LINEAR INDEPENDENCE

Thus far we have defined a mathematical system called a real vector space and noted some of its properties. We further observe that the only real vector space having a finite number of vectors in it is the vector space whose only vector is $\mathbf{0}$, for if $\mathbf{v} \neq \mathbf{0}$ is in a vector space V, then by Exercise T.4 in Section 6.1, $c\mathbf{v} \neq c'\mathbf{v}$, where c and c' are distinct real numbers, and so V has infinitely many vectors in it. However, in this section and the following one we show that most vector spaces V studied here have a set composed of a finite number of vectors that completely describe V. It should be noted that, in general, there is more than one such set describing V. We now turn to a formulation of these ideas.

DEFINITION The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to **span** V if every vector in V is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Moreover, if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, then we also say that the set S **spans** V, or that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ **spans** V, or that V is **spanned by** S, or in the language of Section 6.2, span $S = V$.

The procedure to check if the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ span the vector space V is as follows.

Step 1. Choose an arbitrary vector $\mathbf{v}$ in V.

Step 2. Determine if $\mathbf{v}$ is a linear combination of the given vectors. If it is, then the given vectors span V. If it is not, they do not span V.

Again we investigate the consistency of a linear system, but this time for a right side that represents an arbitrary vector in a vector space V.

EXAMPLE 1 Let V be the vector space R^3 and let

$$\mathbf{v}_1 = (1, 2, 1), \quad \mathbf{v}_2 = (1, 0, 2), \quad \text{and} \quad \mathbf{v}_3 = (1, 1, 0).$$

Do $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ span V?

Solution ***Step 1.*** Let $\mathbf{v} = (a, b, c)$ be any vector in R^3, where a, b, and c are arbitrary real numbers.

Step 2. We must find out whether there are constants c_1, c_2, and c_3 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}.$$

This leads to the linear system (verify)

$$\begin{aligned} c_1 + c_2 + c_3 &= a \\ 2c_1 \qquad + c_3 &= b \\ c_1 + 2c_2 \qquad &= c. \end{aligned}$$

A solution is (verify)

$$c_1 = \frac{-2a + 2b + c}{3}, \qquad c_2 = \frac{a - b + c}{3}, \qquad c_3 = \frac{4a - b - 2c}{3}.$$

Since we have obtained a solution for every choice of a, b, and c, we conclude that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span R^3. This is equivalent to saying that span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = R^3$. ■

EXAMPLE 2 Show that

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

spans the subspace of M_{22} consisting of all symmetric matrices.

Solution *Step 1.* An arbitrary symmetric matrix has the form

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

where a, b, and c are any real numbers.

Step 2. We must find constants d_1, d_2, and d_3 such that

$$d_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = A = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

which leads to a linear system whose solution is (verify)

$$d_1 = a, \qquad d_2 = b, \qquad d_3 = c.$$

Since we have found a solution for every choice of a, b, and c, we c[onclude] that S spans the given subspace.

EXAMPLE 3 Let V be the vector space P_2. Let $S = \{p_1(t), p_2(t)\}$, where $p_1(t) = t$[...] and $p_2(t) = t^2 + 2$. Does S span P_2?

Solution *Step 1.* Let $p(t) = at^2 + bt + c$ be any polynomial in P_2, where a, [...] are any real numbers.

Step 2. We must find out whether there are constants c_1 and c_2 such [that]

$$p(t) = c_1 p_1(t) + c_2 p_2(t)$$

or

$$at^2 + bt + c = c_1(t^2 + 2t + 1) + c_2(t^2 + 2).$$

Thus

$$(c_1 + c_2)t^2 + (2c_1)t + (c_1 + 2c_2) = at^2 + bt + c.$$

Since two polynomials agree for all values of t only if the coefficie[nts of re]spective powers of t agree, we obtain the linear system

$$\begin{aligned} c_1 + c_2 &= a \\ 2c_1 \quad &= b \\ c_1 + 2c_2 &= c. \end{aligned}$$

Using elementary row operations on the augmented matrix of this [sys]tem, we obtain (verify)

$$\left[\begin{array}{cc|c} 1 & 0 & 2a - c \\ 0 & 1 & c - a \\ 0 & 0 & b - 4a + 2c \end{array}\right].$$

If $b - 4a + 2c \neq 0$, then the system is inconsistent and there is no [...] Hence $S = \{p_1(t), p_2(t)\}$ does not span P_2. For example, the p[...]

EXAMPLE 4 The vectors $\mathbf{e}_1 = \mathbf{i} = (1, 0)$ and $\mathbf{e}_2 = \mathbf{j} = (0, 1)$ span R^2, for as was observed in Section 4.1, if $\mathbf{u} = (u_1, u_2)$ is any vector in R^2, then $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2$. As was noted in Section 4.2, every vector $\mathbf{u}$ in R^3 can be written as a linear combination of the vectors $\mathbf{e}_1 = \mathbf{i} = (1, 0, 0)$, $\mathbf{e}_2 = \mathbf{j} = (0, 1, 0)$, and $\mathbf{e}_3 = \mathbf{k} = (0, 0, 1)$. Thus $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ span R^3. Similarly, the vectors $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\mathbf{e}_n = (0, 0, \ldots, 1)$ span R^n, since any vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in R^n can be written as

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \cdots + u_n\mathbf{e}_n.$$

■

EXAMPLE 5 The set $S = \{t^n, t^{n-1}, \ldots, t, 1\}$ spans P_n, since every polynomial in P_n is of the form

$$a_0t^n + a_1t^{n-1} + \cdots + a_{n-1}t + a_n,$$

which is a linear combination of the elements in S. ■

EXAMPLE 6 Consider the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \\ 4 & 4 & -1 & 9 \end{bmatrix}.$$

From Example 8 in Section 6.2, the set of all solutions to $A\mathbf{x} = \mathbf{0}$ forms a subspace of R^4. To determine a spanning set for the solution space of this homogeneous system, we find that the reduced row echelon form of the augmented matrix is (verify)

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution is then given by

$$\begin{aligned} x_1 &= -r - 2s \\ x_2 &= r \\ x_3 &= s \\ x_4 &= s, \end{aligned}$$

where r and s are any real numbers. In matrix form we have that any member of the solution space is given by

$$\mathbf{x} = r\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Hence the vectors $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ span the solution space.

Linear Independence

DEFINITION The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V are said to be **linearly dependent** if there exist constants $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Otherwise, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are called **linearly independent**. That is, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent if whenever $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, we have

$$c_1 = c_2 = \cdots = c_k = 0.$$

That is, the *only* linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ that yields the zero vector is that in which all the coefficients are zero. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, then we also say that the set S is **linearly dependent** or **linearly independent** if the vectors have the corresponding property defined above.

It should be emphasized that for any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, Equation (1) always holds if we choose all the scalars $c_1, c_2, \ldots, c_k$ equal to zero. The important point in this definition is whether or not it is possible to satisfy (1) with at least one of the scalars different from zero.

> The procedure to determine if the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent or linearly independent is as follows.
>
> *Step 1.* Form Equation (1), which leads to a homogeneous system.
>
> *Step 2.* If the homogeneous system obtained in Step 1 has only the trivial solution, then the given vectors are linearly independent; if it has a nontrivial solution, then the vectors are linearly dependent.

EXAMPLE 7 Determine whether the vectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

found in Example 6 as spanning the solution space of $A\mathbf{x} = \mathbf{0}$ are linearly dependent or linearly independent.

Solution Forming Equation (1),

$$c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

we obtain the homogeneous system

$$\begin{aligned} -c_1 - 2c_2 &= 0 \\ c_1 + 0c_2 &= 0 \\ 0c_1 + c_2 &= 0 \\ 0c_1 + c_2 &= 0, \end{aligned}$$

whose only solution is $c_1 = c_2 = 0$. Hence the given vectors are linearly independent.

EXAMPLE 8 Are the vectors $\mathbf{v}_1 = (1, 0, 1, 2)$, $\mathbf{v}_2 = (0, 1, 1, 2)$, and $\mathbf{v}_3 = (1, 1, 1, 3)$ in R^4 linearly dependent or linearly independent?

Solution We form Equation (1),

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0},$$

and solve for c_1, c_2, and c_3. The resulting homogeneous system is (verify)

$$\begin{aligned} c_1 \qquad\quad + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_1 + c_2 + c_3 &= 0 \\ 2c_1 + 2c_2 + 3c_3 &= 0, \end{aligned}$$

which has as its only solution $c_1 = c_2 = c_3 = 0$ (verify), showing that the given vectors are linearly independent. ■

EXAMPLE 9 Consider the vectors

$$\mathbf{v}_1 = (1, 2, -1), \qquad \mathbf{v}_2 = (1, -2, 1), \qquad \mathbf{v}_3 = (-3, 2, -1),$$

and

$$\mathbf{v}_4 = (2, 0, 0) \quad \text{in } R^3.$$

Is $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ linearly dependent or linearly independent?

Solution Setting up Equation (1), we are led to the homogeneous system (verify)

$$\begin{aligned} c_1 + c_2 - 3c_3 + 2c_4 &= 0 \\ 2c_1 - 2c_2 + 2c_3 \qquad &= 0 \\ -c_1 + c_2 - c_3 \qquad &= 0, \end{aligned}$$

a homogeneous system of three equations in four unknowns. By Theorem 1.8, Section 1.5, we are assured of the existence of a nontrivial solution. Hence, S is linearly dependent. In fact, two of the infinitely many solutions are

$$\begin{aligned} c_1 &= 1, & c_2 &= 2, & c_3 &= 1, & c_4 &= 0; \\ c_1 &= 1, & c_2 &= 1, & c_3 &= 0, & c_4 &= -1. \end{aligned}$$ ■

EXAMPLE 10 The vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in R^2, defined in Example 4, are linearly independent, since

$$c_1(1, 0) + c_2(0, 1) = (0, 0)$$

can hold only if $c_1 = c_2 = 0$. Similarly, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ in R^3, and more generally, the vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ in R^n are linearly independent (Exercise T.1). ■

Corollary 6.4 in Section 6.6, to follow, gives another way of testing whether n given vectors in R^n are linearly dependent or linearly independent. We form the matrix A, whose columns are the given n vectors. Then the given vectors are linearly independent if and only if $\det(A) \neq 0$. Thus, in Example 10,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\det(A) = 1$ so that $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent.

EXAMPLE 11 Consider the vectors

$$p_1(t) = t^2 + t + 2, \qquad p_2(t) = 2t^2 + t, \qquad p_3(t) = 3t^2 + 2t + 2.$$

To find out whether $S = \{p_1(t), p_2(t), p_3(t)\}$ is linearly dependent or linearly independent, we set up Equation (1) and solve for c_1, c_2, and c_3. The resulting homogeneous system is (verify)

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ c_1 + c_2 + 2c_3 &= 0 \\ 2c_1 \qquad\quad + 2c_3 &= 0, \end{aligned}$$

which has infinitely many solutions (verify). A particular solution is $c_1 = 1$, $c_2 = 1$, $c_3 = -1$, so

$$p_1(t) + p_2(t) - p_3(t) = 0.$$

Hence S is linearly dependent.

EXAMPLE 12 If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are k vectors in any vector space and $\mathbf{v}_i$ is the zero vector, Equation (1) holds by letting $c_i = 1$ and $c_j = 0$ for $j \neq i$. Thus $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent. *Hence every set of vectors containing the zero vector is linearly dependent.*

Let S_1 and S_2 be finite subsets of a vector space and let S_1 be a subset of S_2. Then (a) if S_1 is linearly dependent, so is S_2; and (b) if S_2 is linearly independent, so is S_1 (Exercise T.2).

We consider next the meaning of linear independence in R^2 and R^3. Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent in R^2. Then there exist scalars c_1 and c_2, not both zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}.$$

If $c_1 \neq 0$, then

$$\mathbf{v}_1 = \left(-\frac{c_2}{c_1}\right)\mathbf{v}_2.$$

If $c_2 \neq 0$, then

$$\mathbf{v}_2 = \left(-\frac{c_1}{c_2}\right)\mathbf{v}_1.$$

Thus one of the vectors is a scalar multiple of the other. Conversely, suppose that $\mathbf{v}_1 = c\mathbf{v}_2$. Then

$$1\mathbf{v}_1 - c\mathbf{v}_2 = \mathbf{0},$$

and since the coefficients of $\mathbf{v}_1$ and $\mathbf{v}_2$ are not both zero, it follows that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent. Thus $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent in R^2 if and only if one of the vectors is a multiple of the other. Hence two vectors in R^2 are linearly dependent if and only if they both lie on the same line passing through the origin [Figure 6.4(a)].

Suppose now that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent in R^3. Then we can write

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0},$$

where c_1, c_2, and c_3 are not all zero, say $c_2 \neq 0$. Then

$$\mathbf{v}_2 = \left(-\frac{c_1}{c_2}\right)\mathbf{v}_1 - \left(\frac{c_3}{c_2}\right)\mathbf{v}_3,$$

Figure 6.4 ►

(a) Linearly dependent vectors in R^2. (b) Linearly independent vectors in R^2.

which means that $\mathbf{v}_2$ is in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_3$.

Now W is either a plane through the origin (when $\mathbf{v}_1$ and $\mathbf{v}_3$ are linearly independent), or a line through the origin (when $\mathbf{v}_1$ and $\mathbf{v}_3$ are linearly dependent), or the origin (when $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3 = \mathbf{0}$). Since a line through the origin always lies in a plane through the origin, we conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ all lie in the same plane through the origin. Conversely, suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ all lie in the same plane through the origin. Then either all three vectors are the zero vector, or all three vectors lie on the same line through the origin, or all three vectors lie in a plane through the origin spanned by two vectors, say $\mathbf{v}_1$ and $\mathbf{v}_3$. Thus, in all these cases, $\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$:

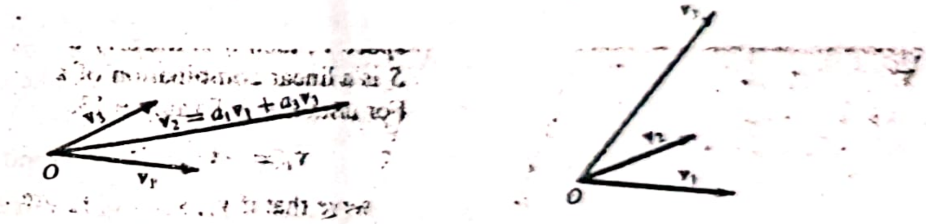
$$\mathbf{v}_2 = a_1\mathbf{v}_1 + a_3\mathbf{v}_3.$$

Then

$$a_1\mathbf{v}_1 - 1\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{0},$$

which means that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent. Hence three vectors in R^3 are linearly dependent if and only if they all lie in the same plane passing through the origin [Figure 6.5(a)].

Figure 6.5 ►



(a) Linearly dependent vectors in R^3. (b) Linearly independent vectors in R^3.

More generally, let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in a vector space V. We can show (Exercise T.13) that $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent if and only if there is a scalar k such that $\mathbf{v} = k\mathbf{u}$. Equivalently, $\mathbf{u}$ and $\mathbf{v}$ are linearly independent if and only if neither vector is a multiple of the other. This approach will not work with sets having three or more vectors. Instead, we use the result given by the following theorem.

THEOREM 6.4 *The nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in a vector space V are linearly dependent if and only if one of the vectors $\mathbf{v}_j$, $j \geq 2$, is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.*

Proof If $\mathbf{v}_j$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$,

$$\mathbf{v}_j = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{j-1}\mathbf{v}_{j-1},$$

then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{j-1}\mathbf{v}_{j-1} + (-1)\mathbf{v}_j + 0\mathbf{v}_{j+1} + \cdots + 0\mathbf{v}_n = \mathbf{0}.$$

Since at least one coefficient, -1, is nonzero, we conclude that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent.

Conversely, suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent. Then there exist scalars $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

Now let j be the largest subscript for which $c_j \neq 0$. If $j > 1$, then

$$\mathbf{v}_j = -\left(\frac{c_1}{c_j}\right)\mathbf{v}_1 - \left(\frac{c_2}{c_j}\right)\mathbf{v}_2 - \cdots - \left(\frac{c_{j-1}}{c_j}\right)\mathbf{v}_{j-1}.$$

If $j = 1$, then $c_1\mathbf{v}_1 = \mathbf{0}$, which implies that $\mathbf{v}_1 = \mathbf{0}$, a contradiction to the hypothesis that none of the vectors are the zero vector. Thus one of the vectors $\mathbf{v}_j$ is a linear combination of the preceding vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}$.

EXAMPLE 13 If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$ are as in Example 9, then we find (verify) that

$$\mathbf{v}_1 + \mathbf{v}_2 + 0\mathbf{v}_3 - \mathbf{v}_4 = \mathbf{0},$$

so $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$ are linearly dependent. We then have

$$\mathbf{v}_4 = \mathbf{v}_1 + \mathbf{v}_2.$$

Remarks

1. We observe that Theorem 6.4 does not say that *every* vector $\mathbf{v}$ is a linear combination of the preceding vectors. Thus, in Example 9, we also have the equation $\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 + 0\mathbf{v}_4 = \mathbf{0}$. We cannot solve, in this equation, for $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$, since its coefficient is zero.
2. We can also prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then S is linearly dependent if and only if one of the vectors in S is a linear combination of all the other vectors in S (see Exercise). For instance, in Example 13,
$$\mathbf{v}_1 = -\mathbf{v}_2 - 0\mathbf{v}_3 + \mathbf{v}_4 \quad \text{and} \quad \mathbf{v}_2 = -\tfrac{1}{2}\mathbf{v}_1 - \tfrac{1}{2}\mathbf{v}_3 - 0\mathbf{v}_4.$$
3. Observe that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent vectors in a vector space, then they must be distinct and none can be the zero vector.

The following result will be used in Section 6.4 as well as in several other places. Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans a vector space V and $\mathbf{v}_j$ is a linear combination of the preceding vectors in S. Then the set

$$S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n\},$$

consisting of S with $\mathbf{v}_j$ deleted, also spans V. To show this result, observe that if $\mathbf{v}$ is any vector in V, then, since S spans V, we can find scalars $a_1, a_2, \ldots, a_n$ such that

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{j-1}\mathbf{v}_{j-1} + a_j\mathbf{v}_j + a_{j+1}\mathbf{v}_{j+1} + \cdots + a_n\mathbf{v}_n.$$

Now if

$$\mathbf{v}_j = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_{j-1}\mathbf{v}_{j-1},$$

then

$$\begin{aligned}\mathbf{v} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_{j-1}\mathbf{v}_{j-1} + a_j(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_{j-1}\mathbf{v}_{j-1}) \\ &\quad + a_{j+1}\mathbf{v}_{j+1} + \cdots + a_n\mathbf{v}_n \\ &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{j-1}\mathbf{v}_{j-1} + c_{j+1}\mathbf{v}_{j+1} + \cdots + c_n\mathbf{v}_n,\end{aligned}$$

which means that span $S_1 = V$.

EXAMPLE 14 Consider the set of vectors $S = \{v_1, v_2, v_3, v_4\}$ in R^4, where

$$v_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad v_2 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \quad \text{and} \quad v_4 = \begin{bmatrix}2\\1\\1\\0\end{bmatrix}.$$

and let $W = \text{span } S$. Since $v_4 = v_1 + v_2$, we conclude that $W = \text{span } S_1$, where $S_1 = \{v_1, v_2, v_3\}$. ■

Exercises

Which of the following vectors span R^2?

(a) (1, 2), (−1, 1).

(b) (0, 0), (1, 1), (−2, −2).

(c) (1, 3), (2, −3), (0, 2).

(d) (2, 4), (−1, 2).

Which of the following sets of vectors span R^3?

(a) {(1, −1, 2), (0, 1, 1)},

(b) {(1, 2, −1), (6, 3, 0), (4, −1, 2), (2, −5, 4)}.

(c) {(2, 2, 3), (−1, −2, 1), (0, 1, 0)}.

(d) {(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)}.

Which of the following vectors span R^4?

(a) (1, 0, 0, 1), (0, 1, 0, 0), (1, 1, 1, 1), (1, 1, 1, 0).

(b) (1, 2, 1, 0), (1, 1, −1, 0), (0, 0, 0, 1).

(c) (6, 4, −2, 4), (2, 0, 0, 1), (3, 2, −1, 2), (5, 6, −3, 2), (0, 4, −2, −1).

(d) (1, 1, 0, 0), (1, 2, −1, 1), (0, 0, 1, 1), (2, 1, 2, 1).

Which of the following sets of polynomials span P_2?

(a) $\{t^2 + 1, t^2 + t, t + 1\}$.

(b) $\{t^2 + 1, t - 1, t^2 + t\}$.

(c) $\{t^2 + 2, 2t^2 - t + 1, t + 2, t^2 + t + 4\}$.

(d) $\{t^2 + 2t - 1, t^2 - 1\}$.

Do the polynomials $t^3 + 2t + 1$, $t^2 - t + 2$, $t^3 + 2$, $-t^3 + t^2 - 5t + 2$ span P_3?

Find a set of vectors spanning the solution space of $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix}1 & 0 & 1 & 0\\1 & 2 & 3 & 1\\2 & 1 & 3 & 1\\1 & 1 & 2 & 1\end{bmatrix}.$$

Find a set of vectors spanning the null space of

$$A = \begin{bmatrix}1 & 1 & 2 & -1\\2 & 3 & 6 & -2\\-2 & 1 & 2 & 2\\0 & -2 & -4 & 0\end{bmatrix}.$$

8. Let

$$x_1 = \begin{bmatrix}2\\-1\\1\end{bmatrix}, \quad x_2 = \begin{bmatrix}4\\-7\\-1\end{bmatrix}, \quad x_3 = \begin{bmatrix}1\\2\\2\end{bmatrix}$$

belong to the solution space of $A\mathbf{x} = \mathbf{0}$. Is $\{x_1, x_2, x_3\}$ linearly independent?

9. Let

$$x_1 = \begin{bmatrix}1\\2\\0\\1\end{bmatrix}, \quad x_2 = \begin{bmatrix}1\\0\\-1\\1\end{bmatrix}, \quad x_3 = \begin{bmatrix}1\\6\\2\\0\end{bmatrix}$$

belong to the null space of A. Is $\{x_1, x_2, x_3\}$ linearly independent?

10. Which of the following sets of vectors in R^3 are linearly dependent? For those that are, express one vector as a linear combination of the rest.

(a) {(1, 2, −1), (3, 2, 5)}.

(b) {(4, 2, 1), (2, 6, −5), (1, −2, 3)}.

(c) {(1, 1, 0), (0, 2, 3), (1, 2, 3), (3, 6, 6)}.

(d) {(1, 2, 3), (1, 1, 1), (1, 0, 1)}.

11. Consider the vector space R^4. Follow the directions of Exercise 10.

(a) {(1, 1, 2, 1), (1, 0, 0, 2), (4, 6, 8, 6), (0, 3, 2, 1)}.

(b) {(1, −2, 3, −1), (−2, 4, −6, 2)}.

(c) {(1, 1, 1, 1), (2, 3, 1, 2), (3, 1, 2, 1), (2, 2, 1, 1)}.

(d) {(4, 2, −1, 3), (6, 5, −5, 1), (2, −1, 3, 5)}.

12. Consider the vector space P_2. Follow the directions of Exercise 10.

(a) $\{t^2 + 1, t - 2, t + 3\}$.

(b) $\{2t^2 + 1, t^2 + 3, t\}$.

(c) $\{3t + 1, 3t^2 + 1, 2t^2 + t + 1\}$.

(d) $\{t^2 - 4, 5t^2 - 5t - 6, 3t^2 - 5t + 2\}$.

13. Consider the vector space M_{22}. Follow the directions of Exercise 10.