can then make the following definition.

Dimension

DEFINITION The **dimension** of a nonzero vector space V is the number of vectors basis for V. We often write **dim** V for the dimension of V. Since the set linearly dependent, it is natural to say that the vector space $\{\mathbf{0}\}$ has dime **zero**.

EXAMPLE 6 The dimension of R^2 is 2; the dimension of R^3 is 3; and in general, the di sion of R^n is n.

EXAMPLE 7 The dimension of P_2 is 3; the dimension of P_3 is 4; and in general, the d sion of P_n is $n+1$.

It can be shown that all finite-dimensional vector spaces of the sa mension differ only in the nature of their elements; their algebraic pro are identical.

It can also be shown that if V is a finite-dimensional vector space, th ery nonzero subspace W of V has a finite basis and $\dim W \leq \dim V$ (Ex T.2).

EXAMPLE 8 The subspace W of R^4 considered in Example 5 has dimension 2.

We might also consider the subspaces of R^2 [recall that R^2 can be as the (x, y)-plane]. First, we have $\{\mathbf{0}\}$ and R^2, the trivial subspaces mension 0 and 2, respectively. Now the subspace V of R^2 spanned by $\mathbf{v} \neq \mathbf{0}$ is a one-dimensional subspace of R^2; V is represented by a line the origin. Thus the subspaces of R^2 are $\{\mathbf{0}\}$, R^2, [illegible] the lines thro origin. Similarly, we ask you to show (Exercise T.8) that the subspaces of R^3 are $\{0\}$, R^3, all lines through the origin, and all planes through the origin.

It can be shown that if a vector space V has dimension n, then any set of $n+1$ vectors in V is necessarily linearly dependent (Exercise T.3). Any set of more than n vectors in R^n is linearly dependent. Thus the four vectors in R^3 considered in Example 9 of Section 6.3 were shown to be linearly dependent. Also, if a vector space V is of dimension n, then no set of $n-1$ vectors in V can span V (Exercise T.4). Thus in Example 3 of Section 6.3, polynomials $p_1(t)$ and $p_2(t)$ do not span P_2, which is of dimension 3.

We now come to a theorem that we shall have occasion to use several times in constructing a basis containing a given set of linearly independent vectors. We shall leave the proof as an exercise (Exercise T.5). The example following the theorem completely imitates the proof.

THEOREM 6.8 *If S is a linearly independent set of vectors in a finite-dimensional vector space V, then there is a basis T for V, which contains S.* ■

Theorem 6.8 says that a linearly independent set of vectors in a vector space V can be extended to a basis for V.

EXAMPLE 9 Suppose that we wish to find a basis for R^4 that contains the vectors $\mathbf{v}_1 = (1, 0, 1, 0)$ and $\mathbf{v}_2 = (-1, 1, -1, 0)$.

We use Theorem 6.8 as follows. First, let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the natural basis for R^4, where

$$\mathbf{e}_1 = (1, 0, 0, 0), \qquad \mathbf{e}_2 = (0, 1, 0, 0), \qquad \mathbf{e}_3 = (0, 0, 1, 0),$$

and

$$\mathbf{e}_4 = (0, 0, 0, 1).$$

Form the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. Since $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ spans R^4, so does S. We now use the alternative proof of Theorem 6.6 to find a subset of S that is a basis for R^4. Thus, we form Equation (3),

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{e}_1 + c_4\mathbf{e}_2 + c_5\mathbf{e}_3 + c_6\mathbf{e}_4 = \mathbf{0},$$

which leads to the homogeneous system

$$\begin{aligned}
c_1 - c_2 + c_3 \qquad\qquad\qquad &= 0\\
c_2 \qquad + c_4 \qquad\quad &= 0\\
c_1 - c_2 \qquad\qquad + c_5 \quad &= 0\\
c_6 &= 0.
\end{aligned}$$

Transforming the augmented matrix to reduced row echelon form, we obtain (verify)

$$\left[\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right].$$

Since the leading 1's appear in columns 1, 2, 3, and 6, we conclude that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_4\}$ is a basis for R^4 containing $\mathbf{v}_1$ and $\mathbf{v}_2$. ■

From the definition of a basis, a set of vectors in a vector space is a basis for V if it spans V and is linearly independent. However, if we are given the additional information that the dimension of V is n, we need only verify one of the two conditions. This is the content of the following theorem.

THEOREM 6.9 *Let V be an n-dimensional vector space, and let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of n vectors in V.*

(a) If S is linearly independent, then it is a basis for V.

(b) If S spans V, then it is a basis for V.

Proof Exercise T.6.

As a particular application of Theorem 6.9, we have the following: To determine if a subset S of R^n is a basis for R^n, first count the number of elements in S. If S has n elements, we can use either part (a) or part (b) of Theorem 6.9 to determine whether S is or is not a basis. If S does not have n elements, it is not a basis for R^n. (Why?) The same line of reasoning applies to any vector space or subspace whose *dimension is known*.

EXAMPLE 10 In Example 5, W = span S is a subspace of R^4, so $\dim W \leq 4$. Since S contains five vectors, we conclude by Corollary 6.1 that S is not a basis for W. In Example 2, since $\dim R^4 = 4$ and the set S contains four vectors, it is possible for S to be a basis for R^4. If S is linearly independent *or* spans R^4, it is a basis; otherwise it is not a basis. Thus, we need only check one of the conditions in Theorem 6.9, not both.

6.4 Exercises

1. Which of the following sets of vectors are bases for R^2?
   (a) $\{(1, 3), (1, -1)\}$.
   (b) $\{(0, 0), (1, 2), (2, 4)\}$.
   (c) $\{(1, 2), (2, -3), (3, 2)\}$.
   (d) $\{(1, 3), (-2, 6)\}$.

2. Which of the following sets of vectors are bases for R^3?
   (a) $\{(1, 2, 0), (0, 1, -1)\}$.
   (b) $\{(1, 1, -1), (2, 3, 4), (4, 1, -1), (0, 1, -1)\}$.
   (c) $\{(3, 2, 2), (-1, 2, 1), (0, 1, 0)\}$.
   (d) $\{(1, 0, 0), (0, 2, -1), (3, 4, 1), (0, 1, 0)\}$.

3. Which of the following sets of vectors are bases for R^4?
   (a) $\{(1, 0, 0, 1), (0, 1, 0, 0), (1, 1, 1, 1), (0, 1, 1, 1)\}$.
   (b) $\{(1, -1, 0, 2), (3, -1, 2, 1), (1, 0, 0, 1)\}$.
   (c) $\{(-2, 4, 6, 4), (0, 1, 2, 0), (-1, 2, 3, 2), (-3, 2, 5, 6), (-2, -1, 0, 4)\}$.
   (d) $\{(0, 0, 1, 1), (-1, 1, 1, 2), (1, 1, 0, 0), (2, 1, 2, 1)\}$.

4. Which of the following sets of vectors are bases for P_2?
   (a) $\{-t^2 + t + 2, 2t^2 + 2t + 3, 4t^2 - 1\}$.
   (b) $\{t^2 + 2t - 1, 2t^2 + 3t - 2\}$.
   (c) $\{t^2 + 1, 3t^2 + 2t, 3t^2 + 2t + 1, 6t^2 + 6t + 3\}$
   (d) $\{3t^2 + 2t + 1, t^2 + t + 1, t^2 + 1\}$.

5. Which of the following sets of vectors are bases for
   (a) $\{t^3 + 2t^2 + 3t, 2t^3 + 1, 6t^3 + 8t^2 + 6t + 4, t^3 + 2t^2 + t + 1\}$.
   (b) $\{t^3 + t^2 + 1, t^3 - 1, t^3 + t^2 + t\}$.
   (c) $\{t^3 + t^2 + t + 1, t^3 + 2t^2 + t + 3, 2t^3 + t^2 + 3t + 2, t^3 + t^2 + 2t + 2\}$.
   (d) $\{t^3 - t, t^3 + t^2 + 1, t - 1\}$.

6. Show that the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   form a basis for the vector space M_{22}.

In Exercises 7 and 8, determine which of the given subsets forms a basis for R^3. Express the vector $(2, 1, 3)$ as a linear combination of the vectors in each subset that is a basis.

7. (a) $\{(1, 1, 1), (1, 2, 3), (0, 1, 0)\}$.
   (b) $\{(1, 2, 3), (2, 1, 3), (0, 0, 0)\}$.

(a) $\{(2, 1, 3), (1, 2, 1), (1, 1, 4), (1, 5, 1)\}$.

(b) $\{(1, 1, 2), (2, 2, 0), (3, 4, -1)\}$.

In Exercises 9 and 10, determine which of the given subsets form a basis for P_2. Express $5t^2 - 3t + 8$ as a linear combination of the vectors in each subset that is a basis.

(a) $\{t^2 + t, t - 1, t + 1\}$.

(b) $\{t^2 + 1, t - 1\}$.

(a) $\{t^2 + t, t^2, t^2 + 1\}$.

(b) $\{t^2 + 1, t^2 - t + 1\}$.

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = (1, 2, 2), \quad \mathbf{v}_2 = (3, 2, 1),$$
$$\mathbf{v}_3 = (11, 10, 7), \quad \text{and} \quad \mathbf{v}_4 = (7, 6, 4).$$

Find a basis for the subspace $W = \operatorname{span} S$ of R^3. What is $\dim W$?

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where

$$\mathbf{v}_1 = (1, 1, 0, -1), \quad \mathbf{v}_2 = (0, 1, 2, 1),$$
$$\mathbf{v}_3 = (1, 0, 1, -1), \quad \mathbf{v}_4 = (1, 1, -6, -3),$$

and $\mathbf{v}_5 = (-1, -5, 1, 0)$. Find a basis for the subspace $W = \operatorname{span} S$ of R^4. What is $\dim W$?

Consider the following subset of P_3:

$$S = \{t^3 + t^2 - 2t + 1, t^2 + 1, t^3 - 2t, 2t^3 + 3t^2 - 4t + 3\}.$$

Find a basis for the subspace $W = \operatorname{span} S$. What is $\dim W$?

Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\}.$$

Find a basis for the subspace $W = \operatorname{span} S$ of M_{22}.

Find a basis for M_{23}. What is the dimension of M_{23}? Generalize to M_{mn}.

Consider the following subset of the vector space of all real-valued functions

$$S = \{\cos^2 t, \sin^2 t, \cos 2t\}.$$

Find a basis for the subspace $W = \operatorname{span} S$. What is $\dim W$?

In Exercises 17 and 18, find a basis for the given subspaces of R^3 and R^4.

(a) All vectors of the form (a, b, c), where $b = a + c$.

(b) All vectors of the form (a, b, c), where $b = a$.

(c) All vectors of the form (a, b, c), where $2a + b - c = 0$.

(a) All vectors of the form (a, b, c), where $a = 0$.

(b) All vectors of the form $(a + c, a - b, b + c, -a + b)$.

(c) All vectors of the form (a, b, c), where $a - b + 5c = 0$.

In Exercises 19 and 20, find the dimensions of the given subspaces of R^4.

19. (a) All vectors of the form (a, b, c, d), where $d = a + b$.

(b) All vectors of the form (a, b, c, d), where $c = a - b$ and $d = a + b$.

20. (a) All vectors of the form (a, b, c, d), where $a = b$.

(b) All vectors of the form $(a + c, -a + b, -b - c, a + b + 2c)$.

21. Find a basis for the subspace of P_2 consisting of all vectors of the form $at^2 + bt + c$, where $c = 2a - 3b$.

22. Find a basis for the subspace of P_3 consisting of all vectors of the form $at^3 + bt^2 + ct + d$, where $a = b$ and $c = d$.

23. Find the dimensions of the subspaces of R^2 spanned by the vectors in Exercise 1.

24. Find the dimensions of the subspaces of R^3 spanned by the vectors in Exercise 2.

25. Find the dimensions of the subspaces of R^4 spanned by the vectors in Exercise 3.

26. Find the dimension of the subspace of P_2 consisting of all vectors of the form $at^2 + bt + c$, where $c = b - 2a$.

27. Find the dimension of the subspace of P_3 consisting of all vectors of the form $at^3 + bt^2 + ct + d$, where $b = 3a - 5d$ and $c = d + 4a$.

28. Find a basis for R^3 that includes the vectors

(a) $(1, 0, 2)$.

(b) $(1, 0, 2)$ and $(0, 1, 3)$.

29. Find a basis for R^4 that includes the vectors $(1, 0, 1, 0)$ and $(0, 1, -1, 0)$.

30. Find all values of a for which $\{(a^2, 0, 1), (0, a, 2), (1, 0, 1)\}$ is a basis for R^3.

31. Find a basis for the subspace W of M_{33} consisting of all symmetric matrices.

32. Find a basis for the subspace of M_{33} consisting of all diagonal matrices.

33. Give an example of a two-dimensional subspace of R^4.

34. Give an example of a two-dimensional subspace of P_3.

In Exercises 35 and 36, find a basis for the given plane.

35. $2x - 3y + 4z = 0$.

36. $x + y - 3z = 0$.

Theoretical Exercises

T.1. Suppose that in the nonzero vector space V, the largest number of vectors in a linearly independent set is m. Show that any set of m linearly independent vectors in V is a basis for V.

T.2. Show that if V is a finite-dimensional vector space, then every nonzero subspace W of V has a finite basis and $\dim W \leq \dim V$.

T.3. Show that if $\dim V = n$, then any $n + 1$ vectors in V are linearly dependent.

T.4. Show that if $\dim V = n$, then no set of $n - 1$ vectors in V can span V.

T.5. Prove Theorem 6.8.

T.6. Prove Theorem 6.9.

T.7. Show that if W is a subspace of a finite-dimensional vector space V and $\dim W = \dim V$, then $W = V$.

T.8. Show that the subspaces of R^3 are $\{\mathbf{0}\}$, R^3, all lines through the origin, and all planes through the origin.

T.9. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V and $c \neq 0$, then $\{c\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is also a basis for V.

T.10. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for vector space V. Then show that $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, where

$$\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3,$$
$$\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3,$$

and

$$\mathbf{w}_3 = \mathbf{v}_3,$$

is also a basis for V.

T.11. Let

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$$

be a set of nonzero vectors in a vector space V ... that every vector in V can be written in one and ... one way as a linear combination of the vectors in ... Show that S is a basis for V.

T.12. Suppose that

$$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$$

is a basis for R^n. Show that if A is an $n \times n$ nonsingular matrix, then

$$\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n\}$$

is also a basis for R^n. (*Hint*: See Exercise T.10 in Section 6.3.)

T.13. Suppose that

$$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$$

is a linearly independent set of vectors in R^n and let ... be a singular matrix. Prove or disprove that

$$\{A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n\}$$

is linearly independent.

T.14. Show that the vector space P of all polynomials is not finite-dimensional. [*Hint*: Suppose that

$$\{p_1(t), p_2(t), \dots, p_k(t)\}$$

is a finite basis for P. Let $d_j = \text{degree } p_j(t)$. Establish a contradiction.]

T.15. Show that the set of vectors $\{t^n, t^{n-1}, \dots, t, 1\}$ in P_n is linearly independent.

MATLAB Exercises

In order to use MATLAB in this section, you should have read Section 12.7. In the following exercises we relate the theory developed in the section to computational procedures in MATLAB that aid in analyzing the situation.

To determine if a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a basis for a vector space V, the definition requires that we show span $S = V$ and S is linearly independent. However, if we know that $\dim V = k$, then Theorem 6.9 implies that we need only show that either span $S = V$ or S is linearly independent. The linear independence, in this special case, is easily analyzed using MATLAB's **rref** *command. Construct the homogeneous system $A\mathbf{x} = \mathbf{0}$ associated with the linear independence/dependence question. Then S is linearly independent if and only if*

$$\mathbf{rref}(\mathbf{A}) = \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix}.$$

In Exercises ML.1 through ML.6, if this special case can be applied, do so; otherwise, determine if S is a basis for V in the conventional manner.

ML.1. $S = \{(1, 2, 1), (2, 1, 1), (2, 2, 1)\}$ in $V = R^3$.

ML.2. $S = \{2t - 2, t^2 - 3t + 1, 2t^2 - 8t + 4\}$ in $V = P_2$

ML.3. $S = \{(1, 1, 0, 0), (2, 1, 1, -1), (0, 0, 1, 1), (1, 2, 1, 2)\}$ in $V = R^4$.

ML.4. $S = \{(1, 2, 1, 0), (2, 1, 3, 1), (2, -2, 4, 2)\}$ in $V = \text{span } S$.

ML.5. $S = \{(1, 2, 1, 0), (2, 1, 3, 1), (2, 2, 1, 2)\}$ in $V = \text{span } S$.

ML.6. $V =$ the subspace of R^3 of all vectors of the form (a, b, c), where $b = 2a - c$ and $S = \{(0, 1, -1), (1, 1, 1)\}$.

In Exercises ML.7 through ML.9, use MATLAB's **rref** *command to determine a subset of S that is a basis for span S. See Example 5.*

ML.7. $S = \{(1, 1, 0, 0), (-2, -2, 0, 0), (1, 0, 2, 1), (2, 1, 2, 1), (0, 1, 1, 1)\}$.
What is dim span S? Does span $S = R^4$?