6.8 ORTHONORMAL BASES IN R^n

From our work with the natural bases for R^2, R^3, and, in general, fo
we know that when these bases are present, the computations are kep
minimum. A subspace W of R^n need not contain any of the natural
vectors, but in this section we want to show that it has a basis with the
properties. That is, we want to show that W contains a basis S such that
vector in S is of unit length and every two vectors in S are orthogonal
method used to obtain such a basis is the Gram–Schmidt process, wh
presented below.

DEFINITION A set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ in R^n is called **orthogonal** if any two of
vectors in S are orthogonal, that is, if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for $i \neq j$. An ortho
set of vectors is an orthogonal set of unit vectors. That is, $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots$
is orthonormal if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for $i \neq j$, and $\mathbf{u}_i \cdot \mathbf{u}_i = 1$ for $i = 1, 2, \ldots$

EXAMPLE 1 If $\mathbf{x}_1 = (1, 0, 2)$, $\mathbf{x}_2 = (-2, 0, 1)$, and $\mathbf{x}_3 = (0, 1, 0)$, then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is an orthogonal set in R^3. The vectors

$$\mathbf{u}_1 = \left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right) \quad \text{and} \quad \mathbf{u}_2 = \left(-\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right)$$

are unit vectors in the directions of $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. Since $\mathbf{x}_3$ is also of unit length, it follows that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{x}_3\}$ is an orthonormal set. Also, span $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is the same as span $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{x}_3\}$. ■

EXAMPLE 2 The natural basis

$$\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

is an orthonormal set in R^3. More generally, the natural basis in R^n is an orthonormal set. ■

An important result about orthogonal sets is the following theorem.

THEOREM 6.16 *Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be an orthogonal set of nonzero vectors in R^n. Then S is linearly independent.*

Proof Consider the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}. \tag{1}$$

Taking the dot product of both sides of (1) with $\mathbf{u}_i$, $1 \le i \le k$, we have

$$(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) \cdot \mathbf{u}_i = \mathbf{0} \cdot \mathbf{u}_i. \tag{2}$$

By properties (c) and (d) of Theorem 4.3, Section 4.2, the left side of (2) is

$$c_1(\mathbf{u}_1 \cdot \mathbf{u}_i) + c_2(\mathbf{u}_2 \cdot \mathbf{u}_i) + \cdots + c_k(\mathbf{u}_k \cdot \mathbf{u}_i),$$

and the right side is 0. Since $\mathbf{u}_j \cdot \mathbf{u}_i = 0$ if $i \ne j$, (2) becomes

$$0 = c_i(\mathbf{u}_i \cdot \mathbf{u}_i) = c_i \|\mathbf{u}_i\|^2. \tag{3}$$

By (a) of Theorem 4.3, Section 4.2; $\|\mathbf{u}_i\| \ne 0$, since $\mathbf{u}_i \ne \mathbf{0}$. Hence (3) implies that $c_i = 0$, $1 \le i \le k$, and S is linearly independent. ■

COROLLARY 6.6 *An orthonormal set of vectors in R^n is linearly independent.*

Proof Exercise T.2. ■

DEFINITION From Theorem 6.9 of Section 6.4 and Corollary 6.6, it follows that an orthonormal set of n vectors in R^n is a basis for R^n (Exercise T.3). An **orthogonal (orthonormal) basis** for a vector space is a basis that is an orthogonal (orthonormal) set.

We have already seen that the computational effort required to solve a given problem is often reduced when we are dealing with the natural basis for R^n. This reduction in computational effort is due to the fact that we are dealing with an orthonormal basis. For example, if $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis for R^n, then if $\mathbf{v}$ is any vector in V, we can write $\mathbf{v}$ as

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n.$$

The coefficients $c_1, c_2, \ldots, c_n$ are obtained by solving a linear system of n equations in n unknowns. (See Section 6.3.)

However, if S is orthonormal, we can obtain the same result with much less work. This is the content of the following theorem.

THEOREM 6.17 *Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for R^n and $\mathbf{v}$ any vector in R^n. Then*

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n,$$

where

$$c_i = \mathbf{v} \cdot \mathbf{u}_i \qquad 1 \leq i \leq n.$$

Proof Exercise T.4.

COROLLARY 6.7 *Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthogonal basis for R^n and $\mathbf{v}$ any vector in R^n. Then*

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n,$$

where

$$c_i = \frac{\mathbf{v} \cdot \mathbf{u}_i}{\mathbf{u}_i \cdot \mathbf{u}_i} \qquad 1 \leq i \leq n.$$

Proof Exercise T.4(b).

EXAMPLE 3 Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for R^3, where

$$\mathbf{u}_1 = \left(\tfrac{2}{3}, -\tfrac{2}{3}, \tfrac{1}{3}\right), \quad \mathbf{u}_2 = \left(\tfrac{2}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\right), \quad \text{and} \quad \mathbf{u}_3 = \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right).$$

Write the vector $\mathbf{v} = (3, 4, 5)$ as a linear combination of the vectors in S.

Solution We have

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3.$$

Theorem 6.17 shows that c_1, c_2, and c_3 can be obtained without having to solve a linear system of three equations in three unknowns. Thus

$$c_1 = \mathbf{v} \cdot \mathbf{u}_1 = 1, \qquad c_2 = \mathbf{v} \cdot \mathbf{u}_2 = 0, \qquad c_3 = \mathbf{v} \cdot \mathbf{u}_3 = 7,$$

and $\mathbf{v} = \mathbf{u}_1 + 7\mathbf{u}_3$.

THEOREM 6.18 **(Gram*–Schmidt** Process)** *Let W be a nonzero subspace of R^n with basis $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$. Then there exists an orthonormal basis $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ for W.*

Proof The proof is constructive; that is, we develop the desired basis T. However, we first find an orthogonal basis $T^* = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ for W.

First, we pick any one of the vectors in S, say $\mathbf{u}_1$, and call it $\mathbf{v}_1$. Thus $\mathbf{v}_1 = \mathbf{u}_1$. We now look for a vector $\mathbf{v}_2$ in the subspace W_1 of W spanned by $\{\mathbf{u}_1, \mathbf{u}_2\}$ that is orthogonal to $\mathbf{v}_1$. Since $\mathbf{v}_1 = \mathbf{u}_1$, W_1 is also the subspace spanned by $\{\mathbf{v}_1, \mathbf{u}_2\}$. Thus

$$\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\mathbf{u}_2.$$

We try to determine c_1 and c_2 so that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Now

$$0 = \mathbf{v}_2 \cdot \mathbf{v}_1 = (c_1\mathbf{v}_1 + c_2\mathbf{u}_2) \cdot \mathbf{v}_1 = c_1(\mathbf{v}_1 \cdot \mathbf{v}_1) + c_2(\mathbf{u}_2 \cdot \mathbf{v}_1).$$

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*Jörgen Pederson Gram (1850–1916) was a Danish actuary.

**Erhard Schmidt (1876–1959) taught at several leading German Universities and was a student of both Hermann Amandus Schwarz and David Hilbert. He made important contributions to the study of integral equations and partial differential equations and, as part of this study, he introduced the method for finding an orthonormal basis in 1907. In 1908 he wrote a paper on infinitely many linear equations in infinitely many unknowns, in which he founded the theory of Hilbert spaces and in which he again used his method.

Since $\mathbf{v}_1 \neq \mathbf{0}$ (why?), $\mathbf{v}_1 \cdot \mathbf{v}_1 \neq 0$, and solving for c_1 and c_2 in (4), we have

$$c_1 = -c_2 \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}.$$

We may assign an arbitrary nonzero value to c_2. Thus, letting $c_2 = 1$, we obtain

$$c_1 = -\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}.$$

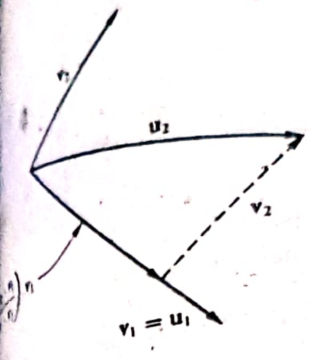


Figure 6.10 ▲

Hence

$$\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\mathbf{u}_2 = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1.$$

Notice that at this point we have an orthogonal subset $\{\mathbf{v}_1, \mathbf{v}_2\}$ of W (see Figure 6.10).

Next, we look for a vector $\mathbf{v}_3$ in the subspace W_2 of W spanned by $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ that is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. Of course, W_2 is also the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{u}_3\}$ (why?). Thus

$$\mathbf{v}_3 = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{u}_3.$$

We now try to find d_1 and d_2 so that

$$\mathbf{v}_3 \cdot \mathbf{v}_1 = 0 \quad \text{and} \quad \mathbf{v}_3 \cdot \mathbf{v}_2 = 0.$$

Now

$$0 = \mathbf{v}_3 \cdot \mathbf{v}_1 = (d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{u}_3) \cdot \mathbf{v}_1 = d_1(\mathbf{v}_1 \cdot \mathbf{v}_1) + d_3(\mathbf{u}_3 \cdot \mathbf{v}_1). \quad (5)$$

$$0 = \mathbf{v}_3 \cdot \mathbf{v}_2 = (d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{u}_3) \cdot \mathbf{v}_2 = d_2(\mathbf{v}_2 \cdot \mathbf{v}_2) + d_3(\mathbf{u}_3 \cdot \mathbf{v}_2). \quad (6)$$

In obtaining the right sides of (5) and (6), we have used the fact that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Observe that $\mathbf{v}_2 \neq \mathbf{0}$ (why?). Solving (5) and (6) for d_1 and d_2, respectively, we obtain

$$d_1 = -d_3 \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \quad \text{and} \quad d_2 = -d_3 \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}.$$

We may assign an arbitrary nonzero value to d_3. Thus, letting $d_3 = 1$, we have

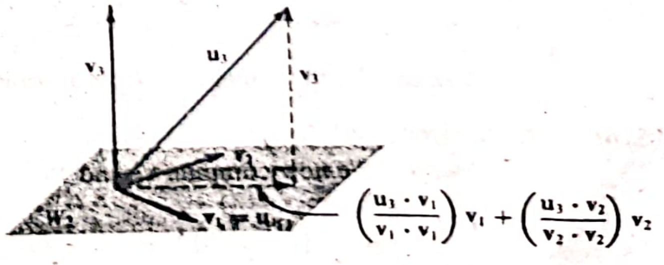
$$d_1 = -\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \quad \text{and} \quad d_2 = -\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}.$$

Hence

$$\mathbf{v}_3 = \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2.$$

Notice that at this point we have an orthogonal subset $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of W (see Figure 6.11).

Figure 6.11 ►



We next seek a vector $\mathbf{v}_4$ in the subspace W_3 of W spanned by the $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$, and thus by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{u}_4\}$, that is orthogonal to $\mathbf{v}_1, \mathbf{v}_2$, $\mathbf{v}_3$. We can then write

$$\mathbf{v}_4 = \mathbf{u}_4 - \left(\frac{\mathbf{u}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \left(\frac{\mathbf{u}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\right)\mathbf{v}_3.$$

We continue in this manner until we have an orthogonal set $T^* = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ of m vectors. It then follows that T^* is a basis for W. If we normalize the $\mathbf{v}_i$, that is, let

$$\mathbf{w}_i = \frac{1}{\|\mathbf{v}_i\|}\mathbf{v}_i \qquad (1 \le i \le m),$$

then $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is an orthonormal basis for W.

We now summarize the Gram–Schmidt process.

The Gram–Schmidt process for computing an orthonormal basis $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ for a nonzero subspace W of R^n with basis $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ is as follows.

Step 1. Let $\mathbf{v}_1 = \mathbf{u}_1$.

Step 2. Compute the vectors $\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m$, successively, one at a time, by the formula

$$\mathbf{v}_i = \mathbf{u}_i - \left(\frac{\mathbf{u}_i \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_i \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 - \cdots - \left(\frac{\mathbf{u}_i \cdot \mathbf{v}_{i-1}}{\mathbf{v}_{i-1} \cdot \mathbf{v}_{i-1}}\right)\mathbf{v}_{i-1}.$$

The set of vectors $T^* = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is an orthogonal set.

Step 3. Let

$$\mathbf{w}_i = \frac{1}{\|\mathbf{v}_i\|}\mathbf{v}_i \qquad (1 \le i \le m).$$

Then $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ is an orthonormal basis for W.

Remark It is not difficult to show that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n such that $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u} \cdot (c\mathbf{v}) = 0$ for any scalar c (Exercise T.7). This result can often be used to simplify hand computations in the Gram–Schmidt process. As soon as a vector $\mathbf{v}_i$ is computed in Step 2, multiply it by a proper scalar to clear any fractions that may be present. We shall use this approach in our computational work with the Gram–Schmidt process.

EXAMPLE 4 Consider the basis $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for R^3, where

$$\mathbf{u}_1 = (1, 1, 1), \quad \mathbf{u}_2 = (-1, 0, -1), \quad \text{and} \quad \mathbf{u}_3 = (-1, 2, 3).$$

Use the Gram–Schmidt process to transform S to an orthonormal basis for

Solution ***Step 1.*** Let $\mathbf{v}_1 = \mathbf{u}_1 = (1, 1, 1)$.

Step 2. We now compute $\mathbf{v}_2$ and $\mathbf{v}_3$:

$$\mathbf{v}_2 = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = (-1, 0, -1) - \left(\tfrac{-2}{3}\right)(1, 1, 1) = \left(-\tfrac{1}{3}, \tfrac{2}{3}, \ldots\right.$$

Multiplying $\mathbf{v}_2$ by 3 to clear fractions, we obtain $(-1, 2, -1)$, which we now use as $\mathbf{v}_2$. Then

$$\mathbf{v}_3 = \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2$$
$$= (-1, 2, 3) - \tfrac{4}{3}(1, 1, 1) - \tfrac{2}{6}(-1, 2, -1) = (-2, 0, 2).$$

Thus

$$T^* = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1, 1, 1), (-1, 2, -1), (-2, 0, 2)\}$$

is an orthogonal basis for R^3.

Step 3. Let

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{3}}(1, 1, 1)$$
$$\mathbf{w}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \frac{1}{\sqrt{6}}(-1, 2, -1)$$
$$\mathbf{w}_3 = \frac{1}{\|\mathbf{v}_3\|}\mathbf{v}_3 = \frac{1}{\sqrt{8}}(-2, 0, 2) = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right).$$

Then

$$T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$$
$$= \left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right\}$$

is an orthonormal basis for R^3. ■

EXAMPLE 5 Let W be the subspace of R^4 with basis $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = (1, -2, 0, 1), \quad \mathbf{u}_2 = (-1, 0, 0, -1), \quad \text{and} \quad \mathbf{u}_3 = (1, 1, 0, 0).$$

Use the Gram–Schmidt process to transform S to an orthonormal basis for W.

Solution *Step 1.* Let $\mathbf{v}_1 = \mathbf{u}_1 = (1, -2, 0, 1)$.

Step 2. We now compute $\mathbf{v}_2$ and $\mathbf{v}_3$:

$$\mathbf{v}_2 = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = (-1, 0, 0, -1) - \left(\tfrac{-2}{6}\right)(1, -2, 0, 1)$$
$$= \left(-\tfrac{2}{3}, -\tfrac{2}{3}, 0, -\tfrac{2}{3}\right).$$

Multiplying $\mathbf{v}_2$ by 3 to clear fractions, we obtain $(-2, -2, 0, -2)$, which we now use as $\mathbf{v}_2$. Then

$$\mathbf{v}_3 = \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2$$
$$= (1, 1, 0, 0) - \left(\tfrac{-1}{6}\right)(1, -2, 0, 1) - \left(\tfrac{-4}{12}\right)(-2, -2, 0, -2)$$
$$= \left(\tfrac{1}{2}, 0, 0, -\tfrac{1}{2}\right).$$

Multiplying $\mathbf{v}_3$ by 2 to clear fractions, we obtain $(1, 0, 0, -1)$, which we now use as $\mathbf{v}_3$. Thus

$$T^* = \{(1, -2, 0, 1), (-2, -2, 0, -2), (1, 0, 0, -1)\}$$

is an orthogonal basis for W.

Step 3. Let

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{6}}(1, -2, 0, 1)$$

$$\mathbf{w}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \frac{1}{\sqrt{12}}(-2, -2, 0, -2) = \frac{1}{\sqrt{3}}(-1, -1, 0, -$$

$$\mathbf{w}_3 = \frac{1}{\|\mathbf{v}_3\|}\mathbf{v}_3 = \frac{1}{\sqrt{2}}(1, 0, 0, -1).$$

Then

$$T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$$
$$= \left\{\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}\right), \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{2}}, 0, 0, \right.\right.$$

is an orthonormal basis for W.

Remarks

1. In solving Example 5, as soon as a vector is computed we multipl an appropriate scalar to eliminate any fractions that may be prese optional step results in simpler computations when working by this approach is taken, the resulting basis, while orthonormal, fer from the orthonormal basis obtained by not clearing fraction computer implementations of the Gram–Schmidt process, includi developed with MATLAB, do not clear fractions.
2. We make one final observation with regard to the Gram–Schmidt In our proof of Theorem 6.18 we first obtained an orthogonal and then normalized all the vectors in T^* to obtain the orthonorm T. Of course, an alternative course of action is to normalize eac as soon as we produce it.

6.8 Exercises

1. Which of the following are orthogonal sets of vectors?
   (a) $\{(1, -1, 2), (0, 2, -1), (-1, 1, 1)\}$.
   (b) $\{(1, 2, -1, 1), (0, -1, -2, 0), (1, 0, 0, -1)\}$.
   (c) $\{(0, 1, 0, -1), (1, 0, 1, 1), (-1, 1, -1, 2)\}$.
2. Which of the following are orthonormal sets of vectors?
   (a) $\left\{\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\right\}$.
   (b) $\left\{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), (0, 1, 0)\right\}$.
   (c) $\{(0, 2, 2, 1), (1, 1, -2, 2), (0, -2, 1, 2)\}$.

In Exercises 3 and 4, let $V = R^3$.

3. Let $\mathbf{u} = (1, 1, -2)$ and $\mathbf{v} = (a, -1, 2)$. For what values of a are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?
4. Let $\mathbf{u} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, and $\mathbf{v} = \left(a, \frac{1}{\sqrt{2}}, -b\right)$. For what values of a and b is $\{\mathbf{u}, \mathbf{v}\}$ an orthonormal set?
5. Use the Gram–Schmidt process to find an ortho basis for the subspace of R^3 with basis $\{(1, -1, 0), (2, 0, 1)\}$.
6. Use the Gram–Schmidt process to find an ortho basis for the subspace of R^3 with basis $\{(1, 0, 2), (-1, 1, 0)\}$.
7. Use the Gram–Schmidt process to find an ortho basis for the subspace of R^4 with basis $\{(1, -1, 0, 1), (2, 0, 0, -1), (0, 0, 1, 0)\}$.
8. Use the Gram–Schmidt process to find an ortho basis for the subspace of R^4 with basis $\{(1, 1, -1, 0), (0, 2, 0, 1), (-1, 0, 0, 1)\}$.
9. Use the Gram–Schmidt process to transform the $\{(1, 2), (-3, 4)\}$ for R^2 into (a) an orthogonal b (b) an orthonormal basis.

...Gram–Schmidt process to transform the ...{(1, 1, 1), (0, 1, 1), (1, 2, 3)} for R^3 into an ...normal basis for R^3.
...Theorem 6.17 to write (2, 3, 1) as a linear ...bination of the basis obtained in part (a).

...orthonormal basis for R^3 containing the vectors ...) and $(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$.

...Gram–Schmidt process to construct an ...mal basis for the subspace W of R^3 *spanned* by ...(2, 2, 2), (0, 0, 1), (1, 2, 3)}.

...Gram–Schmidt process to construct an ...mal basis for the subspace W of R^4 *spanned* by ...0), (2, −1, 0, 1), (3, −3, 0, −2), ...0, −3)}.

...orthonormal basis for the subspace of R^3 ...g of all vectors of the form $(a, a + b, b)$.

...orthonormal basis for the subspace of R^4 ...g of all vectors of the form

$$(a, a + b, c, b + c).$$

...orthonormal basis for the subspace of R^3 ...g of all vectors (a, b, c) such that

$$a + b + c = 0.$$

...orthonormal basis for the subspace of R^4 ...g of all vectors (a, b, c, d) such that

$$a - b - 2c + d = 0.$$

18. Find an orthonormal basis for the solution space of the homogeneous system

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 2x_1 + x_2 + 2x_3 &= 0. \end{aligned}$$

19. Find an orthonormal basis for the solution space of the homogeneous system

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 3 \\ 1 & 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

20. Consider the orthonormal basis

$$S = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$$

for R^2. Using Theorem 6.17 write the vector (2, 3) as a linear combination of the vectors in S.

21. Consider the orthonormal basis

$$S = \left\{\left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right), \left(-\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right), (0, 1, 0)\right\}$$

for R^3. Using Theorem 6.17 write the vector (2, −3, 1) as a linear combination of the vectors in S.

...ical Exercises

...that the natural basis for R^n is an orthonormal ...

...Corollary 6.6.

...that an orthonormal set of n vectors in R^n is a ...for R^n.

...Prove Theorem 6.17.

...Prove Corollary 6.7.

...$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in R^n. Show that if $\mathbf{u}$ ...orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, then $\mathbf{u}$ is orthogonal to ...vector in span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

...be a fixed vector in R^n. Show that the set of all ...in R^n that are orthogonal to $\mathbf{u}$ is a subspace ...

...$\mathbf{u}$ and $\mathbf{v}$ be vectors in R^n. Show that if $\mathbf{u} \cdot \mathbf{v} = 0$, ...$\mathbf{u} \cdot (c\mathbf{v}) = 0$ for any scalar c.

T.8. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal set in R^n. Let the matrix A be given by $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$. Show that A is nonsingular and compute its inverse. Give three different examples of such a matrix in R^2 or R^3.

T.9. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set in R^n. Let A be the matrix whose jth column is $\mathbf{v}_j$, $j = 1, 2, \ldots, n$. Prove or disprove: A is nonsingular.

T.10. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be an orthonormal basis for a subspace W of R^n, where $n > k$. Discuss how to construct an orthonormal basis for V that includes S.

T.11. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for R^n, $S = \text{span}\ \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, and $T = \text{span}\ \{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$. For any $\mathbf{x}$ in S and any $\mathbf{y}$ in T, show that $\mathbf{x} \cdot \mathbf{y} = 0$.

...Exercises

...Schmidt process takes a basis S for a subspace W ...produces an orthonormal basis T for W. The ...to produce the orthonormal basis T that is given ...m is implemented in MATLAB *in routine* ...Type **help gschmidt** *for directions.*

ML.1. Use **gschmidt** to produce an orthonormal basis for R^3 from the basis

$$S = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

Your answer will be in decimal form; rewrite it in terms of $\sqrt{2}$.

ML.2. Use **gschmidt** to produce an orthonormal basis for R^4 from the basis $S = \{(1, 0, 1, 1), (1, 2, 1, 3), (0, 2, 1, 1), (0, 1, 0, 0)\}$.

ML.3. In R^3, $S = \{(0, -1, 1), (0, 1, 1), (1, 1, 1)\}$ is a basis. Find an orthonormal basis T from S and then find $[\mathbf{v}]_T$ for each of the following vectors.

(a) $\mathbf{v} = (1, 2, 0)$. (b) $\mathbf{v} = (1, 1, 1)$.

(c) $\mathbf{v} = (-1, 0, 1)$.

ML.4. Find an orthonormal basis for the subspace of consisting of all vectors of the form

$$(a, 0, a + b, b + c),$$

where a, b, and c are any real numbers.

6.9 ORTHOGONAL COMPLEMENTS

Let W_1 and W_2 be subspaces of a vector space V. Let $W_1 + W_2$ be of all vectors $\mathbf{v}$ in V such that $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is in W_1 and W_2. In Exercise T.10 in Section 6.2, we asked you to show that $W_1 +$ subspace of V. In Exercise T.11 in Section 6.2, we asked you to show $V = W_1 + W_2$ and $W_1 \cap W_2 = \{\mathbf{0}\}$, then V is the direct sum of W_1 and we write $V = W_1 \oplus W_2$. Moreover, in this case every vector in V uniquely written as $\mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is in W_1 and $\mathbf{w}_2$ is in W_2. In this we show that if W is a subspace of R^n, then R^n can be written as a di of W and another subspace of R^n. This subspace will be used to e basic relationship between four vector spaces associated with a matrix

DEFINITION Let W be a subspace of R^n. A vector $\mathbf{u}$ in R^n is said to be **orthogo** if it is orthogonal to every vector in W. The set of all vectors in R^n orthogonal to all the vectors in W is called the **orthogonal compleme** in R^n and is denoted by $W^\perp$ (read "W perp").

EXAMPLE 1 Let W be the subspace of R^3 consisting of all multiples of the vector

$$\mathbf{w} = (2, -3, 4).$$

Thus $W = \text{span}\ \{\mathbf{w}\}$, so W is a one-dimensional subspace of W. Th $\mathbf{u}$ in R^3 belongs to $W^\perp$ if and only if $\mathbf{u}$ is orthogonal to $c\mathbf{w}$, for any can be shown that $W^\perp$ is the plane with normal $\mathbf{w}$.

Observe that if W is a subspace of R^n, then the zero vector alw to $W^\perp$ (Exercise T.1). Moreover, the orthogonal complement of R^n subspace and the orthogonal complement of the zero subspace is (Exercise T.2).

THEOREM 6.19 *Let W be a subspace of R^n. Then*

(a) *$W^\perp$ is a subspace of R^n.*

(b) *$W \cap W^\perp = \{\mathbf{0}\}$.*

Proof (a) Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be in $W^\perp$. Then $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal to ea in W. We now have

$$(\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{w} = \mathbf{u}_1 \cdot \mathbf{w} + \mathbf{u}_2 \cdot \mathbf{w} = 0 + 0 = 0,$$

so $\mathbf{u}_1 + \mathbf{u}_2$ is in $W^\perp$. Also, let $\mathbf{u}$ be in $W^\perp$ and let c be a real s for any vector $\mathbf{w}$ in W, we have

$$(c\mathbf{u}) \cdot \mathbf{w} = c(\mathbf{u} \cdot \mathbf{w}) = c0 = 0,$$

so $c\mathbf{u}$ is in W, which implies that $W^\perp$ is closed under vector addition and scalar multiplication and hence is a subspace of R^n.

(b) Let $\mathbf{u}$ be a vector in $W \cap W^\perp$. Then $\mathbf{u}$ is in both W and $W^\perp$, so $\mathbf{u} \cdot \mathbf{u} = 0$. From Theorem 4.3 in Section 4.2 it follows that $\mathbf{u} = \mathbf{0}$. ■

In Exercise T.3 we ask you to show that if W is a subspace of R^n that is spanned by a set of vectors S, then a vector $\mathbf{u}$ in R^n belongs to $W^\perp$ if and only if $\mathbf{u}$ is orthogonal to every vector in S. This result can be helpful in finding $W^\perp$, as shown in the next example.

EXAMPLE 2 Let W be the subspace of R^4 with basis $\{\mathbf{w}_1, \mathbf{w}_2\}$, where

$$\mathbf{w}_1 = (1, 1, 0, 1) \quad \text{and} \quad \mathbf{w}_2 = (0, -1, 1, 1).$$

Find a basis for $W^\perp$.

Solution Let $\mathbf{u} = (a, b, c, d)$ be a vector in $W^\perp$. Then $\mathbf{u} \cdot \mathbf{w}_1 = 0$ and $\mathbf{u} \cdot \mathbf{w}_2 = 0$. Thus we have

$$\mathbf{u} \cdot \mathbf{w}_1 = a + b + d = 0 \quad \text{and} \quad \mathbf{u} \cdot \mathbf{w}_2 = -b + c + d = 0.$$

Solving the homogeneous system

$$\begin{aligned} a + b + d &= 0 \\ -b + c + d &= 0, \end{aligned}$$

we obtain (verify)

$$a = -r - 2s, \qquad b = r + s, \qquad c = r, \qquad d = s.$$

Then

$$\mathbf{u} = (-r - 2s, r + s, r, s) = r(-1, 1, 1, 0) + s(-2, 1, 0, 1).$$

Hence the vectors $(-1, 1, 1, 0)$ and $(-2, 1, 0, 1)$ span $W^\perp$. Since they are not multiples of each other, they are linearly independent and thus form a basis for $W^\perp$. ■

THEOREM 6.20 *Let W be a subspace of R^n. Then*

$$R^n = W \oplus W^\perp.$$

Proof Let $\dim W = m$. Then W has a basis consisting of m vectors. By the Gram–Schmidt process we can transform this basis to an orthonormal basis. Thus let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be an orthonormal basis for W. If $\mathbf{v}$ is a vector in R^n, let

$$\mathbf{w} = (\mathbf{v} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{v} \cdot \mathbf{w}_2)\mathbf{w}_2 + \cdots + (\mathbf{v} \cdot \mathbf{w}_m)\mathbf{w}_m$$

and

$$\mathbf{u} = \mathbf{v} - \mathbf{w}.$$

Since $\mathbf{w}$ is a linear combination of vectors in S, $\mathbf{w}$ belongs to W. We show that $\mathbf{u}$ lies in $W^\perp$ by showing that $\mathbf{u}$ is orthogonal to every vector in a basis for W. For each $\mathbf{w}_i$ in S, we have

$$\begin{aligned}\mathbf{u}\cdot\mathbf{w}_i &= (\mathbf{v}-\mathbf{w})\cdot\mathbf{w}_i = \mathbf{v}\cdot\mathbf{w}_i - \mathbf{w}\cdot\mathbf{w}_i\\ &= \mathbf{v}\cdot\mathbf{w}_i - [(\mathbf{v}\cdot\mathbf{w}_1)\mathbf{w}_1 + (\mathbf{v}\cdot\mathbf{w}_2)\mathbf{w}_2 + \cdots + (\mathbf{v}\cdot\mathbf{w}_m)\mathbf{w}_m]\cdot\mathbf{w}_i\\ &= \mathbf{v}\cdot\mathbf{w}_i - (\mathbf{v}\cdot\mathbf{w}_i)(\mathbf{w}_i\cdot\mathbf{w}_i)\\ &= 0\end{aligned}$$

since $\mathbf{w}_i\cdot\mathbf{w}_j = 0$ for $i \neq j$ and $\mathbf{w}_i\cdot\mathbf{w}_i = 1$, $1 \leq i \leq m$. Thus $\mathbf{u}$ is ortho[gonal] to every vector in W and so lies in $W^\perp$. Hence

$$\mathbf{v} = \mathbf{w} + \mathbf{u},$$

which means that $R^n = W + W^\perp$. From part (b) of Theorem 6.19, it fo[llows] that

$$R^n = W \oplus W^\perp.$$

Remark As pointed out at the beginning of this section, we also conclude th[at the] vectors $\mathbf{w}$ and $\mathbf{u}$ defined by Equations (1) and (2) are unique.

THEOREM 6.21 *If W is a subspace of R^n, then*

$$(W^\perp)^\perp = W.$$

Proof First, if $\mathbf{w}$ is any vector in W, then $\mathbf{w}$ is orthogonal to every vector $\mathbf{u}$ in [$W^\perp$], so $\mathbf{w}$ is in $(W^\perp)^\perp$. Hence W is a subspace of $(W^\perp)^\perp$. Conversely, let $\mathbf{v}$ [be an] arbitrary vector in $(W^\perp)^\perp$. Then, by Theorem 6.20, $\mathbf{v}$ can be written as

$$\mathbf{v} = \mathbf{w} + \mathbf{u},$$

where $\mathbf{w}$ is in W and $\mathbf{u}$ is in $W^\perp$. Since $\mathbf{u}$ is in $W^\perp$, it is orthogonal to $\mathbf{v}$ [and] $\mathbf{w}$. Thus

$$0 = \mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{w}+\mathbf{u}) = \mathbf{u}\cdot\mathbf{w} + \mathbf{u}\cdot\mathbf{u} = \mathbf{u}\cdot\mathbf{u}$$

or

$$\mathbf{u}\cdot\mathbf{u} = 0,$$

which implies that $\mathbf{u} = \mathbf{0}$. Then $\mathbf{v} = \mathbf{w}$, so $\mathbf{v}$ belongs to W. Hence it fo[llows] that $(W^\perp)^\perp = W$.

Remark Since W is the orthogonal complement of $W^\perp$ and $W^\perp$ is also the ortho[gonal] complement of W, we say that W and $W^\perp$ are **orthogonal complement[s]**.

Relations Among the Fundamental Vector Spaces Associated with a Matrix

If A is a given $m \times n$ matrix, we associate the following four fundam[ental] vector spaces with A: the null space of A, the row space of A, the null s[pace] of A^T, and the column space of A. The following theorem shows that pa[irs of] these four vector spaces are orthogonal complements.