PROOF

Suppose that $\{u_1, u_2, \ldots, u_n\}$ is the set of orthonormal vectors of V. In order to show their linear independence, take $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0v$$

Then $(\alpha_1 v_1 + \alpha_2 v_2 + \ldots, +\alpha_n v_n, v_i)$

$= \alpha_1(v_1, v_i) + \alpha_2(v_2, v_i) + \ldots + \alpha_n(v_n, v_i) = 0v$

By the definition of orthonormality of vectors $\{v_i\}$, the above equation reduces to $\alpha_i = 0$, for each i. Hence v_i's are linearly independent.

COR.

If $w \in V$ and $w = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$, then $(w_i, v_i) = \alpha_i$ and

$\underline{u} = w - (w, v_1)\, v_1 - (w, v_2)\, v_2 - \ldots - (w, v_n)\, v_n$ is orthogonal to each of the vectors of set $\{v_1, v_2, \ldots, v_n\}$, where

$$(u, v_i) = (w, v_i) - (w, v_1)(v_1, v_i) - (w, v_2)(v_2, v_i) - \ldots - (w, v_n)(v_n, v_i)$$

$$= (w, v_i) - (w, v_i) = 0, \text{ for each } i.$$

6.11.3 GRAM-SCHMIDT ORTHONORMALIZATION PROCESS

Now we tend to move to a well-known Gram-Schmidt orthonormalization process in order to get access to the orthonormal basis of a finite dimensional inner product space. The following result finalizes the attainment of the target.

6.11.4 THEOREM

Let V be a finite dimensional inner product space. Then V has an orthonormal set of basis vectors.

PROOF

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of an inner product space V. We form a new basis of V each vector of which is orthonormal in V and they are n in number. We have already ensured that they are all linearly independent in V. We take the following steps an completing the proof of the theorem.

We target at attaining n vectors $u_1, u_2, \ldots, u_n$ of V which are orthonormal in V. We suppose that $\left(\frac{w_i}{\|w_i\|}, \frac{w_i}{\|w_i\|}\right) = 1$ for each i and $(w_i, w_j) = 0$, for $i \neq j$.

(1) As a first step take $\frac{v_1}{\|v_1\|} = u_1 = w_1$, which is of length 1.

(2) Then form $0 \neq u_2 = v_2 - (v_2, w_1)\, w_1$ and note that u_2 is orthogonal to w_1 and take $\frac{u_2}{\| u_2 \|} = w_2$ as an orthonormal vector.

(3) The set $\{w_1, w_2\}$ is orthonormal independent set of V.

(4) We continue to take, $0 \neq u_3 = v_3 - (v_3, w_1)\, w_1 - (v_3, w_2)\, w_2$, which is orthogonal to w_1 and w_2 both.

(5) Take $w_3 = \frac{u_3}{\| u_3 \|}$ and the set $\{w_1, w_2, w_3\}$ is an orthonormal set of V.

(6) If we continue to process to exhaust i, we form, the set $\{w_1, w_2, \ldots, w_i\}$ orthonormal in V, where $0 \neq u_{i+1} = v_{i+1} - (v_{i+1}, w_1)\, w_1 - (v_{i+1}, w_2) w_2 \ldots - (w_{i+1}, w_i) w_i$ is orthogonal to each of $w_1, w_2, \ldots w_i$. By taking $w_{i+1} = \frac{u_{i+1}}{\| u_{i+1} \|}$,

(7) We continue upto $i = n$, the set $\{w_1, w_2, \ldots, w_n\}$ forms an orthonormal basis of V, which is the required target to reach at.

PROOF

The proof is exhibited in the following orthogogalization process:

This procedure makes it possible to obtain a set of orthogonal vectors from a given set of non-orthogonal but linearly independent vectors of a given space.

Let $\{u_1, u_2, \ldots, u_n\}$ be a set of L.I. vectors which will span a subspace of dimension n in a vector space of dimension $d \geq n$. If the vectors $u_i (i = 1, 2, 3, \ldots, n)$ do not form an orthogonal set then Schmidt's procedure is well known to obtain a set of orthogonal and hence the orthonormal vectors.

If $v_1, v_2, \ldots, v_n$ is the required set of orthogonal vectors then we proceed along the following steps:

1. $v_1 = u_1$
2. Let $v_2 = u_2 + a_{21} v_1$ be orthogonal to v_1 where the scalar a_{21} is determined from the condition of orthogonality of v_2 to v_1 i.e. $(v_1, v_2) = 0$.

Thus $(v_1, v_2) = 0 \Rightarrow (v_1, u_2) + a_{21}(v_1, v_2) = 0$ and it gives that

that $a_{21} = -\dfrac{(v_1, u_2)}{(v_1, v_1)}$ and hence $v_2 = u_2 - \dfrac{(v_1, u_2)}{(v_1, v_1)} v_1$ is orthogonal to v_1.

3. Let $v_3 = u_3 + a_{32}v_2 + a_{31}v_1$, where scalars a_{32} and a_{31} are determined from the conditions of orthognality of v_3 to v_2 and v_1 as well.

$$(v_1, v_3) = 0 \Rightarrow (u_3, v_1) + a_{32}(v_2, v_1) + a_{31}(v_1, v_1) = 0 \qquad \cdots\cdots\cdot \text{(i)}$$

and $(v_2, v_3) = 0 \Rightarrow (u_3, v_2) + a_{32}(v_2, v_2) + a_{31}(v_2, v_1) = 0 \qquad \cdots\cdots\cdot$ (ii)

Simultaneous solution of (i) and (ii) gives that

$$a_{31} = \frac{(v_1, u_3)}{(v_1, v_1)} \text{ and } a_{32} = \frac{(v_2, u_3)}{(v_2, v_2)}$$

$$v_4 = u_4 = \frac{(v_3, u_4)}{(v_3, v_3)} v_3 - \frac{(v_2, u_4)}{(v_2, v_2)} v_2 - \frac{(v_1, u_4)}{(v_1, v_1)} v_1$$

$$= (1, 1, 1, -5) - \frac{(15)}{15}(1, 1, 3, -2) - \frac{0}{2}(1, -1, 0, 0) - \frac{(-3)}{3}(1, 1, 0, 1)$$

$$= (1, 1, 1, -5) - (1, 1, 3, -2) + (1, 1, 0, 1)$$

$$= (1, 1, -2, -2)$$

The desired orthogonal set is therefore,

$= \{v_1 = (1, 1, 0, 1), v_2 = (1, -1, 0, 0), v_3 = (1, 1, 3, -2); v_4 = (1, 1, -2, -2)\}$

The corresponding orthonormal set $\{x_i\}$, $i = 1, 2, 3, 4$, is

$$= \left\{x_1 = \frac{1}{\sqrt{3}}(1,1,0,1), x_2 = \frac{1}{\sqrt{2}}(1,-1,0,1), x_3 = \frac{1}{\sqrt{15}}(1,1,3,-2), x_4 = \frac{1}{\sqrt{10}}(1,1,-2,-2)\right\}$$

REMARKS

1. The Schmidt's procedure helps to study vector space configuration with reference to the mutually orthogonal system of coordinate axes.

2. It is important to note that linearly independent coordinate reference system of a vector space may not always be an orthogonal system.

3. The standard basis of vector space $\Re^n$ $(n \geq 2)$ is an orthonormal basis.

4. There are as many orthogonal bases of a finite dimensional vector space as many bases of a vector space.

5. The set of v_i's is not orthogonal to the set of u_i's i.e. $(v_i, v_j) = 0, i \neq j$.

$$v_3 = u_3 - \frac{(v_2, u_3)}{(v_2, v_2)} v_2 - \frac{(v_2, u_3)}{(v_1, v_1)} v_1$$

We form v_1, v_2 and v_3 as orthogonal vectors. By continuing the procedure to ith step we take

$$v_i = u_i - \frac{(v_{i-1}, v_i)}{(v_{i-1}, v_{i-1})} v_{i-1} - \frac{(v_{i-2}, u_i)}{(v_{i-2}, v_{i-2})} v_{i-2} - \cdots - \frac{(v_2, u_i)}{(v_2, v_2)} v_2 - \frac{(v_1, u_i)}{(v_1, v_1)} v_1$$

for $i = 1, 2, 3, \ldots, n$

The vectors $\{v_1, v_2, \ldots, v_n\}$ are L.I. and orthogonal to each other.

Each vector of $\{v_i\}$ is normalized to form the orthonormal set $\{x_i\}$, where

$$x_i = \frac{v_i}{\sqrt{\|v_i\|}}, \quad i = 1, 2, \ldots, n$$

6.11.5 EXAMPLE

Obtain a set of four orthonormial vectors by the Schmidt's procedure from the vectors $u_1 = (1, 1, 0, 1)$, $u_2 = (2, 0, 0, 1)$, $u_3 = (0, 2, 3, -2)$ and $u_4 = (1, 1, 1, -5)$.

SOLUTION

1. Let $v_1 = u_1 = (1, 1, 0, 1)$

2. $v_2 = u_2 - \frac{(v_1, u_2)}{(v_1, v_1)} v_1 = (2, 0, 0, 1) - \frac{3}{3}(1, 1, 0, 1)$

$= (2, 0, 0, 1) - (1, 1, 0, 1) = (1, -1, 0, 0)$

3. $v_3 = u_3 - \frac{(v_2, u_3)}{(v_2, v_2)} v_2 - \frac{(v_1, u_3)}{(v_1, v_1)} v_1$

$= (0, 2, 3, -2) - \left(\frac{-2}{2}\right)(1, -1, 0, 1) - \frac{0}{3}(1, 1, 0, 1)$

$= (0, 2, 3, -2) + (1, -1, 0, 0)$

$= (1, 1, 3, -2)$