where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & -1 & 5 & 7 \\ 0 & 2 & -2 & -2 \end{bmatrix}.$$

BASIS AND DIMENSION

In this section we continue our study of the structure of a vector space V by determining a smallest set of vectors in V that completely describes V.

DEFINITION The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to form a **basis for** V if (a) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ span V and (b) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent.

Remark If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ form a basis for a vector space V, then they must be distinct and nonzero, so we write them as a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

EXAMPLE 1 The vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ form a basis for R^2, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ form a basis for R^3 and, in general, the vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ form a basis for R^n. Each of these sets of vectors is called the **natural basis** or **standard basis** for R^2, R^3, and R^n, respectively. ■

EXAMPLE 2 Show that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where $\mathbf{v}_1 = (1, 0, 1, 0)$, $\mathbf{v}_2 = (0, 1, -1, 2)$, $\mathbf{v}_3 = (0, 2, 2, 1)$, and $\mathbf{v}_4 = (1, 0, 0, 1)$ is a basis for R^4.

Solution To show that S is linearly independent, we form the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$$

and solve for c_1, c_2, c_3, and c_4. Substituting for $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$, we obtain the linear system (verify)

$$\begin{aligned} c_1 \qquad\qquad\quad + c_4 &= 0 \\ c_2 + 2c_3 \qquad &= 0 \\ c_1 - \quad c_2 + 2c_3 \qquad &= 0 \\ 2c_2 + \quad c_3 + c_4 &= 0, \end{aligned}$$

which has as its only solution $c_1 = c_2 = c_3 = c_4 = 0$ (verify), showing that S is linearly independent.

To show that S spans R^4, we let $\mathbf{v} = (a, b, c, d)$ be any vector in R^4. We now seek constants k_1, k_2, k_3, and k_4 such that

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4 = \mathbf{v}.$$

Substituting for $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, and $\mathbf{v}$, we find a solution (verify) for k_1, k_2, k_3, and k_4 to the resulting linear system for any a, b, c, and d. Hence S spans R^4 and is a basis for R^4. ■

EXAMPLE 3 Show that the set $S = \{t^2 + 1, t - 1, 2t + 2\}$ is a basis for the vector

Solution To do this, we must show that S spans V and is linearly independent. To show that it spans V, we take any vector in V, that is, a polynomial $at^2 +$ and must find constants a_1, a_2, and a_3 such that

$$\begin{aligned} at^2 + bt + c &= a_1(t^2 + 1) + a_2(t - 1) + a_3(2t + 2) \\ &= a_1t^2 + (a_2 + 2a_3)t + (a_1 - a_2 + 2a_3). \end{aligned}$$

Since two polynomials agree for all values of t only if the coefficients of respective powers of t agree, we get the linear system

$$\begin{aligned} a_1 \qquad\qquad &= a \\ a_2 + 2a_3 &= b \\ a_1 - a_2 + 2a_3 &= c. \end{aligned}$$

Solving, we have

$$a_1 = a, \quad a_2 = \frac{a + b - c}{2}, \quad a_3 = \frac{c + b - a}{4}.$$

Hence S spans V.

To illustrate this result, suppose that we are given the vector $2t^2 +$ Here $a = 2$, $b = 6$, and $c = 13$. Substituting in the foregoing expressions for a_1, a_2, and a_3, we find that

$$a_1 = 2, \quad a_2 = -\tfrac{5}{2}, \quad a_3 = \tfrac{17}{4}.$$

Hence

$$2t^2 + 6t + 13 = 2(t^2 + 1) - \tfrac{5}{2}(t - 1) + \tfrac{17}{4}(2t + 2).$$

To show that S is linearly independent, we form

$$a_1(t^2 + 1) + a_2(t - 1) + a_3(2t + 2) = 0.$$

Then

$$a_1t^2 + (a_2 + 2a_3)t + (a_1 - a_2 + 2a_3) = 0.$$

Again, this can hold for all values of t only if

$$\begin{aligned} a_1 \qquad\qquad &= 0 \\ a_2 + 2a_3 &= 0 \\ a_1 - a_2 + 2a_3 &= 0. \end{aligned}$$

The only solution to this homogeneous system is $a_1 = a_2 = a_3 =$ implies that S is linearly independent. Thus S is a basis for P_2.

The set of vectors $\{t^n, t^{n-1}, \ldots, t, 1\}$ forms a basis for the vector P_n called the **natural**, or **standard basis**, for P_n. It has already been in Example 5 of Section 6.3 that this set of vectors spans P_n. Linear independence is left as an exercise (Exercise T.15).

EXAMPLE 4 Find a basis for the subspace V of P_2, consisting of all vectors of $at^2 + bt + c$, where $c = a - b$.

Solution Every vector in V is of the form

$$at^2 + bt + a - b$$

which can be written as

$$a(t^2 + 1) + b(t - 1),$$

so the vectors $t^2 + 1$ and $t - 1$ span V. Moreover, these vectors are linearly independent because neither one is a multiple of the other. This conclusion could also be reached (with more work) by writing the equation

$$a_1(t^2 + 1) + a_2(t - 1) = 0$$

or

$$t^2 a_1 + t a_2 + (a_1 - a_2) = 0.$$

Since this equation is to hold for all values of t, we must have $a_1 = 0$ and $a_2 = 0$. ■

A vector space V is called **finite-dimensional** if there is a finite subset of V that is a basis for V. If there is no such finite subset of V, then V is called **infinite-dimensional**.

Almost all the vector spaces considered in this book are finite-dimensional. However, we point out that there are many infinite-dimensional vector spaces that are extremely important in mathematics and physics; their study lies beyond the scope of this book. The vector space P, consisting of all polynomials, and the vector space $C(-\infty, \infty)$ defined in Example 7 of Section 6.2 are not finite-dimensional.

We now establish some results about finite-dimensional vector spaces that will tell about the number of vectors in a basis, compare two different bases, and give properties of bases. First, we observe that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for a vector space V, then $\{c\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is also a basis when $c \neq 0$ (Exercise T.9). Thus a basis for a nonzero vector space is never unique.

THEOREM 6.5 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of the vectors in S.*

Proof First, every vector $\mathbf{v}$ in V can be written as a linear combination of the vectors in S because S spans V. Now let

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \tag{1}$$

and

$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n. \tag{2}$$

Subtracting (2) from (1), we obtain

$$0 = (c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + \cdots + (c_n - d_n)\mathbf{v}_n.$$

Since S is linearly independent, it follows that $c_i - d_i = 0$, $1 \leq i \leq n$, so $c_i = d_i$, $1 \leq i \leq n$. Hence there is only one way to express $\mathbf{v}$ as a linear combination of the vectors in S. ■

We can also show (Exercise T.11) that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a s nonzero vectors in a vector space V such that every vector in V can be w in one and only one way as a linear combination of the vectors in S, the a basis for V.

THEOREM 6.6 *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of nonzero vectors in a vector space V let W = span S. Then some subset of S is a basis for W.*

Proof **Case I.** If S is linearly independent, then since S already spans W, we c clude that S is a basis for W.

Case II. If S is linearly dependent, then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0},$$

where $c_1, c_2, \ldots, c_n$ are not all zero. Thus, some $\mathbf{v}_j$ is a linear combinati the preceding vectors in S (Theorem 6.4). We now delete $\mathbf{v}_j$ from S, ge a subset S_1 of S. Then, by the observation made just before Example 1 Section 6.3, we conclude that $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n\}$ also sp W.

If S_1 is linearly independent, then S_1 is a basis. If S_1 is linearly depende delete a vector of S_1 that is a linear combination of the preceding vector S_1 and get a new set S_2 which spans W. Continuing, since S is a finite set, will eventually find a subset T of S that is linearly independent and spans The set T is a basis for W.

Alternative Constructive Proof When V Is R^m, $n \geq m$. We take the vec in S as $m \times 1$ matrices and form Equation (3) above. This equation lead a homogeneous system in the n unknowns $c_1, c_2, \ldots, c_n$; the columns of $m \times n$ coefficient matrix A are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. We now transform A to a ma B in reduced row echelon form, having r nonzero rows, $1 \leq r \leq m$. With loss of generality, we may assume that the r leading 1's in the r nonzero r of B occur in the first r columns. Thus we have

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & b_{1\,r+1} & \cdots & b_{1n} \\ 0 & 1 & 0 & \cdots & 0 & b_{2\,r+1} & \cdots & b_{2n} \\ 0 & 0 & 1 & \cdots & 0 & b_{3\,r+1} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & b_{r\,r+1} & \cdots & b_{rn} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Solving for the unknowns corresponding to the leading 1's, we that $c_1, c_2, \ldots, c_r$ can be solved for in terms of the other unkno $c_{r+1}, c_{r+2}, \ldots, c_n$. Thus

$$\begin{aligned} c_1 &= -b_{1\,r+1}c_{r+1} - b_{1\,r+2}c_{r+2} - \cdots - b_{1n}c_n, \\ c_2 &= -b_{2\,r+1}c_{r+1} - b_{2\,r+2}c_{r+2} - \cdots - b_{2n}c_n, \\ &\vdots \\ c_r &= -b_{r\,r+1}c_{r+1} - b_{r\,r+2}c_{r+2} - \cdots - b_{rn}c_n, \end{aligned}$$

where $c_{r+1}, c_{r+2}, \ldots, c_n$ can be assigned arbitrary real values. Letting

$$c_{r+1} = 1, \quad c_{r+2} = 0, \ldots, \quad c_n = 0$$

in Equation (4) and using these values in Equation (3), we have

$$-b_{1\,r+1}\mathbf{v}_1 - b_{2\,r+1}\mathbf{v}_2 - \cdots - b_{r\,r+1}\mathbf{v}_r + \mathbf{v}_{r+1} = \mathbf{0},$$

which implies that $\mathbf{v}_{r+1}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. By the observation made just before Example 14 in Section 6.3, the set of vectors obtained from S by deleting $\mathbf{v}_{r+1}$ spans W. Similarly, letting

$$c_{r+1} = 0, \quad c_{r+2} = 1, \quad c_{r+3} = 0, \ldots, \quad c_n = 0,$$

we find that $\mathbf{v}_{r+2}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ and the set of vectors obtained from S by deleting $\mathbf{v}_{r+1}$ and $\mathbf{v}_{r+2}$ spans W. Continuing in this manner, $\mathbf{v}_{r+3}, \mathbf{v}_{r+4}, \ldots, \mathbf{v}_n$ are linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, so it follows that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ spans W.

We next show that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent. Consider the matrix B_D obtained by deleting from B all columns not containing a leading 1. In this case, B_D consists of the first r columns of B. Thus,

$$B_D = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \\ & & & \ddots & \vdots \\ 0 & 0 & & \cdots & 1 \\ 0 & 0 & & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & & \cdots & 0 \end{bmatrix}.$$

Let A_D be the matrix obtained from A by deleting the columns corresponding to the columns that were deleted in B to obtain B_D. In this case, the columns of A_D are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$, the first r columns of A. Since A and B are row equivalent, so are A_D and B_D. Then the homogeneous systems

$$A_D\mathbf{x} = \mathbf{0} \quad \text{and} \quad B_D\mathbf{x} = \mathbf{0}$$

are equivalent. Recall now that the homogeneous system $B_D\mathbf{x} = \mathbf{0}$ can be written equivalently as

$$x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots + x_r\mathbf{w}_r = \mathbf{0}, \tag{5}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix}$$

and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ are the columns of B_D. Since the columns of B_D form a linearly independent set of vectors in R^m, Equation (5) has only the trivial solution. Hence, $A_D\mathbf{x} = \mathbf{0}$ also has only the trivial solution. Thus the columns of A_D are linearly independent. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly independent. ■

The first proof of Theorem 6.6 leads to a simple procedure for finding a subset T of a set S so that T is a basis for span S.

> Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of nonzero vectors in a vector space V. The procedure for finding a subset of S that is a basis for $W = \text{span } S$ is as follows.
>
> ***Step 1.*** Form Equation (3),
>
> $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0},$$
>
> which we solve for $c_1, c_2, \ldots, c_n$. If these are all zero, then S is linearly independent and is then a basis for W.
>
> ***Step 2.*** If $c_1, c_2, \ldots, c_n$ are not all zero, then S is linearly dependent, so one of the vectors in S — say, $\mathbf{v}_j$ — is a linear combination of the preceding vectors in S. Delete $\mathbf{v}_j$ from S, getting the subset S_1, which also spans W.
>
> ***Step 3.*** Repeat Step 1, using S_1 instead of S. By repeatedly deleting vectors of S we obtain a subset T of S that spans W and is linearly independent. Thus T is a basis for W.

This procedure can be rather tedious, since *every time* we delete a vector from S, we must solve a linear system. In Section 6.6 we shall present a much more efficient procedure for finding a basis for $W = \text{span } S$, but the basis is *not* guaranteed to be a subset of S. In many cases this is not a cause for concern, since one basis for $W = \text{span } S$ is as good as any other basis. However, there are cases when the vectors in S have some special properties and we want the basis for $W = \text{span } S$ to have the same properties, so we want the basis to be a subset of S. If $V = R^m$, the alternative proof of Theorem 6.6 yields a very efficient procedure (see Example 5 below) for finding a basis for $W = \text{span } S$ consisting of vectors from S.

> Let $V = R^m$ and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of nonzero vectors in V. The procedure for finding a subset of S that is a basis for $W = \text{span } S$ is as follows.
>
> ***Step 1.*** Form Equation (3),
>
> $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$
>
> ***Step 2.*** Construct the augmented matrix associated with the homogeneous system of Equation (3), and transform it to reduced row echelon form.
>
> ***Step 3.*** The vectors corresponding to the columns containing the leading 1's form a basis for $W = \text{span } S$.

Recall that in the alternative proof of Theorem 6.6 we assumed without loss of generality that the r leading 1's in the r nonzero rows of B occur in the first r columns. Thus, if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_6\}$ and the leading 1's occur in columns 1, 3, and 4, then $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for span S.

Remark In Step 2 of the procedure above, it is sufficient to transform the augmented matrix to row echelon form (see Section 1.5).

EXAMPLE 5 Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ be a set of vectors in R^4, where

$$\mathbf{v}_1 = (1, 2, -2, 1), \qquad \mathbf{v}_2 = (-3, 0, -4, 3),$$
$$\mathbf{v}_3 = (2, 1, 1, -1), \quad \mathbf{v}_4 = (-3, 3, -9, 6), \quad \text{and} \quad \mathbf{v}_5 = (9, 3, 7, -6).$$

Find a subset of S that is a basis for $W = \operatorname{span} S$.

Solution ***Step 1.*** Form Equation (3),

$$c_1(1, 2, -2, 1) + c_2(-3, 0, -4, 3) + c_3(2, 1, 1, -1)$$
$$+ c_4(-3, 3, -9, 6) + c_5(9, 3, 7, -6) = (0, 0, 0, 0).$$

Step 2. Equating corresponding components, we obtain the homogeneous system

$$\begin{aligned} c_1 - 3c_2 + 2c_3 - 3c_4 + 9c_5 &= 0 \\ 2c_1 \qquad\quad + c_3 + 3c_4 + 3c_5 &= 0 \\ -2c_1 - 4c_2 + c_3 - 9c_4 + 7c_5 &= 0 \\ c_1 + 3c_2 - c_3 + 6c_4 - 6c_5 &= 0. \end{aligned}$$

The reduced row echelon form of the associated augmented matrix is (verify)

$$\left[\begin{array}{ccccc|c} 1 & 0 & \frac{1}{2} & \frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 3. The leading 1's appear in columns 1 and 2, so $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $W = \operatorname{span} S$. ■

Remark In the alternative proof of Theorem 6.6 when $V = R^n$, the order of the vectors in the original spanning set S determines which basis for W is obtained. If, for example, we consider Example 5, where $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$ with $\mathbf{w}_1 = \mathbf{v}_4$, $\mathbf{w}_2 = \mathbf{v}_3$, $\mathbf{w}_3 = \mathbf{v}_2$, $\mathbf{w}_4 = \mathbf{v}_1$, and $\mathbf{w}_5 = \mathbf{v}_5$, then the reduced row echelon form of the augmented matrix is (verify)

$$\left[\begin{array}{ccccc|c} 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & -1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

It now follows that $\{\mathbf{w}_1, \mathbf{w}_2\} = \{\mathbf{v}_4, \mathbf{v}_3\}$ is a basis for $W = \operatorname{span} S$.

We shall soon establish a main result (Corollary 6.1) of this section, which will tell us about the number of vectors in two different bases. First, observe that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then $\{c\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is also a basis if $c \neq 0$. Thus a vector space always has infinitely many bases.

THEOREM 6.7 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V and $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a linearly independent set of vectors in V, then $r \leq n$.*

Proof Let $T_1 = \{\mathbf{w}_1, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Since S spans V, so does T_1. Since $\mathbf{w}_1$ is combination of the vectors in S, we find that T_1 is linearly dependent by Theorem 6.4, some $\mathbf{v}_j$ is a linear combination of the preceding T_1. Delete that particular vector $\mathbf{v}_j$.

Let $S_1 = \{\mathbf{w}_1, \mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n\}$. Note that S_1 spans let $T_2 = \{\mathbf{w}_2, \mathbf{w}_1, \mathbf{v}_1, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n\}$. Then T_2 is linearly de and some vector in T_2 is a linear combination of the preceding vectors. Since T is linearly independent, this vector cannot be $\mathbf{w}_1$, so it is $\mathbf{v}$. Repeat this process over and over. If the $\mathbf{v}$ vectors are all eliminated we can run out of $\mathbf{w}$ vectors, then the resulting set of $\mathbf{w}$ vectors, a subset is linearly dependent, which implies that T is also linearly dependent. we have reached a contradiction, we conclude that the number r of $\mathbf{w}$ must be no greater than the number n of $\mathbf{v}$ vectors. That is, $r \leq n$.

COROLLARY 6.1 *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ are bases for a space, then $n = m$.*

Proof Since T is a linearly independent set of vectors, Theorem 6.7 implies $m \leq n$. Similarly, $n \leq m$ because S is linearly independent. Hence $n = m$.

Thus, although a vector space has many bases, we have just shown for a particular vector space V, all bases have the same number of vectors. can then make the following definition.

Dimension