6.7 COORDINATES AND CHANGE OF BASIS

Coordinates

If V is an n-dimensional vector space, we know that V has a basis S w vectors in it; so far we have not paid much attention to the order of the ve in S. However, in the discussion of this section we speak of an ordered $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for V; thus $S_1 = \{\mathbf{v}_2, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a different or basis for V.

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an ordered basis for the n-dimensional v space V, then every vector $\mathbf{v}$ in V can be uniquely expressed in the form

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

where $c_1, c_2, \ldots, c_n$ are real numbers. We shall refer to

$$\left[\mathbf{v}\right]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

as the **coordinate vector of v with respect to the ordered basis S**. The e of $\left[\mathbf{v}\right]_S$ are called the **coordinates of v with respect to S**.

Observe that the coordinate vector $\left[\mathbf{v}\right]_S$ depends on the order in whi vectors in S are listed; a change in the order of this listing may chan coordinates of $\mathbf{v}$ with respect to S. All bases considered in this section are assumed to be ordered bases.

EXAMPLE 1 Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a basis for R^4, where

$$\mathbf{v}_1 = (1, 1, 0, 0), \qquad \mathbf{v}_2 = (2, 0, 1, 0),$$
$$\mathbf{v}_3 = (0, 1, 2, -1), \qquad \mathbf{v}_4 = (0, 1, -1, 0).$$

If

$$\mathbf{v} = (1, 2, -6, 2),$$

compute $[\mathbf{v}]_S$.

Solution To find $[\mathbf{v}]_S$ we need to compute constants c_1, c_2, c_3, and c_4 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{v},$$

which is just a linear combination problem. The previous equation leads to the linear system whose augmented matrix is (verify)

$$\left[\begin{array}{cccc:c} 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 & -6 \\ 0 & 0 & -1 & 0 & 2 \end{array}\right], \tag{1}$$

or, equivalently,

$$\left[\mathbf{v}_1^T \quad \mathbf{v}_2^T \quad \mathbf{v}_3^T \quad \mathbf{v}_4^T \; \vdots \; \mathbf{v}^T\right].$$

Transforming the matrix in (1) to reduced row echelon form, we obtain the solution (verify)

$$c_1 = 3, \qquad c_2 = -1, \qquad c_3 = -2, \qquad c_4 = 1,$$

so the coordinate vector of $\mathbf{v}$ with respect to the basis S is

$$[\mathbf{v}]_S = \begin{bmatrix} 3 \\ -1 \\ -2 \\ 1 \end{bmatrix}.$$

■

EXAMPLE 2 Let $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the natural basis for R^3 and let

$$\mathbf{v} = (2, -1, 3).$$

Compute $[\mathbf{v}]_S$.

Solution Since S is the natural basis,

$$\mathbf{v} = 2\mathbf{e}_1 - 1\mathbf{e}_2 + 3\mathbf{e}_3,$$

so

$$[\mathbf{v}]_S = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

■

Remark In Example 2 the coordinate vector $[\mathbf{v}]_S$ of $\mathbf{v}$ with respect to S agrees with $\mathbf{v}$ because S is the natural basis for R^3.

EXAMPLE 3 Let V be P_1, the vector space of all polynomials of degree ≤ 1, and
$\{\mathbf{v}_1, \mathbf{v}_2\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2\}$ be bases for P_1, where

$$\mathbf{v}_1 = t, \qquad \mathbf{v}_2 = 1, \qquad \mathbf{w}_1 = t + 1, \qquad \mathbf{w}_2 = t - 1.$$

Let $\mathbf{v} = p(t) = 5t - 2$.

(a) Compute $\left[\mathbf{v}\right]_S$.

(b) Compute $\left[\mathbf{v}\right]_T$.

Solution (a) Since S is the standard or natural basis for P_1, we have

$$5t - 2 = 5t + (-2)(1).$$

Hence

$$\left[\mathbf{v}\right]_S = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$$

(b) To compute $\left[\mathbf{v}\right]_T$, we must write $\mathbf{v}$ as a linear combination of w
Thus,

$$5t - 2 = c_1(t + 1) + c_2(t - 1),$$

or

$$5t - 2 = (c_1 + c_2)t + (c_1 - c_2).$$

Equating coefficients of like powers of t, we obtain the linear sy

$$\begin{aligned} c_1 + c_2 &= 5 \\ c_1 - c_2 &= -2, \end{aligned}$$

whose solution is (verify)

$$c_1 = \tfrac{3}{2} \quad \text{and} \quad c_2 = \tfrac{7}{2}.$$

Hence

$$\left[\mathbf{v}\right]_T = \begin{bmatrix} \frac{3}{2} \\ \frac{7}{2} \end{bmatrix}.$$

In some important ways the coordinate vectors of elements in
space behave algebraically in ways that are similar to the way the
themselves behave. For example, it is not difficult to show (see Exer
that if S is a basis for an n-dimensional vector space V, $\mathbf{v}$ and $\mathbf{w}$ are
V, and k is a scalar, then

$$\left[\mathbf{v} + \mathbf{w}\right]_S = \left[\mathbf{v}\right]_S + \left[\mathbf{w}\right]_S.$$

and

$$\left[k\mathbf{v}\right]_S = k\left[\mathbf{v}\right]_S.$$

That is, the coordinate vector of a sum of two vectors is the sum of
dinate vectors, and the coordinate vector of a scalar multiple of a vec
scalar multiple of the coordinate vector. Moreover, the results in E
(2) and (3) can be generalized to show that

$$\left[k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n\right]_S = k_1\left[\mathbf{v}_1\right]_S + k_2\left[\mathbf{v}_2\right]_S + \cdots + k_n\left[\right.$$

That is, the coordinate vector of a linear combination of vectors is
linear combination of the individual coordinate vectors.

Suppose now that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ are bas for the n-dimensional vector space V. We shall examine the relationship b tween the coordinate vectors $[\mathbf{v}]_S$ and $[\mathbf{v}]_T$ of the vector $\mathbf{v}$ in V with resp to the bases S and T, respectively.

If $\mathbf{v}$ is any vector in V, then

$$\mathbf{v} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n$$

so that

$$[\mathbf{v}]_T = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Then

$$\begin{aligned}[\mathbf{v}]_S &= [c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n]_S \\ &= [c_1\mathbf{w}_1]_S + [c_2\mathbf{w}_2]_S + \cdots + [c_n\mathbf{w}_n]_S \\ &= c_1[\mathbf{w}_1]_S + c_2[\mathbf{w}_2]_S + \cdots + c_n[\mathbf{w}_n]_S.\end{aligned}$$

Let the coordinate vector of $\mathbf{w}_j$ with respect to S be denoted by

$$[\mathbf{w}_j]_S = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}.$$

Then

$$[\mathbf{v}]_S = c_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + c_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + c_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

or

$$[\mathbf{v}]_S = P_{S\leftarrow T}[\mathbf{v}]_T,$$

where

$$P_{S\leftarrow T} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \left[[\mathbf{w}_1]_S \;\; [\mathbf{w}_2]_S \;\; \cdots \;\; [\mathbf{w}_n]_S \right]$$

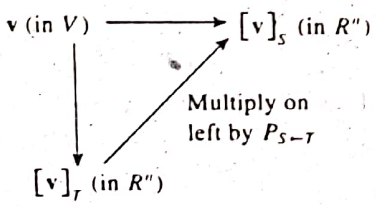

Figure 6.9 ▲

is called the **transition matrix from the T-basis to the S-basis.** Equation (5) says that the coordinate vector of $\mathbf{v}$ with respect to the basis S is the transition matrix $P_{S\leftarrow T}$ times the coordinate vector of $\mathbf{v}$ with respect to the basis T. Figure 6.9 illustrates Equation (5).

We may now summarize the procedure just developed for computing the transition matrix from the T-basis to the S-basis.

> The procedure for computing the transition matrix $P_{S\leftarrow T}$ from the basis $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ for V to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for V is as follows.
>
> ***Step 1.*** Compute the coordinate vector of $\mathbf{w}_j$, $j = 1, 2, \ldots, n$, with respect to the basis S. This means that we first have to express $\mathbf{w}_j$ as a linear combination of the vectors in S:
>
> $$a_{1j}\mathbf{v}_1 + a_{2j}\mathbf{v}_2 + \cdots + a_{nj}\mathbf{v}_n = \mathbf{w}_j, \qquad j = 1, 2, \ldots, n.$$
>
> We now solve for $a_{1j}, a_{2j}, \ldots, a_{nj}$ by Gauss–Jordan reduction, transforming the augmented matrix of this linear system to reduced row echelon form.
>
> ***Step 2.*** The transition matrix $P_{S\leftarrow T}$ from the T-basis to the S-basis is formed by choosing $[\mathbf{w}_j]_S$ as the jth column of $P_{S\leftarrow T}$.

EXAMPLE 4 Let V be R^3 and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be bases for R^3, where

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\mathbf{w}_1 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}.$$

(a) Compute the transition matrix $P_{S \leftarrow T}$ from the T-basis to the S-basis.

(b) Verify Equation (5) for $\mathbf{v} = \begin{bmatrix} 4 \\ -9 \\ 5 \end{bmatrix}$.

Solution (a) To compute $P_{S \leftarrow T}$, we find a_1, a_2, a_3 such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{w}_1.$$

In this case we are led to a linear system of three equations in three unknowns, whose augmented matrix is

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \vdots & \mathbf{w}_1 \end{bmatrix}.$$

That is, the augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 6 \\ 0 & 2 & 1 & 3 \\ 1 & 0 & 1 & 3 \end{array}\right].$$

Similarly, to find b_1, b_2, b_3 and c_1, c_2, c_3 such that

$$b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 = \mathbf{w}_2$$
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}_3,$$

we are led to two linear systems, each of three equations in three unknowns, whose augmented matrices are

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \vdots & \mathbf{w}_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \vdots & \mathbf{w}_3 \end{bmatrix},$$

or, specifically,

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 0 & 2 & 1 & -1 \\ 1 & 0 & 1 & 3 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & 2 & 1 & 5 \\ 1 & 0 & 1 & 2 \end{array}\right].$$

Since the coefficient matrix of all three linear systems is $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$, we can transform the three augmented matrices to reduced row echelon form simultaneously by transforming the partitioned matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \vdots & \mathbf{w}_1 & \vdots & \mathbf{w}_2 & \vdots & \mathbf{w}_3 \end{bmatrix}$$

to reduced row echelon form. Thus we transform

$$\left[\begin{array}{ccc|c|c|c} 2 & 1 & 1 & 6 & 4 & 5 \\ 0 & 2 & 1 & 3 & -1 & 5 \\ 1 & 0 & 1 & 3 & 3 & 2 \end{array}\right]$$

to reduced row echelon form, obtaining (verify)

$$\left[\begin{array}{ccc:c:c:c} 1 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right],$$

which implies that the transition matrix from the T-basis to the S-ba

$$P_{S\leftarrow T} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

(b) If

$$\mathbf{v} = \begin{bmatrix} 4 \\ -9 \\ 5 \end{bmatrix},$$

then to express $\mathbf{v}$ in terms of the T-basis, we use Equation (4). From
associated linear system we find that (verify)

$$\mathbf{v} = \begin{bmatrix} 4 \\ -9 \\ 5 \end{bmatrix} = 1\begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix} = 1\mathbf{w}_1 + 2\mathbf{w}_2 - 2\mathbf{w}_3$$

so

$$\left[\mathbf{v}\right]_T = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

Then by Equation (5) we find that $\left[\mathbf{v}\right]_S$ is

$$P_{S\leftarrow T}\left[\mathbf{v}\right]_T = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}.$$

If we compute $\left[\mathbf{v}\right]_S$ directly by setting up and solving the associated l
system, we find that (verify)

$$\mathbf{v} = \begin{bmatrix} 4 \\ -9 \\ 5 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - 5\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 4\mathbf{v}_1 - 5\mathbf{v}_2 + 1\mathbf{v}_3.$$

so

$$\left[\mathbf{v}\right]_S = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}.$$

Hence

$$\left[\mathbf{v}\right]_S = P_{S\leftarrow T}\left[\mathbf{v}\right]_T.$$

We next show that the transition matrix $P_{S\leftarrow T}$ from the T-basis to th
basis is nonsingular and that $P_{S\leftarrow T}^{-1}$ is the transition matrix from the S-bas
the T-basis.

THEOREM 6.15 *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be bases for th dimensional vector space V. Let $P_{S\leftarrow T}$ be the transition matrix from th basis to the S-basis. Then $P_{S\leftarrow T}$ is nonsingular and $P_{S\leftarrow T}^{-1}$ is the trans matrix from the S-basis to the T-basis.*

Proof We proceed by showing that the null space of $P_{S \leftarrow T}$ contains only the zero vector. Suppose that $P_{S \leftarrow T}\left[\mathbf{v}\right]_T = \mathbf{0}_{R^n}$ for some $\mathbf{v}$ in V. From Equation (5) we have

$$P_{S \leftarrow T}\left[\mathbf{v}\right]_T = \left[\mathbf{v}\right]_S = \mathbf{0}_{R^n}.$$

If $\mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n$, then

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \left[\mathbf{v}\right]_S = \mathbf{0}_{R^n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

so

$$\mathbf{v} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n = \mathbf{0}_V.$$

Hence $\left[\mathbf{v}\right]_T = \mathbf{0}_{R^n}$. Thus the homogeneous system $P_{S \leftarrow T}\mathbf{x} = \mathbf{0}$ has only the trivial solution; it then follows from Theorem 1.13 that $P_{S \leftarrow T}$ is nonsingular. Multiplying both sides of Equation (5) on the left by $P_{S \leftarrow T}^{-1}$, we have

$$\left[\mathbf{v}\right]_T = P_{S \leftarrow T}^{-1}\left[\mathbf{v}\right]_S.$$

That is, $P_{S \leftarrow T}^{-1}$ is then the transition matrix from the S-basis to the T-basis; the jth column of $P_{S \leftarrow T}^{-1}$ is $\left[\mathbf{v}_j\right]_T$. ■

Remark In Exercises T.5 through T.7 we ask you to show that if S and T are bases for the vector space R^n, then

$$P_{S \leftarrow T} = M_S^{-1}M_T,$$

where M_S is the $n \times n$ matrix whose jth column is $\mathbf{v}_j$ and M_T is the $n \times n$ matrix whose jth column is $\mathbf{w}_j$. This formula implies that $P_{S \leftarrow T}$ is nonsingular and it is helpful in solving some of the exercises in this section.

EXAMPLE 5 Let S and T be the bases for R^3 defined in Example 4. Compute the transition matrix $Q_{T \leftarrow S}$ from the S-basis to the T-basis directly and show that $Q_{T \leftarrow S} = P_{S \leftarrow T}^{-1}$.

Solution $Q_{T \leftarrow S}$ is the matrix whose columns are the solution vectors to the linear system obtained from the vector equations

$$\begin{aligned} a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 &= \mathbf{v}_1 \\ b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + b_3\mathbf{w}_3 &= \mathbf{v}_2 \\ c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 &= \mathbf{v}_3. \end{aligned}$$

As in Example 4, we can solve these linear systems simultaneously by transforming the partitioned matrix

$$\left[\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3 \;\vdots\; \mathbf{v}_1 \;\vdots\; \mathbf{v}_2 \;\vdots\; \mathbf{v}_3\right]$$

to reduced row echelon form. That is, we transform

$$\left[\begin{array}{rrr|r|r|r} 6 & 4 & 5 & 2 & 1 & 1 \\ 3 & -1 & 5 & 0 & 2 & 1 \\ 3 & 3 & 2 & 1 & 0 & 1 \end{array}\right]$$

to reduced row echelon form, obtaining (verify)

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right],$$

so

$$Q_{T\leftarrow S} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -1 & 0 & 2 \end{bmatrix}.$$

Multiplying $Q_{T\leftarrow S}$ by $P_{S\leftarrow T}$, we find (verify) that $Q_{T\leftarrow S}P_{S\leftarrow T} = I$ conclude that $Q_{T\leftarrow S} = P_{S\leftarrow T}^{-1}$.

EXAMPLE 6 Let V be P_1 and let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2\}$ be bases for P_1,

$$\mathbf{v}_1 = t, \quad \mathbf{v}_2 = t - 3, \quad \mathbf{w}_1 = t - 1, \quad \mathbf{w}_2 = t + 1.$$

(a) Compute the transition matrix $P_{S\leftarrow T}$ from the T-basis to the S-b

(b) Verify Equation (5) for $\mathbf{v} = 5t + 1$.

(c) Compute the transition matrix $Q_{T\leftarrow S}$ from the S-basis to the T-b show that $Q_{T\leftarrow S} = P_{S\leftarrow T}^{-1}$.

Solution (a) To compute $P_{S\leftarrow T}$, we need to solve the vector equations

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = \mathbf{w}_1$$
$$b_1\mathbf{v}_1 + b_2\mathbf{v}_2 = \mathbf{w}_2$$

simultaneously by transforming the resulting partitioned matrix (

$$\left[\begin{array}{cc|c|c} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & 1 \end{array}\right]$$

to reduced row echelon form. The result is (verify)

$$\left[\begin{array}{cc|c|c} 1 & 0 & \frac{2}{3} & \frac{4}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \end{array}\right],$$

so

$$P_{S\leftarrow T} = \begin{bmatrix} \frac{2}{3} & \frac{4}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}.$$

(b) If $\mathbf{v} = 5t + 1$, then expressing $\mathbf{v}$ in terms of the T-basis, we have

$$\mathbf{v} = 5t + 1 = 2(t - 1) + 3(t + 1),$$

so

$$[\mathbf{v}]_T = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

To verify this, set up and solve the associated linear system resulti the vector equation

$$\mathbf{v} = a_1\mathbf{w}_1 + a_2\mathbf{w}_2.$$

Then

$$[\mathbf{v}]_S = P_{S\leftarrow T}[\mathbf{v}]_T = \begin{bmatrix} \frac{2}{3} & \frac{4}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{16}{3} \\ -\frac{1}{3} \end{bmatrix}.$$

Computing $[\mathbf{v}]_S$ directly from the associated linear system arising from the vector equation

$$\mathbf{v} = b_1\mathbf{w}_1 + b_2\mathbf{w}_2$$

we find that (verify)

$$\mathbf{v} = 5t + 1 = \tfrac{16}{3}t - \tfrac{1}{3}(t - 3),$$

so

$$[\mathbf{v}]_S = \begin{bmatrix} \frac{16}{3} \\ -\frac{1}{3} \end{bmatrix}.$$

Hence

$$[\mathbf{v}]_S = P_{S\leftarrow T}[\mathbf{v}]_T,$$

which is Equation (5).

(c) The transition matrix $Q_{T\leftarrow S}$ from the S-basis to the T-basis is obtained (verify) by transforming the partitioned matrix

$$\left[\begin{array}{cc:c:c} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & -3 \end{array}\right]$$

to reduced row echelon form, obtaining (verify)

$$\left[\begin{array}{cc:c:c} 1 & 0 & \frac{1}{2} & 2 \\ 0 & 1 & \frac{1}{2} & -1 \end{array}\right].$$

Hence

$$Q_{T\leftarrow S} = \begin{bmatrix} \frac{1}{2} & 2 \\ \frac{1}{2} & -1 \end{bmatrix}.$$

Multiplying $Q_{T\leftarrow S}$ by $P_{S\leftarrow T}$, we find (verify) that $Q_{T\leftarrow S}P_{S\leftarrow T} = I_2$, so $Q_{T\leftarrow S} = P_{S\leftarrow T}^{-1}$. ■

6.7 Exercises

bases considered in these exercises are assumed to be ordered bases. In Exercises 1 through 6, compute the coordinate vector of **v** *with respect to the given basis S for V.*

V is R^2, $S = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

V is R^3, $S = \{(1, -1, 0), (0, 1, 0), (1, 0, 2)\}$, $\mathbf{v} = (2, -1, -2)$.

V is P_1, $S = \{t + 1, t - 2\}$, $\mathbf{v} = t + 4$.

V is P_2, $S = \{t^2 - t + 1, t + 1, t^2 + 1\}$, $\mathbf{v} = 4t^2 - 2t + 3$.

5. V is M_{22}, $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$, $\mathbf{v} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$.

6. V is M_{22}, $S = \left\{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\right\}$, $\mathbf{v} = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}$.

In Exercises 7 through 12, compute the vector $\mathbf{v}$ *if the coordinate vector* $[\mathbf{v}]_S$ *is given with respect to the basis S for V.*

7. V is R^2, $S = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$, $[\mathbf{v}]_S = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

8. V is R^3, $S = \{(0, 1, -1), (1, 0, 0), (1, 1, 1)\}$,
$$[\mathbf{v}]_S = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

9. V is P_1, $S = \{t, 2t - 1\}$, $[\mathbf{v}]_S = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

10. V is P_2, $S = \{t^2 + 1, t + 1, t^2 + t\}$, $[\mathbf{v}]_S = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$.

11. V is M_{22}, $S = \left\{ \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} \right\}$, $[\mathbf{v}]_S = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}$.

12. V is M_{22}, $S = \left\{ \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$, $[\mathbf{v}]_S = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$.

13. Let $S = \{(1, 2), (0, 1)\}$ and $T = \{(1, 1), (2, 3)\}$ be bases for R^2. Let $\mathbf{v} = (1, 5)$ and $\mathbf{w} = (5, 4)$.

(a) Find the coordinate vectors of $\mathbf{v}$ and $\mathbf{w}$ with respect to the basis T.

(b) What is the transition matrix $P_{S \leftarrow T}$ from the T- to the S-basis?

(c) Find the coordinate vectors of $\mathbf{v}$ and $\mathbf{w}$ with respect to S using $P_{S \leftarrow T}$.

(d) Find the coordinate vectors of $\mathbf{v}$ and $\mathbf{w}$ with respect to S directly.

(e) Find the transition matrix $Q_{T \leftarrow S}$ from the S- to the T-basis.

(f) Find the coordinate vectors of $\mathbf{v}$ and $\mathbf{w}$ with respect to T using $Q_{T \leftarrow S}$. Compare the answers with those of (a).

14. Let
$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$
and
$$T = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$
be bases for R^3. Let
$$\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -1 \\ 8 \\ -2 \end{bmatrix}.$$
Follow the directions for Exercise 13.

15. Let $S = \{t^2 + 1, t - 2, t + 3\}$ and $T = \{2t^2 + t, t^2 + 3, t\}$ be bases for P_2. Let $\mathbf{v} = 8t^2 - 4t + 6$ and $\mathbf{w} = 7t^2 - t + 9$. Follow the directions for Exercise

16. Let $S = \{t^2 + t + 1, t^2 + 2t + 3, t^2 + 1\}$ and $T = \{t + 1, t^2, t^2 + 1\}$ be bases for P_2. Also let $\mathbf{v} = -t^2 + 4t + 5$ and $\mathbf{w} = 2t^2 - 6$. Follow the directions for Exercise 13.

17. Let
$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$
and
$$T = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$
be bases for M_{22}. Let
$$\mathbf{v} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$
Follow the directions for Exercise 13.

18. Let
$$S = \left\{ \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$
and
$$T = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$
be bases for M_{22}. Let
$$\mathbf{v} = \begin{bmatrix} 0 & 0 \\ 3 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -2 & 3 \\ -1 & 3 \end{bmatrix}.$$
Follow the directions for Exercise 13.

19. Let $S = \{(1, -1), (2, 1)\}$ and $T = \{(3, 0), (4, -1)\}$ be bases for R^2. If $\mathbf{v}$ is in R^2 and
$$[\mathbf{v}]_T = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$
determine $[\mathbf{v}]_S$.

20. Let $S = \{t + 1, t - 2\}$ and $T = \{t - 5, t - 2\}$ be bases for P_1. If $\mathbf{v}$ is in P_1 and
$$[\mathbf{v}]_T = \begin{bmatrix} -1 \\ 3 \end{bmatrix},$$
determine $[\mathbf{v}]_S$.

21. Let $S = \{(-1, 2, 1), (0, 1, 1), (-2, 2, 1)\}$ and $T = \{(-1, 1, 0), (0, 1, 0), (0, 1, 1)\}$ be bases for R^3 is in R^3 and
$$[\mathbf{v}]_S = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$
determine $[\mathbf{v}]_T$.

22. Let $S = \{t^2, t - 1, 1\}$ and $T = \{t^2 + t + 1, t + 1, 1\}$ be bases for P_2. If $\mathbf{v}$ is in P_2 and

$$[\mathbf{v}]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

determine $[\mathbf{v}]_T$.

23. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ be bases for R^3, where

$$\mathbf{v}_1 = (1, 0, 1), \quad \mathbf{v}_2 = (1, 1, 0), \quad \mathbf{v}_3 = (0, 0, 1).$$

If the transition matrix from T to S is

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix},$$

determine T.

24. Let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2\}$ be bases for P_1, where

$$\mathbf{w}_1 = t, \quad \mathbf{w}_2 = t - 1.$$

If the transition matrix from S to T is

$$\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix},$$

determine S.

25. Let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2\}$ be bases for R^2, where

$$\mathbf{v}_1 = (1, 2), \quad \mathbf{v}_2 = (0, 1).$$

If the transition matrix from S to T is

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},$$

determine T.

26. Let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2\}$ be bases for P_1, where

$$\mathbf{w}_1 = t - 1, \quad \mathbf{w}_2 = t + 1.$$

If the transition matrix from T to S is

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix},$$

determine S.

CHAPTER 10

LINEAR TRANSFORMATIONS AND MATRICES

In Section 4.3 we gave the definition, basic properties, and some examples of linear transformations mapping R^n into R^m. In this chapter we consider linear transformations mapping a vector space V into a vector space W.

10.1 DEFINITION AND EXAMPLES

DEFINITION Let V and W be vector spaces. A **linear transformation** L of V into W is a function assigning a unique vector $L(\mathbf{u})$ in W to each $\mathbf{u}$ in V such that:

(a) $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$, for every $\mathbf{u}$ and $\mathbf{v}$ in V.

(b) $L(k\mathbf{u}) = kL(\mathbf{u})$, for every $\mathbf{u}$ in V and every scalar k.

In the definition above, observe that in (a) the $+$ in $\mathbf{u}+\mathbf{v}$ on the left side of the equation refers to the addition operation in V, whereas the $+$ in $L(\mathbf{u}) + L(\mathbf{v})$ on the right side of the equation refers to the addition operation in W. Similarly, in (b) the scalar product $k\mathbf{u}$ is in V, while the scalar product $kL(\mathbf{u})$ is in W.

As in Section 4.3, we shall write the fact that L maps V into W, even if it is not a linear transformation, as

$$L: V \to W.$$

If $V = W$, the linear transformation $L: V \to V$ is also called a **linear operator on** V.

In Section 4.3 we gave a number of examples of linear transformations mapping R^n into R^m. Thus, the following are linear transformations that we have already discussed:

Projection: $L: R^3 \to R^2$ defined by $L(x, y, z) = (x, y)$.

Dilation: $L: R^3 \to R^3$ defined by $L(\mathbf{u}) = r\mathbf{u}, r > 1$.

Contraction: $L: R^3 \to R^3$ defined by $L(\mathbf{u}) = r\mathbf{u}, 0 < r < 1$.

Reflection: $L: R^2 \to R^2$ defined by $L(x, y) = (x, -y)$.

Rotation: $L: R^2 \to R^2$ defined by $L(\mathbf{u}) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\mathbf{u}$

Recall that P_1 is the vector space of all polynomials of degree ≤ 1; in general, P_n is the vector space of all polynomials of degree $\leq n$, and M_{nn} is the vector space of all $n \times n$ matrices.

As in Section 4.3, to verify that a given function is a linear transformation, we have to check that conditions (a) and (b) in the definition above are satisfied.

EXAMPLE 1 Let $L: P_1 \to P_2$ be defined by

$$L(at + b) = t(at + b).$$

Show that L is a linear transformation.

Solution Let $at + b$ and $ct + d$ be vectors in P_1 and let k be a scalar. Then

$$L[(at + b) + (ct + d)] = t[(at + b) + (ct + d)]$$
$$= t(at + b) + t(ct + d) = L(at + b) + L(ct + d)$$

and

$$L[k(at + b)] = t[k(at + b)] = k[t(at + b)] = kL(at + b).$$

Hence L is a linear transformation. ■

EXAMPLE 2 Let $L: P_1 \to P_2$ be defined by

$$L[p(t)] = tp(t) + t^2.$$

Is L a linear transformation?

Solution Let $p(t)$ and $q(t)$ be vectors in P_1 and let k be a scalar. Then

$$L[p(t) + q(t)] = t[p(t) + q(t)] + t^2 = tp(t) + tq(t) + t^2$$

and

$$L[p(t)] + L[q(t)] = [tp(t) + t^2] + [tq(t) + t^2] = t[p(t) + q(t)] + 2t^2.$$

Since $L[p(t) + q(t)] \neq L[p(t)] + L[q(t)]$, we conclude that L is not a linear transformation. ■

EXAMPLE 3 Let $L: M_{mn} \to M_{nm}$ be defined by

$$L(A) = A^T$$

for A in M_{mn}. Is L a linear transformation?

Solution Let A and B be in M_{mn}. Then by Theorem 1.4 in Section 1.4, we have

$$L(A + B) = (A + B)^T = A^T + B^T = L(A) + L(B),$$

and, if k is a scalar,

$$L(kA) = (kA)^T = kA^T = kL(A).$$

Hence L is a linear transformation. ■

EXAMPLE 4 **(Calculus Required)** Let W be the vector space of all real-valued functions and let V be the subspace of all differentiable functions. Let $L: V \to W$ be defined by

$$L(f) = f',$$

where f' is the derivative of f. It is easy to show (Exercise 13), from the properties of differentiation, that L is a linear transformation.

EXAMPLE 5 **(Calculus Required)** Let $V = C[0, 1]$ denote the vector space of all real-valued continuous functions defined on $[0, 1]$. Let $W = R^1$. Define $L: V \to W$ by

$$L(f) = \int_0^1 f(x)\,dx.$$

It is easy to show (Exercise 14), from the properties of integration, that L is a linear transformation.

EXAMPLE 6 Let V be an n-dimensional vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ a basis for V. If $\mathbf{v}$ is a vector in V, then

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

where $c_1, c_2, \ldots, c_n$ are the coordinates of $\mathbf{v}$ with respect to S (see Section [illegible] 6.7). We define $L: V \to R^n$ by

$$L(\mathbf{v}) = \left[\mathbf{v}\right]_S.$$

It is easy to show (Exercise 15) that L is a linear transformation.

EXAMPLE 7 Let A be an $m \times n$ matrix. In Example 7 of Section 4.3 we have observed that if $L: R^n \to R^m$ is defined by

$$L(\mathbf{x}) = A\mathbf{x}$$

for $\mathbf{x}$ in R^n, then L is a linear transformation (see Exercise 16). Specific cases of this type of linear transformation have been seen in Examples 6, 8, 9, and 10 in Section 4.3.

The following two theorems give some basic properties of linear transformations.

THEOREM 10.1 *If $L: V \to W$ is a linear transformation, then*

$$L(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = c_1L(\mathbf{v}_1) + c_2L(\mathbf{v}_2) + \cdots + c_kL(\mathbf{v}_k)$$

for any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in V and any scalars $c_1, c_2, \ldots, c_k$.

Proof Exercise T.1.

THEOREM 10.2 *Let $L: V \to W$ be a linear transformation. Then:*

(a) $L(\mathbf{0}_V) = \mathbf{0}_W$, *where $\mathbf{0}_V$ and $\mathbf{0}_W$ are the zero vectors in V and W, respectively.*

(b) $L(\mathbf{u} - \mathbf{v}) = L(\mathbf{u}) - L(\mathbf{v})$.

Proof (a) We have

$$\mathbf{0}_V = \mathbf{0}_V + \mathbf{0}_V.$$

Then

$$L(\mathbf{0}_V) = L(\mathbf{0}_V + \mathbf{0}_V) = L(\mathbf{0}_V) + L(\mathbf{0}_V). \tag{1}$$

Adding $-L(\mathbf{0}_V)$ to both sides of Equation (1), we obtain

$$L(\mathbf{0}_V) = \mathbf{0}_W.$$

(b) Exercise T.2. ■

The proof of the following corollary is very similar to the proof of the analogous corollary, Corollary 4.1 in Section 4.3.

COROLLARY 10.1 *Let $T: V \to W$ be a function. If $T(\mathbf{0}_V) \neq \mathbf{0}_W$, then T is not a linear transformation.*

Proof Exercise T.3. ■

Remark Example 2 could be solved more easily by using Corollary 10.1 as follows. Since $T(0) = t(0) + t^2 = t^2$, it follows that T is not a linear transformation.

A function f mapping a set V into a set W can be specified by a formula that assigns to every member of V a unique element of W. On the other hand, we can also specify a function by listing next to each member of V its assigned element of W. An example of this would be provided by listing the names of all charge account customers of a department store along with their charge account number. At first glance, it appears impossible to describe a linear transformation $L: V \to W$ of a vector space $V \neq \{\mathbf{0}\}$ into a vector space W in this latter manner, since V has infinitely many members in it. However, the following very useful theorem tells us that once we know what L does to a basis for V, then we have completely determined L. Thus, for a finite-dimensional vector space V, it is possible to describe L by giving only the images of a finite number of vectors in V.

THEOREM 10.3 *Let $L: V \to W$ be a linear transformation of an n-dimensional vector space V into a vector space W. Also, let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V. If $\mathbf{u}$ is any vector in V, then $L(\mathbf{u})$ is completely determined by $\{L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)\}$.*

Proof Since $\mathbf{u}$ is in V, we can write

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n, \tag{2}$$

where $c_1, c_2, \ldots, c_n$ are uniquely determined real numbers. Then

$$L(\mathbf{u}) = L(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1 L(\mathbf{v}_1) + c_2 L(\mathbf{v}_2) + \cdots + c_n L(\mathbf{v}_n),$$

by Theorem 10.1. Thus $L(\mathbf{u})$ has been completely determined by the elements $L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)$. ■

It might be noted that in the proof of Theorem 10.3, the real numbers c_i, $i = 1, 2, \ldots, n$ determined in Equation (2) depend on the basis vectors in S. Thus, if we change S, then we may change the c_i's.

EXAMPLE 8 Let $L: P_1 \to P_2$ be a linear transformation for which we know that

$$L(t+1) = t^2 - 1 \quad \text{and} \quad L(t-1) = t^2 + t.$$

(a) What is $L(7t+3)$?
(b) What is $L(at+b)$?

Solution (a) First, note that $\{t+1, t-1\}$ is a basis for P_1 (verify). Next, we find that (verify)

$$7t + 3 = 5(t+1) + 2(t-1).$$

Then

$$\begin{aligned} L(7t+3) &= L(5(t+1) + 2(t-1)) \\ &= 5L(t+1) + 2L(t-1) \\ &= 5(t^2-1) + 2(t^2+t) = 7t^2 + 2t - 5. \end{aligned}$$

(b) Writing $at+b$ as a linear combination of the given basis vectors, we find that (verify)

$$at + b = \left(\frac{a+b}{2}\right)(t+1) + \left(\frac{a-b}{2}\right)(t-1).$$

Then

$$\begin{aligned} L(at+b) &= L\left(\left(\frac{a+b}{2}\right)(t+1) + \left(\frac{a-b}{2}\right)(t-1)\right) \\ &= \left(\frac{a+b}{2}\right)L(t+1) + \left(\frac{a-b}{2}\right)L(t-1) \\ &= \left(\frac{a+b}{2}\right)(t^2-1) + \left(\frac{a-b}{2}\right)(t^2+t) \\ &= at^2 + \left(\frac{a-b}{2}\right)t - \left(\frac{a+b}{2}\right). \end{aligned}$$

10.1 Exercises

1. Which of the following are linear transformations?
   (a) $L(x, y) = (x+y, x-y)$.
   (b) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+1 \\ y-z \end{bmatrix}$.
   (c) $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

2. Which of the following are linear transformations?
   (a) $L(x, y, z) = (0, 0)$.
   (b) $L(x, y, z) = (1, 2, -1)$.
   (c) $L(x, y, z) = (x^2 + y, y - z)$.

3. Let $L: P_1 \to P_2$ be defined as indicated. Is L a linear transformation? Justify your answer.
   (a) $L[p(t)] = tp(t) + p(0)$.
   (b) $L[p(t)] = tp(t) + t^2 + 1$.
   (c) $L(at+b) = at^2 + (a-b)t$.

4. Let $L: P_2 \to P_1$ be defined as indicated. Is L a linear transformation? Justify your answer.
   (a) $L(at^2 + bt + c) = at + b + 1$.

(b) $L(at^2 + bt + c) = 2at - b$.

(c) $L(at^2 + bt + c) = (a + 2)t + (b - a)$.

Let $L: P_2 \rightarrow P_2$ be defined as indicated. Is L a linear transformation? Justify your answer.

(a) $L(at^2 + bt + c) = (a + 1)t^2 + (b - c)t + (a + c)$.

(b) $L(at^2 + bt + c) = at^2 + (b - c)t + (a - b)$.

(c) $L(at^2 + bt + c) = 0$.

Let C be a fixed $n \times n$ matrix and let $L: M_{nn} \rightarrow M_{nn}$ be defined by $L(A) = CA$. Show that L is a linear transformation.

Let $L: M_{22} \rightarrow M_{22}$ be defined by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} b & c - d \\ c + d & 2a \end{bmatrix}.$$

Is L a linear transformation?

Let $L: M_{22} \rightarrow M_{22}$ be defined by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a - 1 & b + 1 \\ 2c & 3d \end{bmatrix}.$$

Is L a linear transformation?

Let $L: M_{22} \rightarrow R^1$ be defined by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d.$$

Is L a linear transformation?

Let $L: M_{22} \rightarrow R^1$ be defined by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + b - c - d + 1.$$

Is L a linear transformation?

Consider the function $L: M_{34} \rightarrow M_{24}$ defined by

$$L(A) = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & -3 \end{bmatrix} A$$

for A in M_{34}.

(a) Find $L\left(\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 0 & 2 & 3 \\ 4 & 1 & -2 & 1 \end{bmatrix}\right)$.

(b) Show that L is a linear transformation.

12. Let $L: M_{nn} \rightarrow R^1$ be defined by $L(A) = a_{11}a_{22}\cdots a_{nn}$, for an $n \times n$ matrix $A = \left[a_{ij}\right]$. Is L a linear transformation?

13. (*Calculus Required*) Verify that the function in Example 4 is a linear transformation.

14. (*Calculus Required*) Verify that the function in Example 5 is a linear transformation.

15. Verify that the function in Example 6 is a linear transformation.

16. Verify that the function in Example 7 is a linear transformation.

17. Let $L: R^2 \rightarrow R^2$ be a linear transformation for which we know that $L(1, 1) = (1, -2)$, $L(-1, 1) = (2, 3)$.

(a) What is $L(-1, 5)$?

(b) What is $L(a_1, a_2)$?

18. Let $L: P_2 \rightarrow P_3$ be a linear transformation for which we know that $L(1) = 1$, $L(t) = t^2$, and $L(t^2) = t^3 + t$.

(a) Find $L(2t^2 - 5t + 3)$.

(b) Find $L(at^2 + bt + c)$.

19. Let $L: P_1 \rightarrow P_1$ be a linear transformation for which we know that $L(t + 1) = 2t + 3$ and $L(t - 1) = 3t - 2$.

(a) Find $L(6t - 4)$.

(b) Find $L(at + b)$.