CHAPTER 8

EIGENVALUES, EIGENVECTORS, AND DIAGONALIZATION

IGENVALUES AND EIGENVECTORS

In this chapter every matrix considered is a square matrix. Let A be an $n \times n$ matrix. Then, as we have seen in Section 4.3, the function $L: R^n \rightarrow R^n$ defined by $L(\mathbf{x}) = A\mathbf{x}$, for $\mathbf{x}$ in R^n, is a linear transformation. A question of considerable importance in a great many applied problems is the determination of vectors $\mathbf{x}$, if there are any, such that $\mathbf{x}$ and $A\mathbf{x}$ are parallel. Such questions arise in all applications involving vibrations; they arise in aerodynamics, elasticity, nuclear physics, mechanics, chemical engineering, biology, differential equations, and so on. In this section we shall formulate this problem precisely; we also define some pertinent terminology. In the next section we solve this problem for symmetric matrices and briefly discuss the situation in the general case.

DEFINITION Let A be an $n \times n$ matrix. The real number λ is called an **eigenvalue** of A if there exists a nonzero vector $\mathbf{x}$ in R^n such that

$$A\mathbf{x} = \lambda\mathbf{x}. \tag{1}$$

Every nonzero vector $\mathbf{x}$ satisfying (1) is called an **eigenvector of** A **associated with the eigenvalue** λ. We might mention that the word "eigenvalue" is a hybrid one ("eigen" in German means "proper"). Eigenvalues are also called **proper values, characteristic values**, and **latent values**; and eigenvectors are also called **proper vectors**, and so on, accordingly.

Note that $\mathbf{x} = \mathbf{0}$ always satisfies (1), but $\mathbf{0}$ is not an eigenvector, since we insist that an eigenvector be a nonzero vector.

In some applications one encounters matrices with complex entries and vector spaces with scalars that are complex numbers (see Sections A.1 and A.2, respectively). In such a setting the preceding definition of eigenvalue is modified so that an eigenvalue can be a real *or* a complex number. An introduction to this approach, a treatment usually presented in more advanced books, is given in Section A.2. Throughout the rest of this book, unless stated otherwise, we require that an eigenvalue be a real number.

EXAMPLE 1 If A is the identity matrix I_n, then the only eigenvalue is $\lambda = 1$; every vector in R^n is an eigenvector of A associated with the eigenvalue $\lambda =$

$$I_n\mathbf{x} = 1\mathbf{x}.$$

EXAMPLE 2 Let

$$A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}.$$

Then

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so that

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector of A associated with the eigenvalue $\lambda_1 = \frac{1}{2}$. Also,

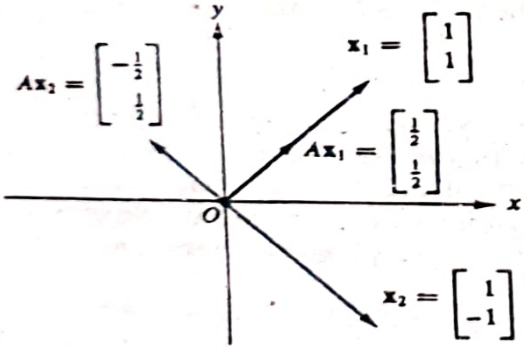
$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so that

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

is an eigenvector of A associated with the eigenvalue $\lambda_2 = -\frac{1}{2}$. Fig shows that $\mathbf{x}_1$ and $A\mathbf{x}_1$ are parallel, and $\mathbf{x}_2$ and $A\mathbf{x}_2$ are parallel also illustrates the fact that if $\mathbf{x}$ is an eigenvector of A, then $\mathbf{x}$ and $A\mathbf{x}$ are pa

Figure 8.1 ►



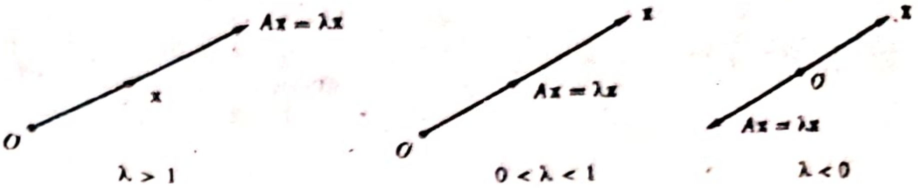
Let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. I 8.2 we show $\mathbf{x}$ and $A\mathbf{x}$ for the cases $\lambda > 1$, $0 < \lambda < 1$, and $\lambda < 0$.

An eigenvalue λ of A can have associated with it many differen vectors. In fact, if $\mathbf{x}$ is an eigenvector of A associated with λ (i.e., A and r is any nonzero real number, then

$$A(r\mathbf{x}) = r(A\mathbf{x}) = r(\lambda\mathbf{x}) = \lambda(r\mathbf{x}).$$

Thus $r\mathbf{x}$ is also an eigenvector of A associated with λ.

Figure 8.2 ►



EXAMPLE 3 Let

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

so that $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue $\lambda_1 = 0$. Also,

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is an eigenvector of A associated with the eigenvalue $\lambda_2 = 1$ (verify). ■

Example 3 points out the fact that although the zero vector, by definition, cannot be an eigenvector, the number zero can be an eigenvalue.

mputing Eigenvalues and Eigenvectors

Thus far we have found the eigenvalues and associated eigenvectors of a given matrix by inspection, geometric arguments, or very simple algebraic approaches. In the following example, we compute the eigenvalues and associated eigenvectors of a matrix by a somewhat more systematic method.

EXAMPLE 4 Let

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

We wish to find the eigenvalues of A and their associated eigenvectors. Thus we wish to find all real numbers λ and all nonzero vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

satisfying (1), that is,

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2}$$

Equation (2) becomes

$$\begin{aligned} x_1 + x_2 &= \lambda x_1 \\ -2x_1 + 4x_2 &= \lambda x_2, \end{aligned}$$

or

$$\begin{aligned} (\lambda - 1)x_1 - x_2 &= 0 \\ 2x_1 + (\lambda - 4)x_2 &= 0. \end{aligned}$$

Equation (3) is a homogeneous system of two equations in two unkn
From Corollary 3.4 of Section 3.2, it follows that the homogeneous sy
in (3) has a nontrivial solution if and only if the determinant of its coeff
matrix is zero; that is, if and only if

$$\begin{vmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 4 \end{vmatrix} = 0.$$

This means that

$$(\lambda - 1)(\lambda - 4) + 2 = 0,$$

or

$$\lambda^2 - 5\lambda + 6 = 0 = (\lambda - 3)(\lambda - 2).$$

Hence

$$\lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 3$$

are the eigenvalues of A. To find all eigenvectors of A associated with λ_1
we form the linear system

$$A\mathbf{x} = 2\mathbf{x},$$

or

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This gives

$$\begin{aligned} x_1 + x_2 &= 2x_1 \\ -2x_1 + 4x_2 &= 2x_2 \end{aligned}$$

or

$$\begin{aligned} (2 - 1)x_1 - x_2 &= 0 \\ 2x_1 + (2 - 4)x_2 &= 0 \end{aligned}$$

or

$$\begin{aligned} x_1 - x_2 &= 0 \\ 2x_1 - 2x_2 &= 0. \end{aligned}$$

Note that we could have obtained this last homogeneous system by m
substituting $\lambda = 2$ in (3). All solutions to this last system are given by

$$\begin{aligned} x_1 &= x_2 \\ x_2 &= \text{any real number } r. \end{aligned}$$

Hence all eigenvectors associated with the eigenvalue $\lambda_1 = 2$ are give
$\begin{bmatrix} r \\ r \end{bmatrix}$, r any nonzero real number. In particular,

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector associated with $\lambda_1 = 2$. Similarly, for $\lambda_2 = 3$ we o
from (3),

$$\begin{aligned} (3 - 1)x_1 - x_2 &= 0 \\ 2x_1 + (3 - 4)x_2 &= 0 \end{aligned}$$

or

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ 2x_1 - x_2 &= 0. \end{aligned}$$

All solutions to this last homogeneous system are given by

$$\begin{aligned} x_1 &= \tfrac{1}{2}x_2 \\ x_2 &= \text{any real number } r. \end{aligned}$$

Hence all eigenvectors associated with the eigenvalue $\lambda_2 = 3$ are given by $\begin{bmatrix} \frac{1}{2}r \\ r \end{bmatrix}$, r any nonzero real number. In particular,

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is an eigenvector associated with the eigenvalue $\lambda_2 = 3$. ■

In Examples 1, 2, and 3 we found eigenvalues and eigenvectors by inspection, whereas in Example 4 we proceeded in a more systematic fashion. We use the procedure of Example 4 as our standard method, as follows.

DEFINITION Let $A = \left[a_{ij}\right]$ be an $n \times n$ matrix. The determinant

$$f(\lambda) = \det(\lambda I_n - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix} \tag{4}$$

is called the **characteristic polynomial of** A. The equation

$$f(\lambda) = \det(\lambda I_n - A) = 0$$

is called the **characteristic equation of** A.

EXAMPLE 5 Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}.$$

The characteristic polynomial of A is (verify)

$$\begin{aligned} f(\lambda) = \det(\lambda I_3 - A) &= \begin{vmatrix} \lambda - 1 & -2 & 1 \\ -1 & \lambda - 0 & -1 \\ -4 & 4 & \lambda - 5 \end{vmatrix} \\ &= \lambda^3 - 6\lambda^2 + 11\lambda - 6. \end{aligned}$$

■

may be complex numbers. The expression involving λ^n in the characteri polynomial of A comes from the product

$$(\lambda - a_{11})(\lambda - a_{22}) \cdots (\lambda - a_{nn}),$$

so the coefficient of λ^n is 1. We can then write

$$f(\lambda) = \det(\lambda I_n - A) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n.$$

If we let $\lambda = 0$ in $\det(\lambda I_n - A)$ as well as in the expression on the right, the we get $\det(-A) = c_n$, which shows that the constant term c_n is $(-1)^n \det(A$ This result can be used to establish the following theorem.

THEOREM 8.1 *An $n \times n$ matrix A is singular if and only if 0 is an eigenvalue of A.*

Proof Exercise T.7(b).

We now extend our List of Nonsingular Equivalences.

List of Nonsingular Equivalences

The following statements are equivalent for an $n \times n$ matrix A.

1. A is nonsingular.
2. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
3. A is row equivalent to I_n.
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.
5. $\det(A) \neq 0$.
6. A has rank n.
7. A has nullity 0.
8. The rows of A form a linearly independent set of n vectors in R^n.
9. The columns of A form a linearly independent set of n vectors in R^n.
10. Zero is *not* an eigenvalue of A.

We now connect the characteristic polynomial of a matrix with its eige values in the following theorem.

THEOREM 8.2 *The eigenvalues of A are the real roots of the characteristic polynomial of*

Proof Let λ be an eigenvalue of A with associated eigenvector $\mathbf{x}$. Then $A\mathbf{x} = \lambda\mathbf{x}$ which can be rewritten as

$$A\mathbf{x} = (\lambda I_n)\mathbf{x}$$

or

$$(\lambda I_n - A)\mathbf{x} = \mathbf{0}, \tag{5}$$

a homogeneous system of n equations in n unknowns. This system has a non trivial solution if and only if the determinant of its coefficient matrix vanishe (Corollary 3.4, Section 3.2), that is, if and only if $\det(\lambda I_n - A) = 0$.

Conversely, if λ is a real root of the characteristic polynomial of A, the $\det(\lambda I_n - A) = 0$, so the homogeneous system (5) has a nontrivial solution Hence λ is an eigenvalue of A.

Thus, to find the eigenvalues of a given matrix A, we must find the real roots of its characteristic polynomial $f(\lambda)$. There are many methods for finding approximations to the roots of a polynomial, some of them more effective than others; indeed, many computer programs are available to find the roots of a polynomial. Two results that are sometimes useful in this connection are as follows: the product of all the roots of the polynomial

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

is $(-1)^n a_n$, and if $a_1, a_2, \ldots, a_n$ are integers, then $f(\lambda)$ cannot have a rational root that is not already an integer. Thus, as possible rational roots of $f(\lambda)$, one need only try the integer factors of a_n. Of course, $f(\lambda)$ might well have irrational roots. However, to minimize the computational effort and as a convenience to the reader, all the characteristic polynomials to be considered in the rest of this chapter have only integer roots, and each of these roots is a factor of the constant term of the characteristic polynomial of A. The corresponding eigenvectors are obtained by substituting the value of λ in Equation (5) and solving the resulting homogeneous system. The solution to this type of problem has already been studied in Section 6.5.

EXAMPLE 6 Consider the matrix of Example 5. The characteristic polynomial is

$$f(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6.$$

The possible integer roots of $f(\lambda)$ are ± 1, ± 2, ± 3, and ± 6. By substituting these values in $f(\lambda)$, we find that $f(1) = 0$, so $\lambda = 1$ is a root of $f(\lambda)$. Hence $(\lambda - 1)$ is a factor of $f(\lambda)$. Dividing $f(\lambda)$ by $(\lambda - 1)$, we obtain (verify)

$$f(\lambda) = (\lambda - 1)(\lambda^2 - 5\lambda + 6).$$

Factoring $\lambda^2 - 5\lambda + 6$, we have

$$f(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda - 3).$$

The eigenvalues of A are then

$$\lambda_1 = 1, \qquad \lambda_2 = 2, \qquad \lambda_3 = 3.$$

To find an eigenvector $\mathbf{x}_1$ associated with $\lambda_1 = 1$, we form the system

$$(1I_3 - A)\mathbf{x} = \mathbf{0},$$

$$\begin{bmatrix} 1-1 & -2 & 1 \\ -1 & 1 & -1 \\ -4 & 4 & 1-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & -2 & 1 \\ -1 & 1 & -1 \\ -4 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

A solution is

$$\begin{bmatrix} -\frac{1}{2}r \\ \frac{1}{2}r \\ r \end{bmatrix}$$

for any real number r. Thus, for $r = 2$,

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

is an eigenvector of A associated with $\lambda_1 = 1$.

To find an eigenvector $\mathbf{x}_2$ associated with $\lambda_2 = 2$, we form the sy

$$(2I_3 - A)\mathbf{x} = \mathbf{0},$$

that is,

$$\begin{bmatrix} 2-1 & -2 & 1 \\ -1 & 2 & -1 \\ -4 & 4 & 2-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ -4 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

A solution is

$$\begin{bmatrix} -\frac{1}{2}r \\ \frac{1}{4}r \\ r \end{bmatrix}$$

for any real number r. Thus, for $r = 4$,

$$\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

is an eigenvector of A associated with $\lambda_2 = 2$.

To find an eigenvector $\mathbf{x}_3$ associated with $\lambda_3 = 3$, we form the sy

$$(3I_3 - A)\mathbf{x} = \mathbf{0},$$

and find that a solution is (verify)

$$\begin{bmatrix} -\frac{1}{4}r \\ \frac{1}{4}r \\ r \end{bmatrix}$$

for any real number r. Thus, for $r = 4$,

$$\mathbf{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

is an eigenvector of A associated with $\lambda_3 = 3$.

EXAMPLE 7 Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then the characteristic polynomial of A is

$$f(\lambda) = \lambda^2 + 1,$$

which has no real roots. (The roots are $\lambda_1 = i$ and $\lambda_2 = -i$.) Thus
that A has no eigenvalues.

The procedure for finding the eigenvalues and associated eigenvectors of a matrix is as follows.

Step 1. Determine the real roots of the characteristic polynomial $p(\lambda) = \det(\lambda I_n - A)$. These are the eigenvalues of A.

Step 2. For each eigenvalue λ, find all the nontrivial solutions to the homogeneous system $(\lambda I_n - A)\mathbf{x} = \mathbf{0}$. These are the eigenvectors of A associated with the eigenvalue λ.

Of course, the characteristic polynomial of a matrix may have some complex roots and it may even have no real roots (see Example 7). However, in the important case of symmetric matrices, all the roots of the characteristic polynomial are real. We shall prove this in Section 8.3 (Theorem 8.6).

Eigenvalues and eigenvectors satisfy many important and interesting properties. For example, if A is an upper (lower) triangular matrix, or a diagonal matrix, then the eigenvalues of A are the elements on the main diagonal of A (Exercise T.3). The set S consisting of all eigenvectors of A associated with λ_j as well as the zero vector is a subspace of R^n (Exercise T.1) called the **eigenspace associated with** λ_j. Other properties are developed in the exercises for this section.

It must be pointed out that the method for finding the eigenvalues of a linear transformation or matrix by obtaining the real roots of the characteristic polynomial is not practical for $n > 4$, since it involves evaluating a determinant. Efficient numerical methods for finding eigenvalues are studied in numerical analysis courses.

We now turn to examine briefly three applications of eigenvalues and eigenvectors. The first two of these applications have already been seen in this book; the third one is new. Chapter 9 is devoted entirely to a deeper study of several additional applications of eigenvalues and eigenvectors.

arkov Chains

We have already discussed Markov processes or chains in Sections 1.4 and 2.3. Let T be a regular transition matrix of a Markov process. In Theorem 2.5 we showed that as $n \to \infty$, T^n approaches a matrix A, all of whose columns are the identical vector $\mathbf{u}$. Also, Theorem 2.6 showed that $\mathbf{u}$ is a steady-state vector, which is the unique probability vector satisfying the matrix equation $T\mathbf{u} = \mathbf{u}$. This means that $\lambda = 1$ is an eigenvalue of T and $\mathbf{u}$ is an associated eigenvector. Finally, since the columns of A add up to 1, it follows from Exercise T.14 that $\lambda = 1$ is an eigenvalue of A.

near Economic Models

In Section 2.4 we discussed the Leontief closed model, consisting of a society made up of a farmer, a carpenter, and a tailor, where each person produces one unit of each commodity during the year. The exchange matrix A gives the portion of each commodity that is consumed by each individual durin the year. The problem facing the economic planner is that of determining prices p_1, p_2, and p_3 of the three commodities so that no one makes mo or loses money, that is, so that we have a state of equilibrium. Let $\mathbf{p}$ deno the price vector. Then the problem is that of finding a solution $\mathbf{p}$ to the lin system $A\mathbf{p} = \mathbf{p}$ whose components p_i will be nonnegative with at least

8.1 Exercises

1. Let $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$.

   (a) Verify that $\lambda_1 = 1$ is an eigenvalue of A and $\mathbf{x}_1 = \begin{bmatrix} r \\ 2r \end{bmatrix}$, $r \neq 0$, is an associated eigenvector.

   (b) Verify that $\lambda_1 = 4$ is an eigenvalue of A and $\mathbf{x}_2 = \begin{bmatrix} r \\ -r \end{bmatrix}$, $r \neq 0$, is an associated eigenvector.

2. Let $A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$.

   (a) Verify that $\lambda_1 = -1$ is an eigenvalue of A and $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an associated eigenvector.

   (b) Verify that $\lambda_2 = 2$ is an eigenvalue of A and $\mathbf{x}_2 = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$ is an associated eigenvector.

   (c) Verify that $\lambda_3 = 4$ is an eigenvalue of A and $\mathbf{x}_3 = \begin{bmatrix} 8 \\ 5 \\ 2 \end{bmatrix}$ is an associated eigenvector.

In Exercises 3 through 5, find the characteristic polynomial of each matrix.

3. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{bmatrix}$.

4. $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$.

5. $\begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

In Exercises 6 through 13, find the characteristic polynomial, eigenvalues, and eigenvectors of each matrix.

6. $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

7. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 3 & 2 & -2 \end{bmatrix}$.

8. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

9. $\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$.

10. $\begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$.

11. $\begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$.

12. $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 4 & 3 \end{bmatrix}$.

13. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

In Exercises 14 and 15, find bases for the eigenspaces (see Exercise T.1) associated with each eigenvalue.

14. $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

15. $\begin{bmatrix} 2 & 2 & 3 & 4 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

In Exercises 16–19, find a basis for the eigenspace (see Exercise T.1) associated with λ.

16. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\lambda = 1$.

17. $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, $\lambda = 2$.

18. $\begin{bmatrix} 3 & 0 & 0 \\ -2 & 3 & -2 \\ 2 & 0 & 5 \end{bmatrix}$, $\lambda = 3$.

19. $\begin{bmatrix} 4 & 2 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, $\lambda = 2$.

20. Let A be the matrix of Exercise 1. Find the eigenvalu and eigenvectors of A^2 and verify Exercise T.5.

21. Consider a living organism that can live to a maximu age of 2 years and whose Leslie matrix is

$$A = \begin{bmatrix} 0 & 0 & 8 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Find a stable age distribution.

22. Consider a living organism that can live to a maxim age of 2 years and whose Leslie matrix is

$$A = \begin{bmatrix} 0 & 4 & 0 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Find a stable age distribution.

Theoretical Exercises

T.1. Let λ_j be a particular eigenvalue of the $n \times n$ matrix A. Show that the subset S of R^n consisting of the zero vector and all eigenvectors of A associated with λ_j is a subspace of R^n, called the **eigenspace** associated with the eigenvalue λ_j.

T.2. In Exercise T.1 why do we have to include the zer vector in the subset S?

T.3. Show that if A is an upper (lower) triangular matri a diagonal matrix, then the eigenvalues of A are th elements on the main diagonal of A.

w that A and A^T have the same eigenvalues. What, ything, can we say about the associated envectors of A and A^T?

λ is an eigenvalue of A with associated eigenvector show that λ^k is an eigenvalue of $A^k = A \cdot A \cdots A$ factors) with associated eigenvector **x**, where k is a sitive integer.

An $n \times n$ matrix A is called **nilpotent** if $A^k = O$ for me positive integer k. Show that if A is nilpotent, hen the only eigenvalue of A is 0. (*Hint*: Use Exercise T.5.)

Let A be an $n \times n$ matrix.

(a) Show that $\det(A)$ is the product of all the roots of the characteristic polynomial of A..

(b) Show that A is singular if and only if 0 is an eigenvalue of A.

Let λ be an eigenvalue of the nonsingular matrix A with associated eigenvector **x**. Show that $1/\lambda$ is an eigenvalue of A^{-1} with associated eigenvector **x**.

Let A be any $n \times n$ real matrix.

(a) Show that the coefficient of λ^{n-1} in the characteristic polynomial of A is given by $-\text{Tr}(A)$, where $\text{Tr}(A)$ denotes the trace of A (see Supplementary Exercise T.1 in Chapter 1).

(b) Show that $\text{Tr}(A)$ is the sum of the eigenvalues of A.

(c) Show that the constant term of the characteristic polynomial of A is ± times the product of the eigenvalues of A.

(d) Show that $\det(A)$ is the product of the eigenvalues of A.

T.10. Let A be an $n \times n$ matrix with eigenvalues λ_1 and λ_2, where $\lambda_1 \neq \lambda_2$. Let S_1 and S_2 be the eigenspaces associated with λ_1 and λ_2, respectively. Explain why the zero vector is the only vector that is in both S_1 and S_2.

T.11. Let λ be an eigenvalue of A with associated eigenvector **x**. Show that $\lambda + r$ is an eigenvalue of $A + rI_n$ with associated eigenvector **x**. Thus, adding a scalar multiple of the identity matrix to A merely shifts the eigenvalues by the scalar multiple.

T.12. Let A be a square matrix.

(a) Suppose that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x} = \mathbf{u}$. Show that **u** is an eigenvector of A.

(b) Suppose that 0 is an eigenvalue of A and **v** is an associated eigenvector. Show that the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

T.13. Let A and B be $n \times n$ matrices such that $A\mathbf{x} = \lambda\mathbf{x}$ and $B\mathbf{x} = \mu\mathbf{x}$. Show that:

(a) $(A + B)\mathbf{x} = (\lambda + \mu)\mathbf{x}$.

(b) $(AB)\mathbf{x} = (\lambda\mu)\mathbf{x}$.

T.14. Show that if A is a matrix all of whose columns add up to 1, then $\lambda = 1$ is an eigenvalue of A. (*Hint*: Consider the product $A^T\mathbf{x}$, where **x** is a vector all of whose entries are 1 and use Exercise T.4.)

TLAB Exercises

AB has a pair of commands that can be used to find the teristic polynomial and eigenvalues of a matrix. nd **poly(A)** *gives the coefficients of the characteristic mial of matrix* A*, starting with the highest-degree If we set* $\mathbf{v} =$ **poly(A)** *and then use command* **roots(v)***, in the roots of the characteristic polynomial of* A*. rocess can also find complex eigenvalues, which are sed in Appendix A.2.*

Once we have an eigenvalue λ of A*, we can use* **rref** *or ln to find a corresponding eigenvector from the linear* $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

1. Find the characteristic polynomial of each of the following matrices using MATLAB.

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 2 & 1 \\ 0 & 4 & 2 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$.

ML.2. Use the **poly** and **roots** commands in MATLAB to find the eigenvalues of the following matrices:

(a) $A = \begin{bmatrix} 1 & -3 \\ 3 & -5 \end{bmatrix}$. (b) $A = \begin{bmatrix} 3 & -1 & 4 \\ -1 & 0 & 1 \\ 4 & 1 & 2 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$. (d) $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$.

ML.3. In each of the following cases, λ is an eigenvalue A. Use MATLAB to find a corresponding eigenvec

(a) $\lambda = 3$, $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$.

(b) $\lambda = -1$, $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & -1 \end{bmatrix}$.