CONTRACTION OF TENSORS

The process of setting two indices in a tensor equal and summing over that repeated index is called contraction. For example, the three possible contractions of a third order tensor A_{ijk} are A_{iik}, A_{iji} and A_{ijj}.

CONTRACTION THEOREM

THEOREM (7.11): The contraction of a tensor of order n ($n \geq 2$) leads to a tensor of order n − 2.

PROOF: We prove this theorem for $n = 3$ i.e. for the tensor A_{ijk}.

By hypotheses, A_{ijk} is a tensor of order 3. Therefore,

$$A'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk}A_{ijk} \qquad (1)$$

Let us contract w.r.t. j and k. Place the corresponding indices q and r equal to each other and sum over this index. Then

$$A'_{pqq} = \ell_{pi}\ell_{qj}\ell_{qk}A_{ijk}$$

$$= \ell_{pi}\delta_{jk}A_{ijk} = \ell_{pi}A_{ijj}$$

which shows that $B_i = A_{ijj}$ is a tensor of rank 1 i.e. a vector.

NOTE: (i) We have seen that contraction can be applied to a tensor of rank 2 or higher.

(ii) We know that contraction of a tensor of order n ($n \geq 2$) leads to a tensor of order n − 2. This tensor of order (n − 2) can then be contracted again (provided that $n \geq 4$), giving a tensor of order n − 4, and so on, until we obtain a tensor of order less than 2. In fact, repeated contraction of a tensor of order n eventually gives a scalar if n is even and a vector if n is odd.

(INNER) MULTIPLICATION OF TENSORS

The process of multiplying tensors (outer multiplication) and then contracting the product w.r.t. indices belonging to different factors is called inner multiplication and the result is called an inner product of the given tensors. For example the expression $A_i B_{jk}$ is the inner product of the tensors A_i and B_{jk}. Similarly $A_i B_i$ is the inner product of two vectors A_i and B_j (i.e. $\vec{A}$ and $\vec{B}$).

EXAMPLE (18): If A_{ijk} and B_{mn} are two tensors of rank 3 and 2 respectively, show that their inner product $A_{ijk}B_{in}$ is a tensor of rank $3 + 2 - 2 = 3$.

SOLUTION: Since A_{ijk} and B_{mn} are tensors, their equations of transformation from the system K to K′ are

$$A'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk}A_{ijk} \qquad (1)$$

$$B'_{st} = \ell_{sm}\ell_{tn}B_{mn} \qquad (2)$$

Multiplying equations (1) and (2), we get

$$A'_{pqr}B'_{st} = \ell_{pi}\ell_{qj}\ell_{rk}\ell_{sm}\ell_{tn}A_{ijk}B_{mn} \qquad (3)$$

ontraction (letting $p = s$ in equation (3) and summing), we get

$$A'_{pqr} B'_{pt} = \ell_{pi} \ell_{qj} \ell_{rk} \ell_{pm} \ell_{tn} A_{ijk} B_{mn}$$

$$= \ell_{qj} \ell_{rk} \ell_{tn} (\ell_{pi} \ell_{pm}) A_{ijk} B_{mn}$$

$$= \ell_{qj} \ell_{rk} \ell_{tn} \delta_{im} A_{ijk} B_{mn}$$

$$= \ell_{qj} \ell_{rk} \ell_{tn} A_{ijk} B_{in}$$

n shows that $C_{jkn} = A_{ijk} B_{in}$ called the inner product of A_{ijk} and B_{mn} is a tensor of rank 3

ontracting w.r.t. j and n or k and m in the product $A_{ijk} B_{mn}$, we can similarly show that any product is a tensor of rank 3 .

OLLARY: The inner product $A_i B_i$ of two vectors A_i and B_j (i.e. $\vec{A}$ and $\vec{B}$) is a tensor of zero i.e. a scalar . For this reason $A_i B_i$ is called the scalar or dot product of $\vec{A}$ and $\vec{B}$.

GENERALIZATION

If $A_{i_1 i_2 \cdots i_m}$ and $B_{j_1 j_2 \cdots j_n}$ are two tensors of rank m and n respectively, then any of inner products is a tensor of rank $m + n - 2$.

MULTIPLICATION OF A TENSOR BY A SCALAR

The multiplication of a tensor of any rank by a scalar yields another tensor of the same rank.

EXAMPLE (16): **Prove that if ϕ is a scalar and A_{ij} is a second order tensor, then $C_{ij} = \phi A_{ij}$ is also a second order tensor.**

SOLUTION: Since A_{ij} is a second order tensor, its equation of transformation from the system K to K$'$ is

$$A'_{mn} = \ell_{mi} \ell_{nj} A_{ij} \tag{1}$$

Multiplying equation (1) by the scalar ϕ, we get

$$\phi A'_{mn} = \ell_{mi} \ell_{nj} (\phi A_{ij})$$

or $$C'_{mn} = \ell_{mi} \ell_{nj} C_{ij} \tag{2}$$

where $C'_{mn} = \phi A'_{mn}$

Equation (2) shows that $C_{ij} = \phi A_{ij}$ is also a second order tensor.

In general, multiplication of a tensor of order n by a scalar gives another tensor of order n.

(OUTER) MULTIPLICATION OF TENSORS

The product of two or more tensors is the tensor whose components are the product of the components of the given tensors. The order of a tensor product is clearly the sum of the orders of the given tensors.

EXAMPLE (17): **If A_{ijk} and B_{mn} are two Cartesian tensors of rank 3 and 2 respectively, prove that $C_{ijkmn} = A_{ijk} B_{mn}$ is also a tensor of rank 5.**

SOLUTION: Since A_{ijk} and B_{mn} are tensors, their equations of transformation from the systems K to K$'$ are

$$A'_{pqr} = \ell_{pi} \ell_{qj} \ell_{rk} A_{ijk} \tag{1}$$

$$B'_{st} = \ell_{sm} \ell_{tn} B_{mn} \tag{2}$$

Multiplying equations (1) and (2), we get

$$A'_{pqr} B'_{st} = \ell_{pi} \ell_{qj} \ell_{rk} \ell_{sm} \ell_{tn} A_{ijk} B_{mn}$$

or $$C'_{pqrst} = \ell_{pi} \ell_{qj} \ell_{rk} \ell_{sm} \ell_{tn} C_{ijkmn} \tag{3}$$

where $C'_{pqrst} = A'_{pqr} B'_{st}$

which shows that $C_{ijkmn} = A_{ijk} B_{mn}$ called the outer product of A_{ijk} and B_{mn} is a tensor of rank $3 + 2 = 5$.

NOTE: (i) The tensor multiplication is non – commutative. For example,

$$C_{ijkmn} = A_{ijk} B_{mn} \neq B_{mn} A_{ijk} = C_{mnijk}$$

(ii) The outer product of two vectors i.e. tensors of the first order, is sometimes called a dyadic tensor or just dyad.

$$A_m B_n C_p = \ell_{mi}\ell_{nj}\ell_{pk} A_i B_j C_k \qquad (5)$$

or $$D'_{mnp} = \ell_{mi}\ell_{nj}\ell_{pk} D_{ijk} \qquad (6)$$

where $D'_{mnp} = A'_m B'_n C'_p$

Equation (6) shows that $D_{ijk} = A_i B_j C_k$ are the components of a third order tensor.

If we set $n = p$ in equation (6), we have

$$D'_{mnn} = \ell_{mi}\ell_{nj}\ell_{nk} D_{ijk}$$

or $$D'_{mnn} = \ell_{mi}\delta_{jk} D_{ijk} = \ell_{mi} D_{ijj}$$

which shows that $D_{ijj} = A_i B_j C_j$ are the components of a first order tensor.

THEOREM (7.10): Prove that the alternating symbol ϵ_{ijk} is a Cartesian tensor of rank 3.

PROOF: Let ϵ_{ijk} and ϵ'_{pqr} be the components of the alternating symbol in the systems and K' respectively. Then

$$\left.\begin{array}{l}\hat{e}_2 \times \hat{e}_3 = \hat{e}_1, \quad \hat{e}_3 \times \hat{e}_1 = \hat{e}_2, \quad \hat{e}_1 \times \hat{e}_2 = \hat{e}_3 \\ \hat{e}_2' \times \hat{e}_3' = \hat{e}_1', \quad \hat{e}_3' \times \hat{e}_1' = \hat{e}_2', \quad \hat{e}_1' \times \hat{e}_2' = \hat{e}_3'\end{array}\right\} \qquad (1)$$

Using the definition of the alternating symbol we can write equations (1) as

$$\left.\begin{array}{l}\hat{e}_i \times \hat{e}_j \cdot \hat{e}_k = \epsilon_{ijk} \\ \hat{e}_p' \times \hat{e}_q' \cdot \hat{e}_r' = \epsilon'_{pqr}\end{array}\right\} \qquad (2)$$

where $(i, j, k, p, q, r = 1, 2, 3)$.

Now $\hat{e}_p' = \ell_{pi}\hat{e}_i$, $\hat{e}_q' = \ell_{qj}\hat{e}_j$, and $\hat{e}_r' = \ell_{rk}\hat{e}_k$ therefore,

$$\epsilon'_{pqr} = \hat{e}_p' \times \hat{e}_q' \cdot \hat{e}_r'$$

$$= (\ell_{pi}\hat{e}_i) \times (\ell_{qj}\hat{e}_j) \cdot (\ell_{rk}\hat{e}_k) = \ell_{pi}\ell_{qj}\ell_{rk}\hat{e}_i \times \hat{e}_j \cdot \hat{e}_k$$

or $$\epsilon'_{pqr} = \ell_{pi}\ell_{qj}\ell_{rk}\epsilon_{ijk} \qquad (3)$$

From equation (3) it is clear that ϵ_{ijk} is a Cartesian tensor of rank 3.

HIGHER ORDER TENSORS

A quantity representable by a set of n suffixes $A_{i_1 i_2 \dots i_n}$ of 3^n numbers (called components) relatively to a coordinate system K is said to be a tensor of order (rank) n, if its components transform under changes of the coordinate system according to the law

$$A'_{j_1 j_2 \dots j_n} = \ell_{j_1 i_1}\ell_{j_2 i_2} \dots \ell_{j_n i_n} A_{i_1 i_2 \dots i_n} \qquad (1)$$

where $A'_{j_1 j_2 \dots j_n}$ are the components of the quantity in the coordinate system K' and $\ell_{j_1 i_1}$, etc. have the usual meanings.

PROBLEM (19): A_{ijk} is a third – order tensor , such that $A_{111} = A_{222} = 1$, $A_{212} = -2$, all other components being zero . Evaluate the components of the vector A_{iji}. The same transformation is made from $Ox_1x_2x_3$ to $Ox'_1x'_2x'_3$ as in problem (18). Evaluate the component A'_{123} of the tensor in the system $Ox'_1x'_2x'_3$.

SOLUTION: Since A_{ijk} is a third order tensor , therefore contracting w.r.t. i and k, i.e. A_{iji} is a first order tensor i.e. vector . We are given that $A_{111} = A_{222} = 1$, $A_{212} = -2$, while all other components are zero .

Let $\quad B_j = A_{iji} \quad (j = 1, 2, 3)$

then
$$B_1 = A_{i1i} = A_{111} + A_{212} + A_{313} = 1 - 2 + 0 = -1$$
$$B_2 = A_{i2i} = A_{121} + A_{222} + A_{323} = 0 + 1 + 0 = 1$$
$$B_3 = A_{i3i} = A_{131} + A_{232} + A_{333} = 0 + 0 + 0 = 0$$

Thus the components of $B_j = A_{iji}$ are $(-1, 1, 0)$. Now we know that

$$\begin{aligned} A'_{123} &= \ell_{1i}\ell_{2j}\ell_{3k}A_{ijk} \\ &= \ell_{11}\ell_{21}\ell_{31}A_{111} + \ell_{12}\ell_{22}\ell_{32}A_{222} + \ell_{12}\ell_{21}\ell_{32}A_{212} \\ &= \left(-\frac{3}{7}\right)\left(-\frac{2}{7}\right)\left(\frac{6}{7}\right) + \left(-\frac{6}{7}\right)\left(\frac{3}{7}\right)\left(-\frac{2}{7}\right) + \left(-\frac{6}{7}\right)\left(-\frac{2}{7}\right)\left(-\frac{2}{7}\right)(-2) \\ &= \frac{36}{343} + \frac{36}{343} + \frac{48}{343} = \frac{120}{343} \end{aligned}$$

PROBLEM (20): For the second – order tensor A_{ij}, show that the quantities

(i) A_{ii} (ii) $A_{ij}A_{ji}$ (iii) $A_{ij}A_{jk}A_{ki}$

are invariant under an orthogonal transformation .

SOLUTION: (i) Since A_{ij} is a second order tensor, therefore $A'_{mn} = \ell_{mi}\ell_{nj}A_{ij}$

If $m = n$, then

$$A'_{mm} = \ell_{mi}\ell_{mj}A_{ij} = \delta_{ij}A_{ij} = A_{ii}$$

or

$$A'_{11} + A'_{22} + A'_{33} = A_{11} + A_{22} + A_{33}$$

showing that A_{ii} is invariant. This result says that the trace of the matrix $[A_{ij}]$ i.e. the sum of its diagonal elements remains invariant under orthogonal transformation. In the mathematical theory of elasticity, the stress tensor σ_{ij} arises and the above result means that the sum of the direct stresses $\sigma_{11} + \sigma_{22} + \sigma_{33}$ is invariant under an orthogonal transformation of axes.

(ii) Since $A'_{mn} = \ell_{mi}\ell_{nj}A_{ij}$ and $A'_{nm} = \ell_{nr}\ell_{ms}A_{rs}$, we have

$$A'_{mn}A'_{nm} = (\ell_{mi}\ell_{nj}A_{ij})(\ell_{nr}\ell_{ms}A_{rs})$$

$$= (\ell_{mi}\ell_{ms})(\ell_{nj}\ell_{nr})A_{ij}A_{rs}$$

$$= \delta_{is}\delta_{jr}A_{ij}A_{rs} = A_{ij}\delta_{is}(\delta_{jr}A_{rs}) = A_{ij}\delta_{is}A_{js} = A_{ij}A_{ji}$$

showing that $A_{ij}A_{ji}$ is invariant.

(iii) $$A'_{mn}A'_{np}A'_{pm} = (\ell_{mi}\ell_{nj}A_{ij})(\ell_{nr}\ell_{ps}A_{rs})(\ell_{pk}\ell_{mv}A_{kv})$$

$$= (\ell_{mi}\ell_{mv})(\ell_{nj}\ell_{nr})(\ell_{ps}\ell_{pk})A_{ij}A_{rs}A_{kv}$$

$$= \delta_{iv}\delta_{jr}\delta_{sk}A_{ij}A_{rs}A_{kv}$$

$$= A_{ij}(\delta_{sk}\delta_{jr}A_{rs})(\delta_{iv}A_{kv})$$

$$= A_{ij}(\delta_{sk}A_{js})(A_{ki}) = A_{ij}A_{jk}A_{ki}$$

showing that $A_{ij}A_{jk}A_{ki}$ is invariant under orthogonal transformation.

PROBLEM (21): Prove that

(i) $$\Delta = \det(A_{ij}) = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \epsilon_{ijk}A_{1i}A_{2j}A_{3k} = \epsilon_{ijk}A_{i1}A_{j2}A_{k3}$$

(ii) $$\epsilon_{rst}\Delta = \begin{vmatrix} A_{r1} & A_{r2} & A_{r3} \\ A_{s1} & A_{s2} & A_{s3} \\ A_{t1} & A_{t2} & A_{t3} \end{vmatrix} = \epsilon_{ijk}A_{ri}A_{sj}A_{tk} = \epsilon_{ijk}A_{ir}A_{js}A_{kt}$$

SOLUTION: (i) Using the definition of ϵ_{ijk}, we can write

$$\epsilon_{ijk}A_{1i}A_{2j}A_{3k} = \epsilon_{123}A_{11}A_{22}A_{33} + \epsilon_{132}A_{11}A_{23}A_{32} + \epsilon_{231}A_{12}A_{23}A_{31} + \epsilon_{213}A_{12}A_{21}A_{33} + \epsilon_{312}A_{13}A_{21}A_{32} + \epsilon_{321}A_{13}A_{22}A_{31}$$

$$= A_{11}(A_{22}A_{33} - A_{23}A_{32}) - A_{12}(A_{21}A_{33} - A_{23}A_{31}) + A_{13}(A_{21}A_{32} - A_{22}A_{31})$$

$$= A_{11}\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12}\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13}\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix}$$

$$= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \Delta$$

Similarly, we can show that $\Delta = \epsilon_{ijk}A_{i1}A_{j2}A_{k3}$.

(ii) $\epsilon_{ijk} A_{ri} A_{sj} A_{tk} = \epsilon_{123} A_{r1} A_{s2} A_{t3} + \epsilon_{132} A_{r1} A_{s3} A_{t2} + \epsilon_{231} A_{r2} A_{s3} A_{t1}$

$+ \epsilon_{213} A_{r2} A_{s1} A_{t3} + \epsilon_{312} A_{r3} A_{s1} A_{t2} + \epsilon_{321} A_{r3} A_{s2} A_{t1}$

$= A_{r1}(A_{s2} A_{t3} - A_{s3} A_{t2}) + A_{r2}(A_{s3} A_{t1} - A_{s1} A_{t3})$

$+ A_{r3}(A_{s1} A_{t2} - A_{s2} A_{t1})$

$$= A_{r1}\begin{vmatrix} A_{s2} & A_{s3} \\ A_{t2} & A_{t3} \end{vmatrix} - A_{r2}\begin{vmatrix} A_{s1} & A_{s3} \\ A_{t1} & A_{t3} \end{vmatrix} + A_{r3}\begin{vmatrix} A_{s1} & A_{s2} \\ A_{t1} & A_{t2} \end{vmatrix}$$

$$= \begin{vmatrix} A_{r1} & A_{r2} & A_{r3} \\ A_{s1} & A_{s2} & A_{s3} \\ A_{t1} & A_{t2} & A_{t3} \end{vmatrix} = \Delta$$

Similarly, we can show that $\epsilon_{rst} \Delta = \epsilon_{ijk} A_{ir} A_{js} A_{kt}$.

PROBLEM (22): **Prove that** $\Delta = \begin{vmatrix} \delta_{m1} & \delta_{m2} & \delta_{m3} \\ \delta_{n1} & \delta_{n2} & \delta_{n3} \\ \delta_{p1} & \delta_{p2} & \delta_{p3} \end{vmatrix} = \epsilon_{mnp}$

and $\epsilon_{ijk}\epsilon_{mnp} = \begin{vmatrix} \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ni} & \delta_{nj} & \delta_{nk} \\ \delta_{pi} & \delta_{pj} & \delta_{pk} \end{vmatrix}$

Hence prove that

(i) $\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$

(ii) $\epsilon_{ijk}\epsilon_{mjk} = 2\delta_{im}$ (iii) $\epsilon_{ijk}\epsilon_{ijk} = 6$

SOLUTION: $\epsilon_{mnp} = \begin{vmatrix} \delta_{m1} & \delta_{m2} & \delta_{m3} \\ \delta_{n1} & \delta_{n2} & \delta_{n3} \\ \delta_{p1} & \delta_{p2} & \delta_{p3} \end{vmatrix}$

$$\epsilon_{123} = \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\epsilon_{231} = \begin{vmatrix} \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \\ \delta_{11} & \delta_{12} & \delta_{13} \end{vmatrix} = \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = 1$$

$$\epsilon_{312} = \begin{vmatrix} \delta_{31} & \delta_{32} & \delta_{33} \\ \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \end{vmatrix} = \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = 1$$

$$\epsilon_{132} = \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{31} & \delta_{32} & \delta_{33} \\ \delta_{21} & \delta_{22} & \delta_{23} \end{vmatrix} = -\begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = -1$$

$$\epsilon_{213} = \begin{vmatrix} \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = -\begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = -1$$

$$\epsilon_{321} = \begin{vmatrix} \delta_{31} & \delta_{32} & \delta_{33} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{11} & \delta_{12} & \delta_{13} \end{vmatrix} = -\begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = -1$$

...i, j, k is not a permutation of $(1, 2, 3)$, so that at least two of i, j, k are equal, the determinant has ...equal rows and so $\Delta = 0$.

...atively, we can prove this result following the procedure of problem (21). We can write

$$\Delta = \epsilon_{ijk}\delta_{mi}\delta_{nj}\delta_{pk}$$

$$= \epsilon_{mjk}\delta_{nj}\delta_{pk} = \epsilon_{mnk}\delta_{pk} = \epsilon_{mnp}$$

...ce $\epsilon_{ijk}\epsilon_{mnp} = \epsilon_{ijk}\Delta = \begin{vmatrix} \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ni} & \delta_{nj} & \delta_{nk} \\ \delta_{pi} & \delta_{pj} & \delta_{pk} \end{vmatrix}$

...ce multiplication of Δ by ϵ_{ijk} is equivalent to interchanging the columns $1, i; 2, j; 3, k$.

(i) Taking $p = k$ in the last result, we get

$$\epsilon_{ijk}\epsilon_{mnk} = \begin{vmatrix} \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ni} & \delta_{nj} & \delta_{nk} \\ \delta_{pi} & \delta_{pj} & \delta_{kk} \end{vmatrix} = \begin{vmatrix} \delta_{mi} & \delta_{mj} & \delta_{mk} \\ \delta_{ni} & \delta_{nj} & \delta_{nk} \\ \delta_{ki} & \delta_{kj} & 3 \end{vmatrix}$$

$$= \delta_{mi}(3\delta_{nj} - \delta_{nk}\delta_{kj}) - \delta_{mj}(3\delta_{ni} - \delta_{nk}\delta_{ki}) + \delta_{mk}(\delta_{ni}\delta_{kj} - \delta_{nj}\delta_{ki})$$

$$= \delta_{mi}(3\delta_{nj} - \delta_{nj}) - \delta_{mj}(3\delta_{ni} - \delta_{ni}) + \delta_{mk}(\delta_{ni}\delta_{kj} - \delta_{nj}\delta_{ki})$$

$$= 3\delta_{mi}\delta_{nj} - \delta_{mi}\delta_{nj} - 3\delta_{ni}\delta_{mj} + \delta_{ni}\delta_{mj} + \delta_{ni}\delta_{mj} - \delta_{nj}\delta_{mi}$$

$$= \delta_{mi}\delta_{nj} - \delta_{ni}\delta_{mj} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$$

(ii) Taking $n = j$ in the result of part (i), we get

$$\epsilon_{ijk}\epsilon_{mjk} = \delta_{im}\delta_{jj} - \delta_{ij}\delta_{jm} = 3\delta_{im} - \delta_{im} = 2\delta_{im}$$

(iii) Taking $m = i$ in the result of part (ii), we get $\epsilon_{ijk}\epsilon_{ijk} = 2\delta_{ii} = 6$

PROBLEM (23): **If $A_i B_i$ is a scalar or invariant, where A_i is an arbitrary vector, then prove that B_i is a vector.**

SOLUTION: Since $A_i B_i$ is a scalar, then by definition

$$A'_j B'_j = A_i B_i \tag{1}$$

where the undashed and dashed symbols have the usual meanings.

Now since A_i is a vector, therefore

$$A'_j = \ell_{ji} A_i$$

or

$$A_i = \ell_{ji} A'_j \tag{2}$$

From equations (1) and (2), we have

$$A'_j B'_j = \ell_{ji} A'_j B_i$$

or

$$A'_j (B'_j - \ell_{ji} B_i) = 0$$

Now A'_j being an arbitrary vector $A'_j \neq 0$ and the above relation will be true only when

$$B'_j = \ell_{ji} B_i$$

which shows that B_i is a vector.