $T_{v_0} \in \text{Hom}(\hat{V}, F)$, since,

$$T_{v_0}(f + g) = (f + g)(v_0) = f(v_0) + g(v_0)$$

$$= T_{v_0}(f) + T_{v_0}(g), \ \forall f, g \in \hat{V}$$

Furthermore, $T_{v_0}(\alpha f) = (\alpha f)(v_0) = \alpha(f(v_0)) = (\alpha T_{v_0})(f)$

Thus $T_{v_0} \in \text{Hom}(\hat{V}, F)$, the dual space $\hat{\hat{V}}$ of $\hat{V}$

In general, given a vector $v \in V$, a vector T_v of $\hat{V}$ is associated with v. This association generates a homomorphism ψ from V(F) into $\hat{V}$, which is defined by,

$$\psi(v) = T_v \in \hat{V} \text{ for every } v \in V. \text{ Then,}$$

$$\psi(v+w) = T_{v+w}, \text{ where, } T_v, T_w \in \hat{V}$$

For $f \in \hat{V}$,

$$\begin{aligned} T_{v+w}(f) &= f(v+w) \\ &= f(v) + f(w) \\ &= T_v(f) + T_w(f) \\ &= (T_v + T_w)(f), \forall f \in \hat{V} \end{aligned}$$

$$\Rightarrow \quad T_{v+w} = T_v + T_w,$$

and hence $\psi(v+w) = \psi(v) + \psi(w) = T_v + T_w$, of V into $\hat{V}$ and $\psi(\lambda v) = \lambda\psi(v)$, $\lambda \in F$.

which proves that ψ is a homomorphism. Infact, ψ is an isomorphism, because,

$v \in \ker \psi$ iff $\psi(v) = T_0$, the zero of Hom(V, F). It implies that $T_v = T_0$.

If $T_v = T_0$, then $T_v(f) = f(v) = 0$, for all $f \in \hat{V}$, then $v = 0v$ and $\ker(\psi) = \{T_0\}$

It proves then that ψ is an isomorphism. Hence $\hat{V} \equiv V$

Note that,

(i) If V is finite dimensional then ψ is onto $\hat{V}$, because dim V = dim $\hat{V}$ = dim $\hat{V}$

(ii) If V is infinite dimensional, then V is not onto

(iii) dimensions of V, $\hat{V}$ and $\hat{V}$ are equal if V is finite dimensional.

(iv) If $0v \neq v \in V(F)$ then there exists $f \in \hat{V}$ such that $f(v) \neq 0v$

6.6 ANNIHILATOR OF A SUBSPACE OF V(F)

We know that any f in Hom(V,W) acts on V and transforms onto vectors of W. The vectors of V(F) which are transformed onto the zero vector of W(F) constitute the kernel of f. It provides information that how many vectors of V(F) are transformed onto a non-zero vector of W(F). They are equal in number to the vector of the quotient space V/ker f. In other words every homomorphism annihilates its kernel which is subspace of the domain space of the homomorphism f.

A fixed subspace W of a vector space V(F) contained in the kernel of a linear functional $f \in Hom(V,F)$ is said to be an **annihilator** of W.

6.6.1 DEFINITION

If W is a subspace of a vectorspace V(F). Then, the set $A(W) = \{f \in \hat{V} : f(w) = 0w$, for all $w \in W\}$ is said to form the set of annihilators of W.

6.6.2 LEMMA:

Let A(W) be the set of all annihilators of W, with W as a subspace of V(F). Then A(W) is a subspace of $\hat{V}$.

PROOF

Let $f, g \in A(W) \subseteq \hat{V}$ and $\alpha, \beta \in F$. Then $f(w) = 0w = g(w)$, for all $w \in W$

$$\text{Then } (\alpha f + \beta g)(w) = (\alpha f)(w) + (\beta g)(w)$$

$$= \alpha(f(w)) + \beta(g(w)) = \alpha 0w + \beta 0w = 0w,$$

which implies that $\alpha f + \beta g \in A(W)$ and hence A (W) forms a subspace of $\hat{V}$

6.6.3 LEMMA

If $U \subset W \subseteq V(F)$, then $A(U) \supset A(W)$.

PROOF

Let $f \in A(W)$. Then $f(w) = 0w, \forall w \in W$

Since $U \subset W$ and $u \in U \Rightarrow u \in W$, therefore $f(u) = 0w$, for all $u \in U$. It implies that $f \in A(U)$ which holds for all $f \in A(W)$. Hence $A(W) \subset A(U)$ or $A(U) \supset A(W)$.

6.6.4 THEOREM

Let W be a subspace of a finite dimensional vector space V(F).

Then $\hat{W} \cong \hat{V}/A(W)$ and dimension of $A(W) = \dim(V) - \dim(W) = \dim\left(\frac{V}{W}\right)$

PROOF

Since V(F) is finite dimensional and $W \subseteq V(F)$ therefore W is finite dimensional.

Let $f \in \hat{V}$ and $\bar{f}$ be the restriction of f to W defined by $f(w) = \bar{f}(w)$, for every $w \in W$. Since $f \in \hat{V}$, therefore $\bar{f} \in \hat{W}$. If $T : \hat{V} \longrightarrow \hat{W}$, then T is a homomorphism defined by,

$$T(f) = \bar{f}.$$

If $f, g \in \hat{V}$, then $T(f + g) = T(f) + T(g)$ and $T(\lambda f) = \lambda T(f)$,

which shows that $T \in Hom(\hat{V}, \hat{W})$ such that $\ker T = A(W)$

We intend to show that T is onto $\hat{W}$. For this purpose, let $h \in \hat{W}$. Then h is the restriction of some $f \in \hat{V}$ i.e., $T(h) = \bar{f}$

Let $\{w_1, w_2, \ldots, w_m\}$ be a basis of W, which can be extended to be a basis $\{w_1, w_2, \ldots, w_m, v_1, \ldots, v_r\}$ of V such that $dm(V) = m + r = n$

Let W_1 be a subspace of V spanned by $\{v_1, v_2, \ldots, v_r\}$ giving $V = W \oplus W_1$. If $h \in \hat{W}$ and $v \in V$ such that $v = w + w_1$, $w \in W$ and $w_1 \in W_1$, then $f(v) = h(w)$, $f \in \hat{V}$ such that $\bar{f} = h = T(f)$. Hence T maps $\hat{V}$ onto $\hat{W}$.

Since $\ker T = A(W)$, therefore, by the fundamental theorem of homomorphsims (Chapter 4),

$$\hat{V}/A(W) \cong \hat{W}$$

So the isomorphic spaces have same dimension. Thus,

$$\dim \hat{V} - \dim A(W) = \dim(\hat{W})$$

$\Rightarrow \quad \dim V - \dim A(W) = \dim W$, where $\dim \hat{V} = \dim V$ and $\dim \hat{W} = \dim W$

$\Rightarrow \quad n - m = \dim A(W) \Rightarrow \dim V - \dim W = \dim(V/W)$,

giving that $\dim A(W) = \dim(V/W)$.

COR

$A(A(W)) = W$, for each subspace W of $V(F)$.

PROOF

Since $W \subset V$ and $A(A(W)) \subset \hat{\hat{V}} \equiv V$, therefore $W \subset A(A(W))$, for if $w \in W$ then T_w acts on V by $T_w(f) = f(w) = 0w$, $\forall f \in A(W)$

However, $\dim A(A(W)) = \dim \hat{V} - \dim A(W)$

or $\dim A(A(W)) = \dim V - (\dim V - \dim W)$

$= \dim W$

Since $W \subset A(A(W))$, and $\dim W = \dim(A(A(W)))$, therefore $W = A(A(W))$.

6.7 MATRIX REPRESENTATIONS OF LINEAR TRANSFORMATIONS

We have shown in this chapter that the set $Hom_F(V, W)$ of all vector space homomorphisms (vector spaces being over the same field) from a vector space $V(F)$ into a vector $W(F)$. The elements of $Hom_F(V, W)$ act as linear transforms from $V(F)$ into $W(F)$ respecting their binary operations of addition and scalar multiplication. i.e.

If $T \in Hom_F(V, W)$ and $v_1, v_2 \in V$, then

(a) $T(v_1 + v_2) = T(v_1) + T(v_2)$

and (b) $T(\alpha v_1) = \alpha T(v_1)$, for all $\alpha \in F$.

(a) and (b) imply that T preserves the geometrical figures of parallelogram of $V(F)$ into $W(F)$, if v_1 and v_2 are not collinear vectors of $V(F)$.
If the vectors v_1 and v_2 are collinear then T preserves the collinearity. T may stretch or contract or reverse a vector along a line.

At this point one desires to develop a connection between the space $Hom_F(V, W)$ linear transforms and the space $M_{m\times n}$ of m×n matrices over the same field as of V and W finite dimensions n and m respectively. We are aware of the fact the space $M_{m\times n}(F)$ same dimension mn as that of space $Hom_F(V, W)$. For the sake of establishing relationship we shall assume that

(1) V and W are finite dimensional vector spaces over the same field F($\mathfrak{R}$

(2) $\dim V = n$, $\dim W = m$

(3) $V = \mathfrak{R}^n$ and $W = \mathfrak{R}^m$

(4) The elements of $\mathfrak{R}^n$ and $\mathfrak{R}^m$ are Coordinated Column Vectors having n components respectively.