$$L(A + B) \neq L(A) + L(B).$$

Use MATLAB to do the computations.

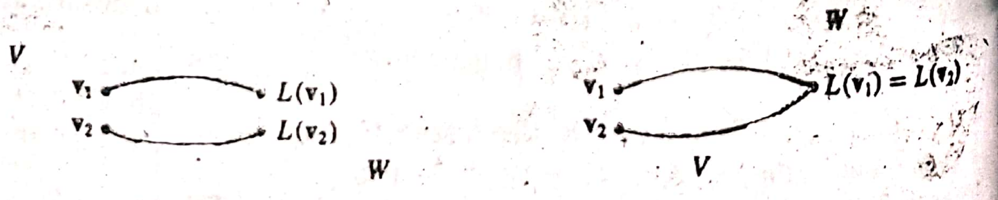
10.2 THE KERNEL AND RANGE OF A LINEAR TRANSFORMATION

In this section we study special types of linear transformations; we formulate the notions of one-to-one linear transformations and onto linear transformations. We also develop methods for determining when a linear transformation is one-to-one or onto.

DEFINITION A linear transformation $L: V \to W$ is said to be **one-to-one** if for all $\mathbf{v}_1, \mathbf{v}_2$ in V, $\mathbf{v}_1 \neq \mathbf{v}_2$ implies that $L(\mathbf{v}_1) \neq L(\mathbf{v}_2)$. An equivalent statement is that L is one-to-one if for all $\mathbf{v}_1, \mathbf{v}_2$ in V, $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies that $\mathbf{v}_1 = \mathbf{v}_2$.

This definition says that L is one-to-one if $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$ are distinct whenever $\mathbf{v}_1$ and $\mathbf{v}_2$ are distinct (Figure 10.1).

Figure 10.1 ▶



(a) L is one-to-one. (b) L is not one-to-one.

EXAMPLE 1 Let $L: R^2 \to R^2$ be defined by

$$L(x, y) = (x + y, x - y).$$

To determine whether L is one-to-one, we let

$$\mathbf{v}_1 = (a_1, a_2) \quad \text{and} \quad \mathbf{v}_2 = (b_1, b_2).$$

Then if

$$L(\mathbf{v}_1) = L(\mathbf{v}_2),$$

we have

$$\begin{aligned} a_1 + a_2 &= b_1 + b_2 \\ a_1 - a_2 &= b_1 - b_2. \end{aligned}$$

Adding these equations, we obtain $2a_1 = 2b_1$, or $a_1 = b_1$, which implies that $a_2 = b_2$. Hence, $\mathbf{v}_1 = \mathbf{v}_2$ and L is one-to-one. ■

EXAMPLE 2 Let $L: R^3 \to R^2$ be the linear transformation defined in Example 1 of Section 4.3 (the projection function) by

$$L(x, y, z) = (x, y).$$

Since $(1, 3, 3) \neq (1, 3, -2)$ but

$$L(1, 3, 3) = L(1, 3, -2) = (1, 3),$$

we conclude that L is not one-to-one. ■

We shall now develop some more efficient ways of determining whether or not a linear transformation is one-to-one.

Let $L: V \to W$ be a linear transformation. The **kernel** of L, $\ker L$, is the subset of V consisting of all vectors $\mathbf{v}$ such that $L(\mathbf{v}) = \mathbf{0}_W$.

We observe that property (a) of Theorem 10.2 in Section 10.1 assures us that $\ker L$ is never an empty set, since $\mathbf{0}_V$ is in $\ker L$.

EXAMPLE 3 Let $L: R^3 \to R^2$ be as defined in Example 2. The vector $(0, 0, 2)$ is in $\ker L$, since $L(0, 0, 2) = (0, 0)$. However, the vector $(2, -3, 4)$ is not in $\ker L$, since $L(2, -3, 4) = (2, -3)$. To find $\ker L$, we must determine all $\mathbf{x}$ in R^3 so that $L(\mathbf{x}) = \mathbf{0}$. That is, we seek $\mathbf{x} = (x_1, x_2, x_3)$ so that

$$L(\mathbf{x}) = L(x_1, x_2, x_3) = \mathbf{0} = (0, 0).$$

However, $L(\mathbf{x}) = (x_1, x_2)$. Thus $(x_1, x_2) = (0, 0)$, so $x_1 = 0$, $x_2 = 0$, and x_3 can be any real number. Hence, $\ker L$ consists of all vectors in R^3 of the form $(0, 0, r)$, where r is any real number. It is clear that $\ker L$ consists of the z-axis in three-dimensional space R^3. ■

EXAMPLE 4 If L is as defined in Example 1, then $\ker L$ consists of all vectors $\mathbf{x}$ in R^2 such that $L(\mathbf{x}) = \mathbf{0}$. Thus we must solve the linear system

$$\begin{aligned} x + y &= 0 \\ x - y &= 0 \end{aligned}$$

for x and y. The only solution is $\mathbf{x} = \mathbf{0}$, so $\ker L = \{\mathbf{0}\}$.

EXAMPLE 5 If $L: R^4 \to R^2$ is defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x+y \\ z+w \end{bmatrix},$$

then ker L consists of all vectors $\mathbf{u}$ in R^4 such that $L(\mathbf{u}) = \mathbf{0}$. This leads t[o the]
linear system

$$\begin{aligned} x + y \quad\quad &= 0 \\ z + w &= 0. \end{aligned}$$

Thus ker L consists of all vectors of the form

$$\begin{bmatrix} r \\ -r \\ s \\ -s \end{bmatrix},$$

where r and s are any real numbers.

THEOREM 10.4 *If $L: V \to W$ is a linear transformation, then* ker L *is a subspace of* V.

Proof First, observe that ker L is not an empty set, since $\mathbf{0}_V$ is in ker L. Also, [let u]
and $\mathbf{v}$ be in ker L. Then since L is a linear transformation,

$$L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v}) = \mathbf{0}_W + \mathbf{0}_W = \mathbf{0}_W,$$

so $\mathbf{u}+\mathbf{v}$ is in ker L. Also, if c is a scalar, then since L is a linear transforma[tion,]

$$L(c\mathbf{u}) = cL(\mathbf{u}) = c\mathbf{0}_W = \mathbf{0}_W,$$

so $c\mathbf{u}$ is in ker L. Hence ker L is a subspace of V.

EXAMPLE 6 If L is as in Example 1, then ker L is the subspace $\{\mathbf{0}\}$; its dimension is z[ero.]

EXAMPLE 7 If L is as in Example 2, then a basis for ker L is

$$\{(0, 0, 1)\}$$

and dim(ker L) = 1. Thus ker L consists of the z-axis in three-dimensi[onal]
space R^3.

EXAMPLE 8 If L is as in Example 5, then a basis for ker L consists of the vectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix};$$

thus dim(ker L) = 2.

If $L: R^n \to R^m$ is a linear transformation defined by $L(\mathbf{x}) = A\mathbf{x}$, w[here]
A is an $m \times n$ matrix, then the kernel of L is the solution space of the ho[mo]geneous system $A\mathbf{x} = \mathbf{0}$.

An examination of the elements in ker L allows us to decide whether [L is]
or is not one-to-one.

THEOREM 10.5 *A linear transformation $L: V \to W$ is one-to-one if and only if* $\ker L = \{\mathbf{0}_V\}$.

Proof Let L be one-to-one. We show that $\ker L = \{\mathbf{0}_V\}$. Let $\mathbf{x}$ be in $\ker L$. Then $L(\mathbf{x}) = \mathbf{0}_W$. Also, we already know that $L(\mathbf{0}_V) = \mathbf{0}_W$. Thus $L(\mathbf{x}) = L(\mathbf{0}_V)$. Since L is one-to-one, we conclude that $\mathbf{x} = \mathbf{0}_V$. Hence $\ker L = \{\mathbf{0}_V\}$.

Conversely, suppose that $\ker L = \{\mathbf{0}_V\}$. We wish to show that L is one-to-one. Assume that $L(\mathbf{u}) = L(\mathbf{v})$, for $\mathbf{u}$ and $\mathbf{v}$ in V. Then

$$L(\mathbf{u}) - L(\mathbf{v}) = \mathbf{0}_W,$$

so by Theorem 10.2, $L(\mathbf{u} - \mathbf{v}) = \mathbf{0}_W$, which means that $\mathbf{u} - \mathbf{v}$ is in $\ker L$. Therefore, $\mathbf{u} - \mathbf{v} = \mathbf{0}_V$, so $\mathbf{u} = \mathbf{v}$. Thus L is one-to-one. ■

Note that we can also state Theorem 10.5 as: *L is one-to-one if and only if* $\dim(\ker L) = 0$.

The proof of Theorem 10.5 has also established the following result, which we state as Corollary 10.2.

COROLLARY 10.2 *If $L(\mathbf{x}) = \mathbf{b}$ and $L(\mathbf{y}) = \mathbf{b}$, then $\mathbf{x} - \mathbf{y}$ belongs to* $\ker L$. *In other words, any two solutions to $L(\mathbf{x}) = \mathbf{b}$ differ by an element of the kernel of L.*

Proof Exercise T.1. ■

EXAMPLE 9 The linear transformation in Example 1 is one-to-one; the one in Example 2 is not. ■

In Section 10.3 we shall prove that for every linear transformation $L: R^n \to R^m$, we can find a unique $m \times n$ matrix A so that if $\mathbf{x}$ is in R^n, then $L(\mathbf{x}) = A\mathbf{x}$. It then follows that to find $\ker L$, we need to find the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$. Hence to find $\ker L$ we need only use techniques with which we are already familiar.

If $L: V \to W$ is a linear transformation, then the **range** of L, denoted by range L, is the set of all vectors in W that are images, under L, of vectors in V. Thus a vector $\mathbf{w}$ is in range L if there exists some vector $\mathbf{v}$ in V such that $L(\mathbf{v}) = \mathbf{w}$. If range $L = W$, we say that L is **onto**.

THEOREM 10.6 *If $L: V \to W$ is a linear transformation, then* range L *is a subspace of W.*

Proof First, observe that range L is not an empty set, since $\mathbf{0}_W = L(\mathbf{0}_V)$, so $\mathbf{0}_W$ is in range L. Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be in range L. Then $\mathbf{w}_1 = L(\mathbf{v}_1)$ and $\mathbf{w}_2 = L(\mathbf{v}_2)$ for some $\mathbf{v}_1$ and $\mathbf{v}_2$ in V. Now

$$\mathbf{w}_1 + \mathbf{w}_2 = L(\mathbf{v}_1) + L(\mathbf{v}_2) = L(\mathbf{v}_1 + \mathbf{v}_2),$$

which implies that $\mathbf{w}_1 + \mathbf{w}_2$ is in range L. Also, if c is a scalar, then $c\mathbf{w}_1 = cL(\mathbf{v}_1) = L(c\mathbf{v}_1)$, so $c\mathbf{w}_1$ is in range L. Hence range L is a subspace of W. ■

EXAMPLE 10 Let L be the linear transformation defined in Example 2. To find out whether L is onto, we choose any vector $\mathbf{y} = (y_1, y_2)$ in R^2 and seek a vector $\mathbf{x} = (x_1, x_2, x_3)$ in R^3 such that $L(\mathbf{x}) = \mathbf{y}$. Since $L(\mathbf{x}) = (x_1, x_2)$, we find that if $x_1 = y_1$ and $x_2 = y_2$, then $L(\mathbf{x}) = \mathbf{y}$. Therefore, L is onto and the dimension of range L is 2. ■

EXAMPLE 11 Let $L: R^3 \to R^3$ be defined by

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

(a) Is L onto?
(b) Find a basis for range L.
(c) Find ker L.
(d) Is L one-to-one?

Solution (a) Given any

$$\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

in R^3, where a, b, and c are any real numbers, can we find

$$\mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

so that $L(\mathbf{v}) = \mathbf{w}$? We seek a solution to the linear system

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

and we find the reduced row echelon form of the augmented matrix to be (verify)

$$\left[\begin{array}{ccc:c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-b-a \end{array}\right].$$

Thus a solution exists only for $c - b - a = 0$, so L is not onto.

(b) To find a basis for range L, we note that

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 + a_3 \\ a_1 + a_2 + 2a_3 \\ 2a_1 + a_2 + 3a_3 \end{bmatrix}$$

$$= a_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

This means that

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

spans range L. That is, range L is the subspace of R^3 spanned by the columns of the matrix defining L.

The first two vectors in this set are linearly independent, since they are not constant multiples of each other. The third vector is the sum of the first two. Therefore, the first two vectors form a basis for range L, and dim(range L) = 2.

(c) To find $\ker L$, we wish to find all $\mathbf{v}$ in R^3 so that $L(\mathbf{v}) = \mathbf{0}_{R^3}$. Solving the resulting homogeneous system, we find (verify) that $a_1 = -a_3$ and $a_2 = -a_3$. Thus $\ker L$ consists of all vectors of the form

$$\begin{bmatrix} -a \\ -a \\ a \end{bmatrix} = a \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},$$

where a is any real number. Moreover, $\dim(\ker L) = 1$.

(d) Since $\ker L \neq \{\mathbf{0}_{R^3}\}$, it follows from Theorem 10.5 that L is not one-to-one. ■

> The problem of finding a basis for $\ker L$ always reduces to the problem of finding a basis for the solution space of a homogeneous system; this latter problem has been solved in Example 1 of Section 6.5.
>
> If range L is a subspace of R^m, then a basis for range L can be obtained by the method discussed in the alternative constructive proof of Theorem 6.6 or by the procedure given in Section 6.6. Both approaches are illustrated in the next example.

EXAMPLE 12 Let $L: R^4 \to R^3$ be defined by

$$L(a_1, a_2, a_3, a_4) = (a_1 + a_2, a_3 + a_4, a_1 + a_3).$$

Find a basis for range L.

Solution We have

$$L(a_1, a_2, a_3, a_4) = a_1(1, 0, 1) + a_2(1, 0, 0) + a_3(0, 1, 1) + a_4(0, 1, 0).$$

Thus

$$S = \{(1, 0, 1), (1, 0, 0), (0, 1, 1), (0, 1, 0)\}$$

spans range L. To find a subset of S that is a basis for range L, we proceed as in Theorem 6.6 by first writing

$$a_1(1, 0, 1) + a_2(1, 0, 0) + a_3(0, 1, 1) + a_4(0, 1, 0) = (0, 0, 0).$$

The reduced row echelon form of the augmented matrix of this homogeneous system is (verify)

$$\left[\begin{array}{cccc:c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right].$$

Since the leading 1's appear in columns 1, 2, and 3, we conclude that the first three vectors in S form a basis for range L. Thus

$$\{(1, 0, 1), (1, 0, 0), (0, 1, 1)\}$$

is a basis for range L.

Alternatively, we may proceed as in Section 6.6 to form the matrix whose rows are the given vectors

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Transforming this matrix to reduced row echelon form, we obtain (verify)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis for range L.

To determine if a linear transformation is one-to-one or onto, we mu solve a linear system. This is one further demonstration of the frequenc with which linear systems must be solved to answer many questions in line algebra. Finally, from Example 11, where $\dim(\ker L) = 1$, $\dim(\text{range } L) =$ and $\dim(\text{domain } L) = 3$, we saw that

$$\dim(\ker L) + \dim(\text{range } L) = \dim(\text{domain } L).$$

This very important result is always true and we now prove it in the follow theorem.

THEOREM 10.7 *If $L: V \to W$ is a linear transformation of an n-dimensional vector space into a vector space W, then*

$$\dim(\ker L) + \dim(\text{range } L) = \dim V.$$

Proof Let $k = \dim(\ker L)$. If $k = n$, then $\ker L = V$ (Exercise T.7, Section 6.4 which implies that $L(\mathbf{v}) = \mathbf{0}_W$ for every $\mathbf{v}$ in V. Hence range $L = \{\mathbf{0}_W$ $\dim(\text{range } L) = 0$, and the conclusion holds. Next, suppose that $1 \leq k <$ We shall prove that $\dim(\text{range } L) = n - k$. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis f ker L. By Theorem 6.8 we can extend this basis to a basis

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$$

for V. We prove that the set

$$T = \{L(\mathbf{v}_{k+1}), L(\mathbf{v}_{k+2}), \ldots, L(\mathbf{v}_n)\}$$

is a basis for range L.

First, we show that T spans range L. Let $\mathbf{w}$ be any vector in range L. The $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v}$ in V. Since S is a basis for V, we can find a unique s of real numbers $a_1, a_2, \ldots, a_n$ such that

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

Then

$$\begin{aligned} \mathbf{w} &= L(\mathbf{v}) \\ &= L(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k + a_{k+1}\mathbf{v}_{k+1} + \cdots + a_n\mathbf{v}_n) \\ &= a_1L(\mathbf{v}_1) + a_2L(\mathbf{v}_2) + \cdots + a_kL(\mathbf{v}_k) + a_{k+1}L(\mathbf{v}_{k+1}) + \cdots + a_nL(\mathbf{v}_n) \\ &= a_{k+1}L(\mathbf{v}_{k+1}) + \cdots + a_nL(\mathbf{v}_n) \end{aligned}$$

because $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are in ker L. Hence T spans range L.

Now we show that T is linearly independent. Suppose that

$$a_{k+1}L(\mathbf{v}_{k+1}) + a_{k+2}L(\mathbf{v}_{k+2}) + \cdots + a_nL(\mathbf{v}_n) = \mathbf{0}_W.$$

Then by (b) of Theorem 10.2

$$L(a_{k+1}\mathbf{v}_{k+1} + a_{k+2}\mathbf{v}_{k+2} + \cdots + a_n\mathbf{v}_n) = \mathbf{0}_W.$$

Hence the vector $a_{k+1}\mathbf{v}_{k+1} + a_{k+2}\mathbf{v}_{k+2} + \cdots + a_n\mathbf{v}_n$ is in ker L, and we can write

$$a_{k+1}\mathbf{v}_{k+1} + a_{k+2}\mathbf{v}_{k+2} + \cdots + a_n\mathbf{v}_n = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k.$$

where $b_1, b_2, \ldots, b_k$ are uniquely determined real numbers. We then have

$$b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k - a_{k+1}\mathbf{v}_{k+1} - a_{k+2}\mathbf{v}_{k+2} - \cdots - a_n\mathbf{v}_n = \mathbf{0}_V.$$

Since S is linearly independent, we find that

$$b_1 = b_2 = \cdots = b_k = a_{k+1} = a_{k+2} = \cdots = a_n = 0.$$

Hence T is linearly independent and forms a basis for range L.

If $k = 0$, then ker L has no basis; we let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V. The proof now proceeds as above. ■

The dimension of ker L is also called the **nullity** of L, and the dimension of range L is called the **rank** of L. With this terminology the conclusion of Theorem 10.7 is very similar to that of Theorem 6.12. This is not a coincidence, since in the next section we shall show how to attach a unique $m \times n$ matrix to L, whose properties reflect those of L.

The following example illustrates Theorem 10.7 graphically.

EXAMPLE 13 Let $L: R^3 \to R^3$ be the linear transformation defined by

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 + a_3 \\ a_1 + a_2 \\ a_2 - a_3 \end{bmatrix}.$$

A vector $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ is in ker L if

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We must then find a basis for the solution space of the homogeneous system

$$\begin{aligned} a_1 + a_3 &= 0 \\ a_1 + a_2 &= 0 \\ a_2 - a_3 &= 0. \end{aligned}$$

We find (verify) that a basis for ker L is $\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right\}$, so dim(ker L) $= 1$, and ker L is a line through the origin.

Next, every vector in range L is of the form $\begin{bmatrix} a_1 + a_3 \\ a_1 + a_2 \\ a_2 - a_3 \end{bmatrix}$, which can be written as

$$a_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Then a basis for range L is

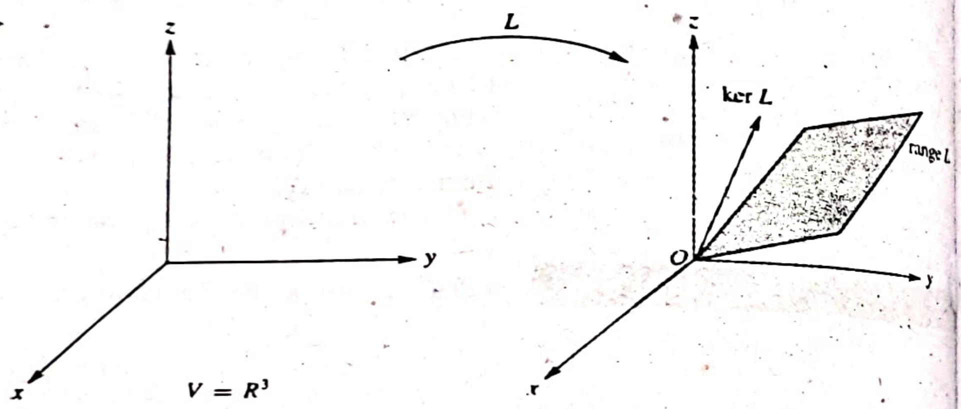
$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(explain), so dim(range L) $= 2$ and range L is a plane passing through the origin. These results are illustrated in Figure 10.2. Moreover,

$$\dim R^3 = 3 = \dim(\ker L) + \dim(\text{range } L) = 1 + 2,$$

verifying Theorem 10.7.

Figure 10.2 ►



We have seen that a linear transformation may be one-to-one and not onto or onto and not one-to-one. However, the following corollary shows that each of these properties implies the other if the vector spaces V and W have the same dimensions.

COROLLARY 10.3 *Let $L: V \to W$ be a linear transformation and let* $\dim V = \dim W$.

(a) *If L is one-to-one, then it is onto.*

(b) *If L is onto, then it is one-to-one.*

Proof Exercise T.2.

EXAMPLE 14 Let $L: P_2 \to P_2$ be the linear transformation defined by

$$L(at^2 + bt + c) = (a + 2b)t + (b + c).$$

(a) Is $-4t^2 + 2t - 2$ in ker L?

(b) Is $t^2 + 2t + 1$ in range L?

(c) Find a basis for ker L.

(d) Is L one-to-one?

(e) Find a basis for range L.

(f) Is L onto?

(g) Verify Theorem 10.7.

Solution (a) Since

$$L(-4t^2 + 2t - 2) = (-4 + 2 \cdot 2)t + (-2 + 2) = 0,$$

we conclude that $-4t^2 + 2t - 2$ is in ker L.

(b) The vector $t^2 + 2t + 1$ is in range L if we can find a vector $at^2 + bt + c$ in P_2 such that

$$L(at^2 + bt + c) = t^2 + 2t + 1.$$

Since $L(at^2 + bt + c) = (a + 2b)t + (b + c)$, we have

$$(a + 2b)t + (b + c) = t^2 + 2t + 1.$$

The left side of this equation can also be written as $0t^2+(a+2b)t+(b+c)$. Thus

$$0t^2 + (a + 2b)t + (b + c) = t^2 + 2t + 1.$$

We must then have

$$\begin{aligned} 0 &= 1 \\ a + 2b &= 2 \\ b + c &= 1. \end{aligned}$$

Since this linear system has no solution, the given vector is not in range L.

(c) The vector $at^2 + bt + c$ is in ker L if

$$L(at^2 + bt + c) = 0,$$

that is, if

$$(a + 2b)t + (b + c) = 0.$$

Then

$$\begin{aligned} a + 2b \quad &= 0 \\ b + c &= 0. \end{aligned}$$

Transforming the augmented matrix of this linear system to reduced row echelon form, we find (verify) that a basis for the solution space is

$$\left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\},$$

so a basis for ker L is $\{2t^2 - t + 1\}$.

(d) Since ker L does not consist only of the zero vector, L is not one-to-one.

(e) Every vector in range L has the form

$$(a + 2b)t + (b + c),$$

so the vectors t and 1 span range L. Since these vectors are also linearly independent, they form a basis for range L.

(f) The dimension of P_2 is 3, while range L is a subspace of P_2 of dimension 2, so range $L \neq P_2$. Hence, L is not onto.

(g) From (c), dim(ker L) = 1, and from (e), dim(range L) = 2, so

$$3 = \dim P_2 = \dim(\ker L) + \dim(\text{range } L).$$

If $L: R^n \to R^n$ is a linear transformation defined by $L(\mathbf{x}) = A\mathbf{x}$, ... A is an $n \times n$ matrix, then using Theorem 10.7, Equation (1), and Coro... 6.2, we can show (Exercise T.4) that L is one-to-one if and only if det(A)...

We now make one final remark for a linear system $A\mathbf{x} = \mathbf{b}$, where ... $n \times n$. We again consider the linear transformation $L: R^n \to R^n$ defined by $L(\mathbf{x}) = A\mathbf{x}$, for $\mathbf{x}$ in R^n. If A is a nonsingular matrix, then dim(range ... rank $A = n$, so dim(ker L) = 0. Thus L is one-to-one and hence onto... means that the given linear system has a unique solution (of course, we al... knew this result from other considerations). Now assume that A is sing... Then rank $A < n$. This means that dim(ker L) = n − rank $A > 0$, so L i... one-to-one and not onto. Therefore, there exists a vector $\mathbf{b}$ in R^n, for w... the system $A\mathbf{x} = \mathbf{b}$ has no solution. Moreover, since A is singular, $A\mathbf{x} =$ ... a nontrivial solution $\mathbf{x}_0$. If $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{y}$, then $\mathbf{x}_0 + \mathbf{y}$ is a sol... to $A\mathbf{x} = \mathbf{b}$ (verify). Thus, for A singular, if a solution to $A\mathbf{x} = \mathbf{b}$ exists, it is not unique.

10.2 Exercises

1. Let $L: R^2 \to R^2$ be the linear transformation defined by $L(a_1, a_2) = (a_1, 0)$.
   (a) Is (0, 2) in ker L?
   (b) Is (2, 2) in ker L?
   (c) Is (3, 0) in range L?
   (d) Is (3, 2) in range L?
   (e) Find ker L.
   (f) Find range L.

2. Let $L: R^2 \to R^2$ be the linear transformation defined by
$$L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$
   (a) Is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in ker L?
   (b) Is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ in ker L?
   (c) Is $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ in range L?
   (d) Is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in range L?
   (e) Find ker L.
   (f) Find a set of vectors spanning range L.

3. Let $L: R^2 \to R^3$ be defined by
$$L(x, y) = (x, x + y, y).$$
   (a) Find ker L.
   (b) Is L one-to-one?
   (c) Is L onto?

4. Let $L: R^4 \to R^3$ be defined by
$$L(x, y, z, w) = (x + y, z + w, x + z).$$
   (a) Find a basis for ker L.
   (b) Find a basis for range L.
   (c) Verify Theorem 10.7.

5. Let $L: R^5 \to R^4$ be defined by
$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -1 & 3 & -1 \\ 1 & 0 & 0 & 2 & -1 \\ 2 & 0 & -1 & 5 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$
   (a) Find a basis for ker L.
   (b) Find a basis for range L.
   (c) Verify Theorem 10.7.

6. Let $L: R^3 \to R^3$ be defined by
$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 3 & -1 \\ -1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$
   (a) Is L one-to-one?
   (b) Find the dimension of range L.

7. Let $L: R^4 \to R^3$ be defined by
$$L\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x + y \\ y - z \\ z - w \end{bmatrix}.$$
   (a) Is L onto?
   (b) Find the dimension of ker L.
   (c) Verify Theorem 10.7.

8. Let $L: R^3 \to R^3$ be defined by
$$L(x, y, z) = (x - y, x + 2y, z).$$
   (a) Find a basis for ker L.

(b) Find a basis for range L.

(c) Verify Theorem 10.7.

Verify Theorem 10.7 for the following linear transformations.

(a) $L(x, y) = (x + y, y)$.

(b) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & -1 & -1 \\ 2 & 2 & 3 \\ 2 & -3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(c) $L(x, y, z) = (x + y - z, x + y, y + z)$.

Let $L: R^4 \to R^4$ be defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & -1 & 2 \\ 1 & 0 & 0 & -1 \\ 4 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

(a) Find a basis for ker L.

(b) Find a basis for range L.

(c) Verify Theorem 10.7.

Let $L: P_2 \to P_2$ be the linear transformation defined by $L(at^2 + bt + c) = (a + c)t^2 + (b + c)t$.

(a) Is $t^2 - t - 1$ in ker L?

(b) Is $t^2 + t - 1$ in ker L?

(c) Is $2t^2 - t$ in range L?

(d) Is $t^2 - t + 2$ in range L?

(e) Find a basis for ker L.

(f) Find a basis for range L.

Let $L: P_3 \to P_3$ be the linear transformation defined by

$$L(at^3 + bt^2 + ct + d) = (a - b)t^3 + (c - d)t.$$

(a) Is $t^3 + t^2 + t - 1$ in ker L?

(b) Is $t^3 - t^2 + t - 1$ in ker L?

(c) Is $3t^3 + t$ in range L?

(d) Is $3t^3 - t^2$ in range L?

(e) Find a basis for ker L.

(f) Find a basis for range L.

Let $L: M_{22} \to M_{22}$ be the linear transformation defined by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + b & b + c \\ a + d & b + d \end{bmatrix}.$$

(a) Find a basis for ker L.

(b) Find a basis for range L.

14. Let $L: P_2 \to R^2$ be the linear transformation defined by $L(at^2 + bt + c) = (a, b)$.

(a) Find a basis for ker L.

(b) Find a basis for range L.

15. Let $L: M_{22} \to M_{22}$ be the linear transformation defined by

$$L(\mathbf{v}) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \mathbf{v} - \mathbf{v} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

(a) Find a basis for ker L.

(b) Find a basis for range L.

16. Let $L: M_{22} \to M_{22}$ be the linear transformation defined by $L(A) = A^T$.

(a) Find a basis for ker L.

(b) Find a basis for range L.

17. ***(Calculus Required)*** Let $L: P_2 \to P_1$ be the linear transformation defined by

$$L[p(t)] = p'(t).$$

(a) Find a basis for ker L.

(b) Find a basis for range L.

18. ***(Calculus Required)*** Let $L: P_2 \to R^1$ be the linear transformation defined by

$$L[p(t)] = \int_0^1 p(t)\,dt.$$

(a) Find a basis for ker L.

(b) Find a basis for range L.

19. Let $L: R^4 \to R^6$ be a linear transformation.

(a) If dim(ker L) = 2, what is dim(range L)?

(b) If dim(range L) = 3, what is dim(ker L)?

20. Let $L: V \to R^5$ be a linear transformation.

(a) If L is onto and dim(ker L) = 2, what is dim V?

(b) If L is one-to-one and onto, what is dim V?

Theoretical Exercises

T.1. Prove Corollary 10.2.

T.2. Prove Corollary 10.3.

T.3. Let A be an $m \times n$ matrix and let $L: R^n \to R^m$ be defined by $L(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x}$ in R^n. Show that the column space of A is the range of L.

T.4. Let $L: R^n \to R^n$ be a linear transformation defined by $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ matrix. Show t[hat L] is one-to-one if and only if $\det(A) \neq 0$. [Hint: U[se] Theorem 10.7, Equation (1), and Corollary 6.2

T.5. Let $L: V \to W$ be a linear transformation. I[f] $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ spans V, show that $\{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_k)\}$ spans range L.

T.6. Let $L: V \to W$ be a linear transformation.
(a) Show that $\dim(\text{range } L) \leq \dim V$.
(b) Show that if L is onto, then $\dim W \leq \dim V$.

T.7. Let $L: V \to W$ be a linear transformation, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of vectors in V. Show that if $T = \{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)\}$ is linearly independent, then so is S. (*Hint*: Assume that S is linearly dependent. What can we say about T?)

T.8. Let $L: V \to W$ be a linear transformation. Show that L is one-to-one if and only if $\dim(\text{range } L) = \dim V$.

T.9. Let $L: V \to W$ be a linear transformation. Show that L is one-to-one if and only if the image of every linearly independent set of vectors in V is a linearly independent set of vectors in W.

T.10. Let $L: V \to W$ be a linear transformation, and let $\dim V = \dim W$. Show that L is one-to-one if and only if the image under L of a basis for V is a basis for W.

T.11. Let V be an n-dimensional vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V. Let $L: V \to R^n$ be defined by $L(\mathbf{v}) = [\mathbf{v}]_S$. Show that
(a) L is a linear transformation.
(b) L is one-to-one.
(c) L is onto.

MATLAB Exercises

In order to use MATLAB in this section, you should first read Section 12.8. Find a basis for the kernel and range of the linear transformation $L(\mathbf{x}) = A\mathbf{x}$ for each of the following matrices A.

ML.1. $A = \begin{bmatrix} 1 & 2 & 5 & 5 \\ -2 & -3 & -8 & -7 \end{bmatrix}$.

ML.2. $A = \begin{bmatrix} -3 & 2 & -7 \\ 2 & -1 & 4 \\ 2 & -2 & 6 \end{bmatrix}$.

ML.3. $A = \begin{bmatrix} 3 & 3 & -3 & 1 & 11 \\ -4 & -4 & 7 & -2 & -19 \\ 2 & 2 & -3 & 1 & 9 \end{bmatrix}$.

10.3 THE MATRIX OF A LINEAR TRANSFORMATION

We have shown in Theorem 4.8 that if $L: R^n \to R^m$ is a linear transformation, then there is a unique $m \times n$ matrix A so that $L(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x}$ in R^n. In this section we generalize this result for a linear transformation $L: V \to W$ of a finite-dimensional vector space V into a finite-dimensional vector space W.

The Matrix of a Linear Transformation

THEOREM 10.8 *Let $L: V \to W$ be a linear transformation of an n-dimensional vector space V into an m-dimensional vector space W ($n \neq 0$ and $m \neq 0$) and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be bases for V and W, respectively. Then the $m \times n$ matrix A, whose jth column is the coordinate vector $[L(\mathbf{v}_j)]_T$ of $L(\mathbf{v}_j)$ with respect to T, is associated with L and has the following property: If $\mathbf{x}$ is in V, then*

$$[L(\mathbf{x})]_T = A[\mathbf{x}]_S, \tag{1}$$

where $[\mathbf{x}]_S$ and $[L(\mathbf{x})]_T$ are the coordinate vectors of $\mathbf{x}$ and $L(\mathbf{x})$ with respect to the respective bases S and T. Moreover, A is the only matrix with this property.

Proof The proof is a constructive one; that is, we show how to construct the matrix A. It is more complicated than the proof of Theorem 4.8. Consider the vector $\mathbf{v}_j$ in V for $j = 1, 2, \ldots, n$. Then $L(\mathbf{v}_j)$ is a vector in W, and since T is a basis for W, we can express this vector as a linear combination of the vectors in T in a unique manner. Thus

$$L(\mathbf{v}_j) = c_{1j}\mathbf{w}_1 + c_{2j}\mathbf{w}_2 + \cdots + c_{mj}\mathbf{w}_m \quad (1 \leq j \leq n). \tag{2}$$

This means that the coordinate vector of $L(\mathbf{v}_j)$ with respect to T is

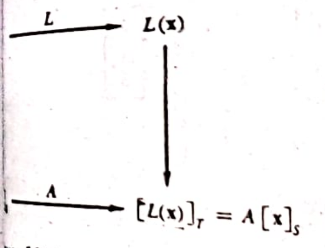
$$\left[L(\mathbf{v}_j)\right]_T = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix}.$$

We now define the $m \times n$ matrix A by choosing $\left[L(\mathbf{v}_j)\right]_T$ as the jth column of A and show that this matrix satisfies the properties stated in the theorem. We leave the rest of the proof as Exercise T.1 and amply illustrate it in the examples below. ■

DEFINITION The matrix A of Theorem 10.8 is called the **matrix representing L with respect to the bases S and T**, or the **matrix of L with respect to S and T**.

We now summarize the procedure given in Theorem 10.8.

> The procedure for computing the matrix of a linear transformation $L: V \rightarrow W$ with respect to the bases $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ for V and W, respectively, is as follows.
>
> ***Step 1.*** Compute $L(\mathbf{v}_j)$ for $j = 1, 2, \ldots, n$.
>
> ***Step 2.*** Find the coordinate vector $\left[L(\mathbf{v}_j)\right]_T$ of $L(\mathbf{v}_j)$ with respect to the basis T. This means that we have to express $L(\mathbf{v}_j)$ as a linear combination of the vectors in T [see Equation (2)].
>
> ***Step 3.*** The matrix A of L with respect to S and T is formed by choosing $\left[L(\mathbf{v}_j)\right]_T$ as the jth column of A.



ure 10.3 ▲

Figure 10.3 gives a graphical interpretation of Equation (1), that is, of Theorem 10.8. The top horizontal arrow represents the linear transformation L from the n-dimensional vector space V into the m-dimensional vector space W and takes the vector $\mathbf{x}$ in V to the vector $L(\mathbf{x})$ in W. The bottom horizontal line represents the matrix A. Then $\left[L(\mathbf{x})\right]_T$, a coordinate vector in R^m, is obtained simply by multiplying $\left[\mathbf{x}\right]_S$, a coordinate vector in R^n, by the matrix A. We can thus always work with matrices rather than with linear transformations.

Physicists and others who deal at great length with linear transformations perform most of their computations with the matrices of the linear transformations.

EXAMPLE 1 Let $L: R^3 \rightarrow R^2$ be defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y \\ y - z \end{bmatrix}. \quad (2$$

Let

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \quad \text{and} \quad T = \{\mathbf{w}_1, \mathbf{w}_2\}$$

be bases for R^3 and R^2, respectively, where

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix},$$

$$\mathbf{w}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix}0\\1\end{bmatrix}.$$

We now find the matrix A of L with respect to S and T. We have

$$L(\mathbf{v}_1) = \begin{bmatrix}1+0\\0-0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix},$$

$$L(\mathbf{v}_2) = \begin{bmatrix}0+1\\1-0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix},$$

$$L(\mathbf{v}_3) = \begin{bmatrix}0+0\\0-1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}.$$

Since T is the natural basis for R^2, the coordinate vectors of $L(\mathbf{v}_1)$, $L(\mathbf{v}_2)$, and $L(\mathbf{v}_3)$ with respect to T are the same as $L(\mathbf{v}_1)$, $L(\mathbf{v}_2)$, and $L(\mathbf{v}_3)$, respectively. That is,

$$[L(\mathbf{v}_1)]_T = \begin{bmatrix}1\\0\end{bmatrix}, \quad [L(\mathbf{v}_2)]_T = \begin{bmatrix}1\\1\end{bmatrix}, \quad [L(\mathbf{v}_3)]_T = \begin{bmatrix}0\\-1\end{bmatrix}.$$

Hence

$$A = \begin{bmatrix}1 & 1 & 0\\0 & 1 & -1\end{bmatrix}.$$

EXAMPLE 2 Let $L: R^3 \to R^2$ be defined as in Example 1. Now let

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \quad \text{and} \quad T = \{\mathbf{w}_1, \mathbf{w}_2\}$$

be bases for R^3 and R^2, respectively, where

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix},$$

$$\mathbf{w}_1 = \begin{bmatrix}1\\2\end{bmatrix}, \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix}-1\\1\end{bmatrix}.$$

Find the matrix of L with respect to S and T.

Solution We have

$$L(\mathbf{v}_1) = \begin{bmatrix}1\\-1\end{bmatrix}, \quad L(\mathbf{v}_2) = \begin{bmatrix}1\\0\end{bmatrix}, \quad L(\mathbf{v}_3) = \begin{bmatrix}2\\0\end{bmatrix}.$$

To find the coordinate vectors $[L(\mathbf{v}_1)]_T$, $[L(\mathbf{v}_2)]_T$, and $[L(\mathbf{v}_3)]_T$, we write

$$L(\mathbf{v}_1) = \begin{bmatrix}1\\-1\end{bmatrix} = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 = a_1\begin{bmatrix}1\\2\end{bmatrix} + a_2\begin{bmatrix}-1\\1\end{bmatrix},$$

$$L(\mathbf{v}_2) = \begin{bmatrix}1\\0\end{bmatrix} = b_1\mathbf{w}_1 + b_2\mathbf{w}_2 = b_1\begin{bmatrix}1\\2\end{bmatrix} + b_2\begin{bmatrix}-1\\1\end{bmatrix},$$

$$L(\mathbf{v}_3) = \begin{bmatrix}2\\0\end{bmatrix} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 = c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix}.$$

That is, we must solve three linear systems, each of two equations in two unknowns. Since their coefficient matrix is the same, we solve them all at once, as in Example 4 of Section 6.7. Thus we form the matrix

$$\left[\begin{array}{rr:r:r:r} 1 & -1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 0 & 0 \end{array}\right],$$

which we transform to reduced row echelon form, obtaining (verify)

$$\left[\begin{array}{rr:r:r:r} 1 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & -1 & -\frac{2}{3} & -\frac{4}{3} \end{array}\right].$$

Hence the matrix A of L with respect to S and T is

$$A = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ -1 & -\frac{2}{3} & -\frac{4}{3} \end{bmatrix}.$$

Equation (1) is then

$$[L(\mathbf{x})]_T = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ -1 & -\frac{2}{3} & -\frac{4}{3} \end{bmatrix} [\mathbf{x}]_S. \tag{4}$$

To illustrate Equation (4), let

$$\mathbf{x} = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}.$$

Then from the definition of L, Equation (3), we have

$$L(\mathbf{x}) = \begin{bmatrix} 1+6 \\ 6-3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}.$$

Now (verify)

$$[\mathbf{x}]_S = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}.$$

Then from (4),

$$[L(\mathbf{x})]_T = A[\mathbf{x}]_S = \begin{bmatrix} \frac{10}{3} \\ -\frac{11}{3} \end{bmatrix}.$$

Hence

$$L(\mathbf{x}) = \frac{10}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{11}{3}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix},$$

which agrees with the previous value for $L(\mathbf{x})$. ■

Notice that the matrices obtained in Examples 1 and 2 are different even though L is the same in both cases. It can be shown that there is a relationship between these two matrices. The study of this relationship is beyond the scope of this book.

The procedure used in Example 2 can be used to find the matrix representing a linear transformation $L: R^n \to R^m$ with respect to given bases S and T for R^n and R^m, respectively.

The procedure for computing the matrix of a linear transformation $L: R^n \to R^m$ with respect to the bases $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ for R^n and R^m, respectively, is as follows.

Step 1. Compute $L(\mathbf{x}_j)$ for $j = 1, 2, \ldots, n$.

Step 2. Form the matrix

$$\left[\mathbf{w}_1 \quad \mathbf{w}_2 \quad \cdots \quad \mathbf{w}_m \;\vdots\; L(\mathbf{v}_1) \;\vdots\; L(\mathbf{v}_2) \;\vdots\; \cdots \;\vdots\; L(\mathbf{v}_n)\right],$$

which we transform to reduced row echelon form, obtaining the matrix

$$\left[I_n \;\vdots\; A\right].$$

Step 3. The matrix A is the matrix representing L with respect to bases S and T.

EXAMPLE 3 Let $L: R^3 \to R^2$ be as defined in Example 1. Now let

$$S = \{\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1\} \quad \text{and} \quad T = \{\mathbf{w}_1, \mathbf{w}_2\},$$

where $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{w}_1$, and $\mathbf{w}_2$ are as in Example 2. Then the matrix of L with respect to S and T is

$$A = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ -\frac{4}{3} & -\frac{2}{3} & -1 \end{bmatrix}.$$

Note that if we change the order of the vectors in the bases S and T, then the matrix A of L may change.

EXAMPLE 4 Let $L: R^3 \to R^2$ be defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Let

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \quad \text{and} \quad T = \{\mathbf{w}_1, \mathbf{w}_2\}$$

be the natural bases for R^3 and R^2, respectively. Find the matrix of L with respect to S and T.

Solution We have

$$L(\mathbf{v}_1) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\mathbf{w}_1 + 1\mathbf{w}_2, \quad \text{so} \quad \left[L(\mathbf{v}_1)\right]_T =$$

$$L(\mathbf{v}_2) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\mathbf{w}_1 + 2\mathbf{w}_2, \quad \text{so} \quad \left[L(\mathbf{v}_2)\right]_T =$$

Also,

$$\left[L(\mathbf{v}_3)\right]_T = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{(verify).}$$

Then the matrix of L with respect to S and T is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

Of course, the reason that A is the same matrix as the one involved in the definition of L is that the natural bases are being used for R^3 and R^2. ■

EXAMPLE 5 Let $L: R^3 \to R^2$ be defined as in Example 4. Now let

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \quad \text{and} \quad T = \{\mathbf{w}_1, \mathbf{w}_2\},$$

where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Find the matrix of L with respect to S and T.

Solution We have

$$L(\mathbf{v}_1) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad L(\mathbf{v}_2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad L(\mathbf{v}_3) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

We now form (verify)

$$\left[\mathbf{w}_1 \;\; \mathbf{w}_2 \;\vdots\; L(\mathbf{v}_1) \;\vdots\; L(\mathbf{v}_2) \;\vdots\; L(\mathbf{v}_3)\right] = \left[\begin{array}{cc:c:c:c} 1 & 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 2 & 3 \end{array}\right].$$

Transforming this matrix to reduced row echelon form, we obtain (verify)

$$\left[\begin{array}{cc:c:c:c} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 & 1 \end{array}\right],$$

so the matrix of L with respect to S and T is

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

This matrix is, of course, far different from the one that defined L. Thus, although a matrix A may be involved in the definition of a linear transformation L, we cannot conclude that it is necessarily the matrix representing L with respect to two given bases S and T. ■

EXAMPLE 6 Let $L: P_1 \to P_2$ be defined by $L[p(t)] = tp(t)$.

(a) Find the matrix of L with respect to the bases $S = \{t, 1\}$ and $T = \{t^2, t, 1\}$ for P_1 and P_2, respectively.

(b) If $p(t) = 3t - 2$, compute $L[p(t)]$ directly and using the matrix obtained in (a).

Solution (a) We have

$$L(t) = t \cdot t = t^2 = 1(t^2) + 0(t) + 0(1), \quad \text{so} \quad [L(t)]_T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$$

$$L(1) = t \cdot 1 = t = 0(t^2) + 1(t) + 0(1), \quad \text{so} \quad [L(1)]_T = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Hence the matrix of L with respect to S and T is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(b) Computing $L[p(t)]$ directly, we have

$$L[p(t)] = tp(t) = t(3t - 2) = 3t^2 - 2t.$$

To compute $L[p(t)]$ using A, we first write

$$p(t) = 3 \cdot t + (-2)1, \quad \text{so} \quad [p(t)]_S = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Then

$$[L[p(t)]]_T = A\,[p(t)]_S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}.$$

Hence

$$L[p(t)] = 3t^2 + (-2)t + 0(1) = 3t^2 - 2t.$$

EXAMPLE 7 Let $L\colon P_1 \to P_2$ be as defined in Example 6.

(a) Find the matrix of L with respect to the bases $S = \{t, 1\}$ and $T = \{t^2, t - 1, t + 1\}$ for P_1 and P_2, respectively.

(b) If $p(t) = 3t - 2$, compute $L[p(t)]$ using the matrix obtained in (a).

Solution (a) We have (verify)

$$L(t) = t^2 = 1(t^2) + 0(t - 1) + 0(t + 1), \quad \text{so} \quad [L(t)]_T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$$

$$L(1) = t = 0(t^2) + \tfrac{1}{2}(t - 1) + \tfrac{1}{2}(t + 1), \quad \text{so} \quad [L(1)]_T = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Then the matrix of L with respect to S and T is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}.$$

(b) We have

$$[L[p(t)]]_T = A[p(t)]_S = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}.$$

Hence

$$L[p(t)] = 3t^2 + (-1)(t-1) + (-1)(t+1) = 3t^2 - 2t. \quad \blacksquare$$

Suppose that $L: V \to W$ is a linear transformation and that A is the matrix of L with respect to bases for V and W. Then the problem of finding ker L reduces to the problem of finding the solution space of $A\mathbf{x} = \mathbf{0}$. Moreover, the problem of finding range L reduces to the problem of finding the column space of A. •

If $L: V \to V$ is a linear operator (a linear transformation from V to W, where $V = W$) and V is an n-dimensional vector space, then to obtain a matrix representing L, we fix bases S and T for V and obtain the matrix of L with respect to S and T. However, it is often convenient in this case to choose $S = T$. To avoid redundancy in this case, we refer to A as the **matrix of L with respect to S**. If $L: R^n \to R^n$ is a linear operator, then the matrix representing L with respect to the natural basis for R^n has already been discussed in Theorem 6.3 in Section 6.1, where it was called the **standard matrix** representing L.

Let $I: V \to V$ be the identity linear operator on an n-dimensional vector space defined by $I(\mathbf{v}) = \mathbf{v}$ for every $\mathbf{v}$ in V. If S is a basis for V, then the matrix of I with respect to S is I_n (Exercise T.2). Let T be another basis for V. Then the matrix of I with respect to S and T is the transition matrix (see Section 6.7) from the S-basis to the T-basis (Exercise T.5).

If $L: R^n \to R^n$ is a linear operator defined by $L(\mathbf{x}) = A\mathbf{x}$, for $\mathbf{x}$ in R^n, then we can show that L is one-to-one and onto if and only if A is nonsingular.

We can now extend our List of Nonsingular Equivalences.

List of Nonsingular Equivalences

The following statements are equivalent for an $n \times n$ matrix A.

1. A is nonsingular.
2. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
3. A is row equivalent to I_n.
4. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ matrix $\mathbf{b}$.
5. $\det(A) \neq 0$.
6. A has rank n.
7. A has nullity 0.
8. The rows of A form a linearly independent set of n vectors in R^n.
9. The columns of A form a linearly independent set of n vectors in R^n.
10. Zero is *not* an eigenvalue of A.
11. The linear operator $L: R^n \to R^n$ defined by $L(\mathbf{x}) = A\mathbf{x}$, for $\mathbf{x}$ in R^n, is one-to-one and onto.