4.15 CURL OF A VECTOR POINT FUNCTION

Let $\vec{A}(x, y, z)$ be a differentiable vector point function in a certain region of space. Then the curl or rotation of $\vec{A}$, written as $\nabla \times \vec{A}$ or curl $\vec{A}$, is defined by

$$\nabla \times \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}$$

Note that in the expansion of the determinant the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ must precede A_1, A_2, A_3.

If $\vec{A}$ is a constant vector, then $\nabla \times \vec{A} = \vec{0}$. If $\nabla \times \vec{A} = \vec{0}$ in some region R, then $\vec{A}$ is called an **irrotational** vector point function in that region.

EXAMPLE (10): If $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$, find $\nabla \times \vec{A}$ at the point $(1,-1,1)$.

SOLUTION: We have

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix} = (2z^4 + 2x^2y)\hat{i} + 3xz^2\hat{j} - 4xyz\hat{k}$$

$$(\nabla \times \vec{A})_{(1,-1,1)} = 3\hat{j} + 4\hat{k}$$

4.16 PROPERTIES OF THE CURL

THEOREM (4.9): If $\vec{A}$ and $\vec{B}$ are differentiable vector point functions, and ϕ is a differentiable scalar point function, then prove that

(i) $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$

(ii) $\nabla \times (\phi \vec{A}) = \phi(\nabla \times \vec{A}) + (\nabla \phi) \times \vec{A}$

(iii) $\nabla \times (\nabla \phi) = \vec{0}$ (curl grad $\phi = \vec{0}$)

(iv) $\nabla \cdot (\nabla \times \vec{A}) = 0$ (div curl $\vec{A} = 0$)

PROOF: Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

(i) $\vec{A} + \vec{B} = (A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}$

Hence, $$\nabla \times (A + \vec{B}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 + B_1 & A_2 + B_2 & A_3 + B_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix} = \nabla \times \vec{A} + \nabla \times \vec{B}$$

(ii) $\phi \vec{A} = \phi A_1\hat{i} + \phi A_2\hat{j} + \phi A_3\hat{k}$, then

$$\nabla \times (\phi \vec{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(\phi A_3) - \frac{\partial}{\partial z}(\phi A_2)\right]\hat{i} + \left[\frac{\partial}{\partial z}(\phi A_1) - \frac{\partial}{\partial x}(\phi A_3)\right]\hat{j} + \left[\frac{\partial}{\partial x}(\phi A_2) - \frac{\partial}{\partial y}(\phi A_1)\right]$$

$$= \left[\phi\frac{\partial A_3}{\partial y}+\frac{\partial \phi}{\partial y}A_3-\phi\frac{\partial A_2}{\partial z}-\frac{\partial \phi}{\partial z}A_2\right]\hat{i}+\left[\phi\frac{\partial A_1}{\partial z}+\frac{\partial \phi}{\partial z}A_1-\phi\frac{\partial A_3}{\partial x}-\frac{\partial \phi}{\partial x}A_3\right]\hat{j}$$

$$+\left[\phi\frac{\partial A_2}{\partial x}+\frac{\partial \phi}{\partial x}A_2-\phi\frac{\partial A_1}{\partial y}-\frac{\partial \phi}{\partial y}A_1\right]\hat{k}$$

$$= \phi\left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\hat{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\hat{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\hat{k}\right]$$

$$+\left[\left(\frac{\partial \phi}{\partial y}A_3-\frac{\partial \phi}{\partial z}A_2\right)\hat{i}+\left(\frac{\partial \phi}{\partial z}A_1-\frac{\partial \phi}{\partial x}A_3\right)\hat{j}+\left(\frac{\partial \phi}{\partial x}A_2-\frac{\partial \phi}{\partial y}A_3\right)\hat{k}\right]$$

$$= \phi(\nabla\times\vec{A})+\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z}\\ A_1 & A_2 & A_3\end{vmatrix} = \phi(\nabla\times\vec{A})+(\nabla\phi)\times\vec{A}$$

(iii) $$\nabla\times(\nabla\phi) = \nabla\times\left(\frac{\partial \phi}{\partial x}\hat{i}+\frac{\partial \phi}{\partial y}\hat{j}+\frac{\partial \phi}{\partial z}\hat{k}\right) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z}\end{vmatrix}$$

$$= \left(\frac{\partial^2 \phi}{\partial y\partial z}-\frac{\partial^2 \phi}{\partial z\partial y}\right)\hat{i}+\left(\frac{\partial^2 \phi}{\partial z\partial x}-\frac{\partial^2 \phi}{\partial x\partial z}\right)\hat{j}+\left(\frac{\partial^2 \phi}{\partial x\partial y}-\frac{\partial^2 \phi}{\partial y\partial x}\right)\hat{k}$$

$$= \vec{0}$$

provided we assume that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial ,

i.e. $$\frac{\partial^2 \phi}{\partial y\partial z}=\frac{\partial^2 \phi}{\partial z\partial y}\ ,\quad \frac{\partial^2 \phi}{\partial z\partial x}=\frac{\partial^2 \phi}{\partial x\partial z}\ ,\quad \frac{\partial^2 \phi}{\partial x\partial y}=\frac{\partial^2 \phi}{\partial y\partial x}.$$

(iv) Since $\nabla\times\vec{A} = \left[\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\hat{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\hat{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\hat{k}\right]$

Hence $$\nabla\cdot(\nabla\times\vec{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)$$

$$= \frac{\partial^2 A_3}{\partial x\partial y}-\frac{\partial^2 A_2}{\partial x\partial z}+\frac{\partial^2 A_1}{\partial y\partial z}-\frac{\partial^2 A_3}{\partial y\partial x}+\frac{\partial^2 A_2}{\partial z\partial x}-\frac{\partial^2 A_1}{\partial z\partial y}=0$$

assuming that $\vec{A}$ has continuous second partial derivatives .

THEOREM (4.10): Prove that

(i) $\nabla\times\vec{r} = \vec{0}$ (ii) $\nabla\times[f(r)\,\vec{r}\,] = \vec{0}$

(iii) $\nabla\times(r^n\vec{r}) = \vec{0}$ (iv) $\nabla\times\left(\frac{\vec{r}}{r^2}\right) = \vec{0}$

where $\vec{r}$ is the position vector .

PROOF: Since $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, therefore

(i) $$\nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

(ii) Using theorem (4.9) part (ii), we have

$$\nabla \times (\phi \vec{A}) = \phi (\nabla \times \vec{A}) + \nabla \phi \times \vec{A}$$, therefore

$$\nabla \times [f(r)\vec{r}] = f(r)(\nabla \times \vec{r}) + [\nabla f(r)] \times \vec{r}$$

$$= f(r)\vec{0} + \frac{f'(r)}{r}\vec{r} \times \vec{r} = \vec{0} \quad (\text{since } \vec{r} \times \vec{r} = \vec{0}).$$

(iii) Setting $f(r) = r^n$ in part (ii), we get

$$\nabla \times (r^n \vec{r}) = \vec{0}$$

(iv) Let $n = -2$ in part (iii), we get

$$\nabla \times \left(\frac{\vec{r}}{r^2} \right) = \vec{0}.$$

4.17 GEOMETRICAL INTERPRETATION OF THE CURL

To find a possible interpretation of the curl, let us consider a body rotating with uniform angular speed ω about an axis ℓ. Let us define the angular velocity vector $\vec{\omega}$ to be a vector of length ω extending along ℓ in the direction in which a right–handed screw would move if given the same rotation as the body. Finally, let $\vec{r}$ be the vector drawn from any point O on the axis ℓ to an arbitrary point P (x , y , z) on the body as shown in figure (4.9).

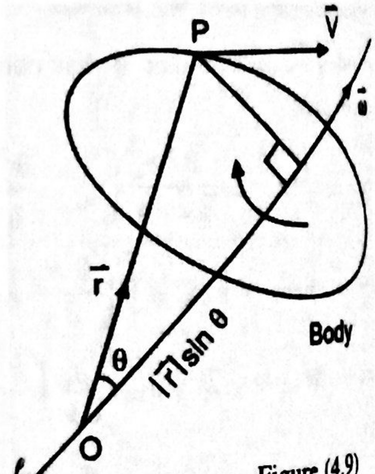

Figure (4.9)

It is clear that the radius at which P rotates is $|\vec{r}| \sin\theta$.

Hence, the linear speed of P is

$$|\vec{V}| = \omega |\vec{r}| \sin\theta$$

$$= |\vec{\omega}||\vec{r}| \sin\theta = |\vec{\omega} \times \vec{r}|$$

Moreover, the velocity vector $\vec{V}$ is directed perpendicular to the plane of $\vec{\omega}$ and $\vec{r}$, so that $\vec{\omega}$, $\vec{r}$, and $\vec{V}$ form a right handed system. Hence, the cross product $\vec{\omega} \times \vec{r}$ gives not only the magnitude of $\vec{V}$ but the direction as well, i.e. $\vec{V} = \vec{\omega} \times \vec{r}$

If we now take the point O as the origin of coordinates, we can write

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \quad \text{and} \quad \vec{\omega} = \omega_1\,\hat{i} + \omega_2\,\hat{j} + \omega_3\,\hat{k}$$

Hence, the equation $\vec{V} = \vec{\omega} \times \vec{r}$ can be written as

$$\vec{V} = (\omega_2 z - \omega_3 y)\,\hat{i} - (\omega_1 z - \omega_3 x)\,\hat{j} + (\omega_1 y - \omega_2 x)\,\hat{k}$$

If we take the curl of $\vec{V}$, we have

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & -(\omega_1 z - \omega_3 x) & \omega_1 y - \omega_2 x \end{vmatrix}$$

Expanding this, remembering that $\vec{\omega}$ is a constant vector, we find

$$\nabla \times \vec{V} = 2\,\omega_1\,\hat{i} + 2\,\omega_2\,\hat{j} + 2\,\omega_3\,\hat{k} = 2\,\vec{\omega}$$

or $$\vec{\omega} = \frac{1}{2}\,\nabla \times \vec{V}$$

which says that the angular velocity at any point of a uniformly rotating body is equal to one–half the curl of the linear velocity at that point of the body. This justifies the name rotation used for curl. It is also motivation of the term **irrotational** for a vector field whose curl is the zero vector. In fluid dynamics, $\nabla \times \vec{V}$ is called vorticity vector and measures the degree to which a fluid swirls, or rotates about a given direction – much as the angular velocity vector measures the rate of rotation of a rigid body.

4.18 OPERATIONS WITH ∇

Here we consider the various combinations of the operator ∇ with vector and scalar functions.

THEOREM (4.11): If $\vec{A}$ and $\vec{B}$ are two vector point functions and ϕ a scalar point function, then show that

(i) $(\vec{A}\cdot\nabla)\,\phi = \vec{A}\cdot\nabla\phi$

(ii) $(\vec{A}\times\nabla)\,\phi = \vec{A}\times\nabla\phi$

(iii) $(\vec{A}\cdot\nabla)\,\vec{B} = A_1\dfrac{\partial \vec{B}}{\partial x} + A_2\dfrac{\partial \vec{B}}{\partial y} + A_3\dfrac{\partial \vec{B}}{\partial z}$

(iv) $(\vec{A}\cdot\nabla)\,\vec{r} = \vec{A}$

(v) **Give possible meaning to $(\vec{A}\times\nabla)\,\vec{B}$.**

PROOF: Let $\vec{A} = A_1\,\hat{i} + A_2\,\hat{j} + A_3\,\hat{k}$, then

$$\vec{A}\cdot\nabla = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial y}\right) = A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z}$$

(i) $$(\vec{A}.\nabla)\phi = \left(A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}\right)\phi$$

$$= A_1\frac{\partial\phi}{\partial x}+A_2\frac{\partial\phi}{\partial y}+A_3\frac{\partial\phi}{\partial z}$$

$$= (A_1\hat{i}+A_2\hat{j}+A_3\hat{k}).\left(\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}\right)$$

$$= \vec{A}.\nabla\phi$$

(ii) $$(\vec{A}\times\nabla)\phi = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_1 & A_2 & A_3\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{vmatrix}\phi$$

$$= \left[\hat{i}\left(A_2\frac{\partial}{\partial z}-A_3\frac{\partial}{\partial y}\right)+\hat{j}\left(A_3\frac{\partial}{\partial x}-A_1\frac{\partial}{\partial z}\right)+\hat{k}\left(A_1\frac{\partial}{\partial y}-A_2\frac{\partial}{\partial x}\right)\right]\phi$$

$$= \left(A_2\frac{\partial\phi}{\partial z}-A_3\frac{\partial\phi}{\partial y}\right)\hat{i}+\left(A_3\frac{\partial\phi}{\partial x}-A_1\frac{\partial\phi}{\partial z}\right)\hat{j}+\left(A_1\frac{\partial\phi}{\partial y}-A_2\frac{\partial\phi}{\partial x}\right)\hat{k}$$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_1 & A_2 & A_3\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix} = \vec{A}\times\nabla\phi$$

(iii) $$(\vec{A}.\nabla)\vec{B} = \left(A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}\right)\vec{B}$$

$$= A_1\frac{\partial\vec{B}}{\partial x}+A_2\frac{\partial\vec{B}}{\partial y}+A_3\frac{\partial\vec{B}}{\partial z}$$

(iv) $$(\vec{A}.\nabla)\vec{r} = \left(A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}\right)(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= A_1\hat{i}+A_2\hat{j}+A_3\hat{k} = \vec{A}$$

(v) No definition or meaning can be assigned to $(\vec{A}\times\nabla)\vec{B}$, because it is a kind of differential operator with vector quantities.

EXAMPLE (11): If $\vec{A} = 2yz\hat{i}-x^2y\hat{j}+xz^2\hat{k}$, $\vec{B} = x^2\hat{i}+yz\hat{j}-xy\hat{k}$ and $\phi = 2x^2yz^3$, find

(i) $(\vec{A}.\nabla)\phi$ (ii) $(\vec{A}\times\nabla)\phi$

(iii) $(\vec{A}.\nabla)\vec{B}$

SOLUTION: Since $\phi = 2x^2yz^3$, we have

$$\nabla\phi = 4xyz^3\hat{i}+2x^2z^3\hat{j}+6x^2yz^2\hat{k}$$

(i) $(\vec{A}.\nabla)\phi = \vec{A}.\nabla\phi$

$= (2yz\hat{i} - x^2y\hat{j} + xz^2\hat{k}).(4xyz^3\hat{i} + 2x^2z^3\hat{j} + 6x^2yz^2\hat{k})$

$= 8xy^2z^4 - 2x^4yz^3 + 6x^3yz^4$

(ii) $(\vec{A}\times\nabla)\phi = \vec{A}\times\nabla\phi$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2yz & -x^2y & xz^2 \\ 4xyz^3 & 2x^2z^3 & 6x^2yz^2 \end{vmatrix}$$

$= -(6x^4y^2z^2 + 2x^3z^5)\hat{i} + (4x^2yz^5 - 12x^2y^2z^3)\hat{j}$

$+ (4x^2yz^4 + 4x^3y^2z^3)\hat{k}$

(iii) Since $\vec{A}.\nabla = 2yz\frac{\partial}{\partial x} - x^2y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}$, therefore

$(\vec{A}.\nabla)\vec{B} = \left(2yz\frac{\partial}{\partial x} - x^2y\frac{\partial}{\partial y} + xz^2\frac{\partial}{\partial z}\right)(x^2\hat{i} + yz\hat{j} - xy\hat{k})$

$= 2yz(2x\hat{i} - y\hat{k}) - x^2y(z\hat{j} - x\hat{k}) + xz^2(y\hat{j})$

$= 4xyz\hat{i} + (xyz^2 - x^2yz)\hat{j} + (x^3y - 2y^2z)\hat{k}$

THEOREM (4.12): If $\vec{A}$ and $\vec{B}$ are two vector functions, prove that

$$(\vec{A}\times\nabla).\vec{B} = \vec{A}.(\nabla\times\vec{B})$$

PROOF: We have

$\vec{A}\times\nabla = \left(A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y}\right)\hat{i} + \left(A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z}\right)\hat{j} + \left(A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x}\right)\hat{k}$, therefore

$(\vec{A}\times\nabla).\vec{B} = \left(A_2\frac{\partial}{\partial z} - A_3\frac{\partial}{\partial y}\right)(\hat{i}.\vec{B}) + \left(A_3\frac{\partial}{\partial x} - A_1\frac{\partial}{\partial z}\right)(\hat{j}.\vec{B}) + \left(A_1\frac{\partial}{\partial y} - A_2\frac{\partial}{\partial x}\right)(\hat{k}.\vec{B})$

Since $\hat{i}.\vec{B} = B_1$, $\hat{j}.\vec{B} = B_2$, $\hat{k}.\vec{B} = B_3$, therefore

$(\vec{A}\times\nabla).\vec{B} = \left(A_2\frac{\partial B_1}{\partial z} - A_3\frac{\partial B_1}{\partial y}\right) + \left(A_3\frac{\partial B_2}{\partial x} - A_1\frac{\partial B_2}{\partial z}\right) + \left(A_1\frac{\partial B_3}{\partial y} - A_2\frac{\partial B_3}{\partial x}\right)$

$= A_1\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) + A_2\left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right) + A_3\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)$

$= (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}).\left[\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)\hat{i} + \left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right)\hat{j} + \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)\hat{k}\right]$

$= \vec{A}.(\nabla\times\vec{B})$

4.19 VECTOR IDENTITIES

THEOREM (4.13): If $\vec{A}$ and $\vec{B}$ are two differentiable vector point functions, prove that

(i) $\nabla.(\vec{A}\times\vec{B}) = \vec{B}.(\nabla\times\vec{A}) - \vec{A}.(\nabla\times\vec{B})$

(ii) $\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}.\nabla)\vec{A} - \vec{B}(\nabla.\vec{A}) - (\vec{A}.\nabla)\vec{B} + \vec{A}(\nabla.\vec{B})$

(iii) $\nabla(\vec{A}.\vec{B}) = (\vec{B}.\nabla)\vec{A} + (\vec{A}.\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + \vec{A}\times(\nabla\times\vec{B})$

(iv) $\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla.\vec{A}) - \nabla^2\vec{A}$

PROOF: Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

(i) $\vec{A}\times\vec{B} = (A_2B_3 - A_3B_2)\hat{i} + (A_3B_1 - A_1B_3)\hat{j} + (A_1B_2 - A_2B_1)\hat{k}$

and $\nabla.(\vec{A}\times\vec{B}) = \frac{\partial}{\partial x}(A_2B_3 - A_3B_2) + \frac{\partial}{\partial y}(A_3B_1 - A_1B_3) + \frac{\partial}{\partial z}(A_1B_2 - A_2B_1)$

$$= A_2\frac{\partial B_3}{\partial x} + B_3\frac{\partial A_2}{\partial x} - A_3\frac{\partial B_2}{\partial x} - B_2\frac{\partial A_3}{\partial x}$$

$$+ A_3\frac{\partial B_1}{\partial y} + B_1\frac{\partial A_3}{\partial y} - A_1\frac{\partial B_3}{\partial y} - B_3\frac{\partial A_1}{\partial y}$$

$$+ A_1\frac{\partial B_2}{\partial z} + B_2\frac{\partial A_1}{\partial z} - A_2\frac{\partial B_1}{\partial z} - B_1\frac{\partial A_2}{\partial z}$$

$$= B_1\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + B_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + B_3\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$- A_1\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right) - A_2\left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right) - A_3\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)$$

$$= (B_1\hat{i} + B_2\hat{j} + B_3\hat{k}).\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{k}\right]$$

$$- (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}).\left[\left(\frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z}\right)\hat{i} + \left(\frac{\partial B_1}{\partial z} - \frac{\partial B_3}{\partial x}\right)\hat{j} + \left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right)\hat{k}\right]$$

$$= \vec{B}.(\nabla\times\vec{A}) - \vec{A}.(\nabla\times\vec{B})$$

(ii) We know that $\nabla\times\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\vec{V}$

$$= \hat{i}\times\frac{\partial\vec{V}}{\partial x} + \hat{j}\times\frac{\partial\vec{V}}{\partial y} + \hat{k}\times\frac{\partial\vec{V}}{\partial z} = \sum\hat{i}\times\frac{\partial\vec{V}}{\partial x}$$

$$\nabla\times(\vec{A}\times\vec{B}) = \sum\left[\hat{i}\times\frac{\partial}{\partial x}(\vec{A}\times\vec{B})\right]$$

$$= \sum\left[\hat{i}\times\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x} + \frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)\right]$$

$$= \sum \left[\hat{i} \times \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x} \right) \right] + \sum \left[\hat{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B} \right) \right]$$

$$= \sum \left[\hat{i} \times \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x} \right) \right] - \sum \left[\hat{i} \times \left(\vec{B} \times \frac{\partial \vec{A}}{\partial x} \right) \right] \qquad (1)$$

Now $\hat{i} \times \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x} \right) = \left(\hat{i} \cdot \frac{\partial \vec{B}}{\partial x} \right) \vec{A} - (\hat{i} \cdot \vec{A}) \frac{\partial \vec{B}}{\partial x}$

and so $\sum \left[\hat{i} \times \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x} \right) \right] = \left[\sum \left(\hat{i} \cdot \frac{\partial \vec{B}}{\partial x} \right) \right] \vec{A} - \vec{A} \cdot \left(\sum \hat{i} \frac{\partial}{\partial x} \right) \vec{B}$

$$= (\nabla \cdot \vec{B}) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} \qquad (2)$$

Similarly, on interchanging $\vec{A}$ and $\vec{B}$ in equation (2), we get

$$\sum \left[\hat{i} \times \left(\vec{B} \times \frac{\partial \vec{A}}{\partial x} \right) \right] = (\nabla \cdot \vec{A}) \vec{B} - (\vec{B} \cdot \nabla) \vec{A} \qquad (3)$$

Substitution of equations (2) and (3) in equation (1) gives

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} + \vec{A} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \vec{A})$$

(iii) $\nabla (\vec{A} \cdot \vec{B}) = \sum \left[\hat{i} \frac{\partial}{\partial x} (\vec{A} \cdot \vec{B}) \right]$

$$= \sum \left[\hat{i} \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} + \frac{\partial \vec{A}}{\partial x} \cdot \vec{B} \right) \right]$$

$$= \sum \left[\left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right) \hat{i} \right] + \sum \left[\left(\vec{B} \cdot \frac{\partial \vec{A}}{\partial x} \right) \hat{i} \right] \qquad (4)$$

We know that

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$$

or $(\vec{A} \cdot \vec{B}) \vec{C} = (\vec{A} \cdot \vec{C}) \vec{B} - \vec{A} \times (\vec{B} \times \vec{C})$

Thus $\left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right) \hat{i} = (\vec{A} \cdot \hat{i}) \frac{\partial \vec{B}}{\partial x} - \vec{A} \times \left(\frac{\partial \vec{B}}{\partial x} \times \hat{i} \right)$

$$= (\vec{A} \cdot \hat{i}) \frac{\partial \vec{B}}{\partial x} + \vec{A} \times \left(\hat{i} \times \frac{\partial \vec{B}}{\partial x} \right)$$

and so $\sum \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right) \hat{i} = \vec{A} \cdot \left(\sum \hat{i} \frac{\partial}{\partial x} \right) \vec{B} + \vec{A} \times \sum \left(\hat{i} \times \frac{\partial \vec{B}}{\partial x} \right)$

$$= (\vec{A} \cdot \nabla) \vec{B} + \vec{A} \times (\nabla \times \vec{B}) \qquad (5)$$

Similarly , interchanging $\vec{A}$ and $\vec{B}$ in equation (5) , we get

$$\sum\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\hat{i}=(\vec{B}\cdot\nabla)\vec{A}+\vec{B}\times(\nabla\times\vec{A}) \qquad (6)$$

Substitution of equations (5) and (6) in equation (4) gives

$$\nabla(\vec{A}\cdot\vec{B})=(\vec{B}\cdot\nabla)\vec{A}+(\vec{A}\cdot\nabla)\vec{B}+\vec{B}\times(\nabla\times\vec{A})+\vec{A}\times(\nabla\times\vec{B})$$

(iv) We know that $\vec{B}\times(\vec{C}\times\vec{D})=(\vec{B}\cdot\vec{D})\vec{C}-(\vec{B}\cdot\vec{C})\vec{D}$

Setting $\vec{B}=\vec{C}=\nabla$ and $\vec{D}=\vec{A}$, we get

$$\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-(\nabla\cdot\nabla)\vec{A}$$

$$=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$$

Note that rather than writing $(\nabla\cdot\vec{A})\nabla$, we must write $\nabla(\nabla\cdot\vec{A})$ to make sure that ∇ operates on $\nabla\cdot\vec{A}$.

EXAMPLE (12): If $\vec{A}$ is a constant vector , prove that

(i) $\nabla(\vec{A}\cdot\vec{r})=\vec{A}$ **(ii)** $\nabla\cdot(\vec{A}\times\vec{r})=0$

(iii) $\nabla\times(\vec{A}\times\vec{r})=2\vec{A}$

SOLUTION: By theorem [4.13 (iii)] , we have

(i) $\nabla(\vec{A}\cdot\vec{r})=(\vec{r}\cdot\nabla)\vec{A}+(\vec{A}\cdot\nabla)\vec{r}+\vec{r}\times(\nabla\times\vec{A})+\vec{A}\times(\nabla\times\vec{r})$

Since $\vec{A}$ is constant vector , we have

$(\vec{r}\cdot\nabla)\vec{A}=\vec{0}$, $\nabla\times\vec{A}=\vec{0}$. Furthermore , $\nabla\times\vec{r}=\vec{0}$.

Thus $\nabla(\vec{A}\cdot\vec{r})=(\vec{A}\cdot\nabla)\vec{r}=\vec{A}$ [using theorem (4.11) (iv)]

ALTERNATIVE METHOD

Let $\vec{A}=A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$ and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then

$$\vec{A}\cdot\vec{r}=A_1x+A_2y+A_3z$$

and $$\nabla(\vec{A}\cdot\vec{r})=\hat{i}\frac{\partial}{\partial x}(A_1x+A_2y+A_3z)+\hat{j}\frac{\partial}{\partial y}(A_1x+A_2y+A_3\hat{z})+\hat{k}\frac{\partial}{\partial z}(A_1x+A_2y+A_3\hat{z})$$

$$=A_1\hat{i}+A_2\hat{j}+A_3\hat{k}=\vec{A}$$

By theorem [4.13 (i)]

$$\nabla.(\bar{A} \times \bar{r}) = \bar{r}.(\nabla \times \bar{A}) - \bar{A}.(\nabla \times \bar{r}) = \bar{r}\,.\,\bar{0} - \bar{A}\,.\,\bar{0} = 0$$

By theorem [4.13 (ii)]

$$\nabla \times (\bar{A} \times \bar{r}) = (\bar{r}.\nabla)\bar{A} - \bar{r}(\nabla.\bar{A}) - (\bar{A}.\nabla)\bar{r} + \bar{A}(\nabla.\bar{r})$$

$$= -(\bar{A}.\nabla)\bar{r} + \bar{A}(\nabla.\bar{r})$$

$$= -\bar{A} + 3\bar{A} = 2\bar{A} \qquad [\text{since } (\bar{A}.\nabla)\bar{r} = \bar{A}]$$