Chapter

GRADIENT, DIVERGENCE, AND CURL

4.1 INTRODUCTION

In this chapter, we shall discuss three physically and geometrically important concepts related to scalar and vector fields, namely, the gradient, the divergence, and the curl. These concepts of vector calculus play an important role in engineering, physics, and several branches of applied mathematics, for example, mechanics, fluid mechanics, elasticity, and electromagnetic theory.

4.2 SCALAR AND VECTOR FIELDS

SCALAR POINT FUNCTION AND SCALAR FIELD

If to each point (x, y, z) of a region R in space there corresponds a unique number or scalar $\phi(x, y, z)$, then ϕ is called a scalar function of position or scalar point function in R. The set of all values of ϕ in R constitutes a scalar field.

Examples of scalar fields are :

(i) $\phi(x, y, z) = x^3 y - z^2$ defines a scalar point function and hence is a scalar field.

(ii) The temperature $T(x, y, z)$ within a body B is a scalar point function because at each point of the body there is one and only one temperature. Hence it defines a scalar field, namely, the temperature field in B.

VECTOR POINT FUNCTION AND VECTOR FIELD

If to each point (x, y, z) of a region R in space there corresponds a unique vector $\overline{A}(x, y, z)$, then $\overline{A}$ is called a vector function of position or vector point function in R and is written as

$$\overline{A} = \overline{A}(x, y\ z)$$
$$= A_1(x, y, z)\,\hat{i} + A_2(x, y, z)\,\hat{j} + A_3(x, y, z)\,\hat{k}$$

The set of all values of $\overline{A}$ in R constitutes a vector field as shown in figure (4.1).

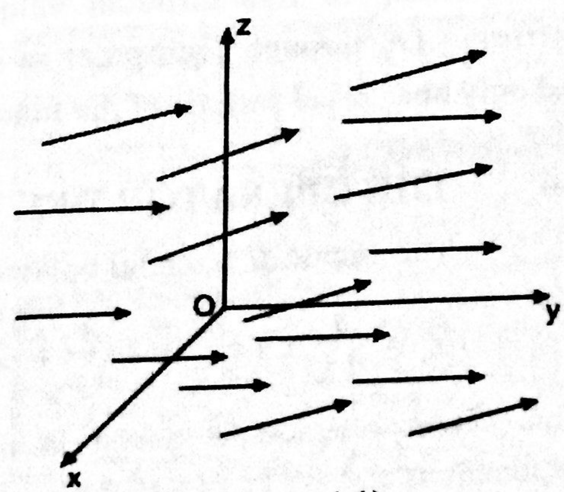

Figure (4.1)

Examples of vector fields are :

(i)
$$\overline{A}(x, y, z) = x^2 y\,\hat{i} - 2yz^3\,\hat{j} + x^2 z\,\hat{k}$$
defines a vector point function and hence is a vector field.

(ii) The set of tangent vectors of a curve C and th... fields as shown in figure (4.2) .

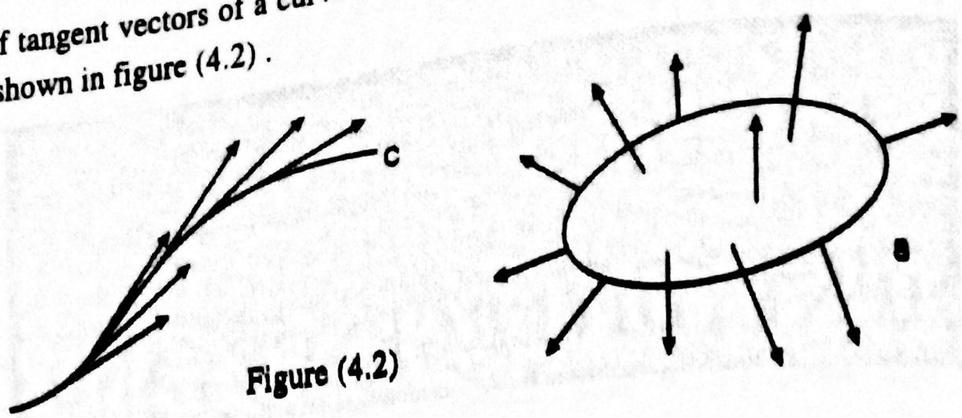

Figure (4.2)

4.3 LEVEL SURFACES

Let $\phi(x, y, z)$ be a scalar point function in a region R of space . Then for each fixed value of the constant C , the equation

$$\phi(x, y, z) = C$$

represents a surface in three–dimensional space , and if C is allowed to take a variety of different values , we obtain a family of surfaces , which are called the level surfaces of the function ϕ . On each level surface the function ϕ has the same value at each point on it . For example , for any fixed positive value of C , the equation

$$\phi = x^2 + y^2 + z^2 = C$$

represents a sphere with centre at the origin and radius $\sqrt{C}$. If C varies , the level surfaces of ϕ are the concentric spheres .

If ϕ represents the potential then the level surfaces are called the **equipotential** surfaces and if ϕ represents the temperature then the level surfaces are called the **isothermal** surfaces .

Now we show that one and only one level surface will pass through each point in space . Let $\phi(x, y, z) = C$ and $\phi(x, y, z) = C'$ be two level surfaces of ϕ corresponding to two different values C and C' respectively . If these surfaces intersect each other , then at each point of intersection , the value of ϕ is C and C' as shown in the figure (4.3) .

Since by definition $\phi(x, y, z)$ is a single valued function it is possible only if $C = C'$. Thus no two level surfaces corresponding to two different values of the constant can intersect , i.e. through each point in space there passes one , and only one , level surface of the function ϕ .

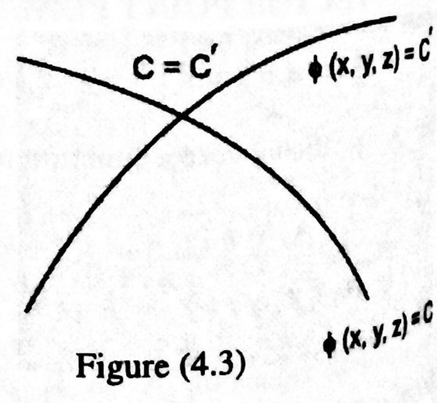

Figure (4.3)

4.4 THE OPERATOR DEL

The vector differential operator del , written ∇ , is defined by

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} .$$

This vector operator is useful in defining three quantities which arise in geometrical and physical applications and are known as the gradient , the divergence , and the curl .

4.5 GRADIENT OF A SCALAR POINT FUNCTION

Let $\phi(x, y, z)$ be a differentiable scalar point function in a certain region R of space. Then the gradient of ϕ, written $\nabla\phi$ or grad ϕ is defined by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Note that $\nabla\phi$ is a vector quantity. If ϕ is constant, then $\nabla\phi = 0$.

EXAMPLE (1): **If $\phi(x, y, z) = 3x^2y - y^3z^2$, find $\nabla\phi$ at the point $(1, -2, -1)$.**

SOLUTION: We have

$$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= 6xy\hat{i} + (3x^2 - 3y^2z^2)\hat{j} - 2y^3z\hat{k}$$

$$(\nabla\phi)_{(1,-2,-1)} = -12\hat{i} - 9\hat{j} - 16\hat{k}$$

4.6 PROPERTIES OF THE GRADIENT

THEOREM (4.1): **If ϕ and ψ are differentiable scalar point functions and C is a constant, then**

(i) $\nabla(C\phi) = C\nabla\phi$ (ii) $\nabla(\phi^n) = n\phi^{n-1}\nabla\phi$

(iii) $\nabla(\phi + \psi) = \nabla\phi + \nabla\psi$ (iv) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(v) $\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}, \ \psi \neq 0$

PROOF: We have

(i) $$\nabla(C\phi) = \hat{i}\frac{\partial}{\partial x}(C\phi) + \hat{j}\frac{\partial}{\partial y}(C\phi) + \hat{k}\frac{\partial}{\partial z}(C\phi)$$

$$= \hat{i}C\frac{\partial\phi}{\partial x} + \hat{j}C\frac{\partial\phi}{\partial y} + \hat{k}C\frac{\partial\phi}{\partial z}$$

$$= C\left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) = C\nabla\phi$$

(ii) $$\nabla(\phi^n) = \frac{\partial}{\partial x}(\phi^n)\hat{i} + \frac{\partial}{\partial y}(\phi^n)\hat{j} + \frac{\partial}{\partial z}(\phi^n)\hat{k}$$

$$= n\phi^{n-1}\frac{\partial\phi}{\partial x}\hat{i} + n\phi^{n-1}\frac{\partial\phi}{\partial y}\hat{j} + n\phi^{n-1}\frac{\partial\phi}{\partial z}\hat{k}$$

$$= n\phi^{n-1}\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) = n\phi^{n-1}\nabla\phi$$

(iii) $$\nabla(\phi + \psi) = \hat{i}\frac{\partial}{\partial x}(\phi + \psi) + \hat{j}\frac{\partial}{\partial y}(\phi + \psi) + \hat{k}\frac{\partial}{\partial z}(\phi + \psi)$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) + \left(\hat{i}\frac{\partial\psi}{\partial x} + \hat{j}\frac{\partial\psi}{\partial y} + \hat{k}\frac{\partial\psi}{\partial z}\right)$$

$$= \nabla\phi + \nabla\psi$$

(iv) $\nabla(\phi\psi) = \hat{i}\frac{\partial}{\partial x}(\phi\psi)+\hat{j}\frac{\partial}{\partial y}(\phi\psi)+\hat{k}\frac{\partial}{\partial z}(\phi\psi)$

$$= \hat{i}\left(\phi\frac{\partial\psi}{\partial x}+\psi\frac{\partial\phi}{\partial x}\right)+\hat{j}\left(\phi\frac{\partial\psi}{\partial y}+\psi\frac{\partial\phi}{\partial y}\right)+\hat{k}\left(\phi\frac{\partial\psi}{\partial z}+\psi\frac{\partial\phi}{\partial z}\right)$$

$$= \phi\left(\hat{i}\frac{\partial\psi}{\partial x}+\hat{j}\frac{\partial\psi}{\partial y}+\hat{k}\frac{\partial\psi}{\partial z}\right)+\psi\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \phi\nabla\psi+\psi\nabla\phi$$

(v) $\nabla\left(\frac{\phi}{\psi}\right) = \nabla\left(\phi\frac{1}{\psi}\right) = \phi\nabla\left(\frac{1}{\psi}\right)+\frac{1}{\psi}\nabla\phi$

$$= \phi\left(-\frac{1}{\psi^2}\right)\nabla\psi+\frac{1}{\psi}\nabla\phi$$

$$= \frac{\psi\nabla\phi-\phi\nabla\psi}{\psi^2}, \quad \psi \neq 0$$

THEOREM (4.2): Prove that

(i) $\nabla f(r) = \frac{f'(r)\,\bar{r}}{r}$

(ii) $\nabla(r^n) = n r^{n-2}\,\bar{r}$, **where n is any real number.**

(iii) $\nabla r = \hat{r}$ (iv) $\nabla\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r^3}$

PROOF: We know that

(i) $\nabla f(r) = \frac{\partial}{\partial x}f(r)\hat{i}+\frac{\partial}{\partial y}f(r)\hat{j}+\frac{\partial}{\partial z}f(r)\hat{k}$

$$= \frac{\partial f}{\partial r}\frac{\partial r}{\partial x}\hat{i}+\frac{\partial f}{\partial r}\frac{\partial r}{\partial y}\hat{j}+\frac{\partial f}{\partial r}\frac{\partial r}{\partial z}\hat{k}$$

$$= \frac{df}{dr}\frac{\partial r}{\partial x}\hat{i}+\frac{df}{dr}\frac{\partial r}{\partial y}\hat{j}+\frac{df}{dr}\frac{\partial r}{\partial z}\hat{k}$$

Since $r=\sqrt{x^2+y^2+z^2}$, it follows that $\frac{\partial r}{\partial x}=\frac{x}{r}$, $\frac{\partial r}{\partial y}=\frac{y}{r}$, $\frac{\partial r}{\partial z}=\frac{z}{r}$

$$\nabla f(r) = f'(r)\frac{x}{r}\hat{i}+f'(r)\frac{y}{r}\hat{j}+f'(r)\frac{z}{r}\hat{k}$$

$$= \frac{f'(r)}{r}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= \frac{f'(r)\,\bar{r}}{r}$$

(ii) Using (i) $\nabla(r^n) = \frac{n r^{n-1}\,\bar{r}}{r} = n r^{n-2}\,\bar{r}$

Note that if $\bar{r} = r\hat{r}$, where $\hat{r}$ is a unit vector in the direction of $\bar{r}$, then $\nabla r^n = n r^{n-1}\hat{r}$.

ALTERNATIVE METHOD

$$\nabla(r^n) = \nabla(x^2+y^2+z^2)^{n/2}$$
$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2} + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2} + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2}$$
$$= n(x^2+y^2+z^2)^{(n/2)-1}(x\hat{i}+y\hat{j}+z\hat{k})$$
$$= n(r^2)^{(n/2)-1}\vec{r}$$
$$= n r^{n-2}\vec{r}$$

(iii) Let $n = 1$ in part (ii) then $\nabla r = \frac{\vec{r}}{r} = \hat{r}$,

(iv) Let $n = -1$ in part (ii) then $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

EXAMPLE (2): **Show that**

(i) $\nabla(\ln r) = \frac{\vec{r}}{r^2}$ (ii) $\nabla(e^{r^2}) = 2e^{r^2}\vec{r}$

(iii) $\nabla(r^3) = 3r\vec{r}$

SOLUTION: (i) We know that

$$\nabla f(r) = \frac{f'(r)\vec{r}}{r} \qquad (1)$$

Let $f(r) = \ln r$ in equation (1), we get

$$\nabla(\ln r) = \frac{\frac{1}{r}\vec{r}}{r} = \frac{\vec{r}}{r^2}$$

(ii) Let $f(r) = e^{r^2}$ in equation (1), we get

$$\nabla(e^{r^2}) = \frac{e^{r^2}2r\vec{r}}{r} = 2e^{r^2}\vec{r}$$

(iii) We know that

$$\nabla(r^n) = n r^{n-2}\vec{r} \qquad (2)$$

Let $n = 3$ in equation (2) we get

$$\nabla(r^3) = 3r\vec{r}$$

4.7 GEOMETRICAL INTERPRETATION OF GRADIENT

GRADIENT AS A NORMAL VECTOR TO A SURFACE

THEOREM (4.3): **Prove that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x, y, z) = C$, where C is a constant.**

PROOF:

Let Γ be a curve through any point $P(x, y, z)$ which lies on the level surface $\phi(x, y, z) = C$ (1)

Let the parametric equations of the curve Γ be $x = x(s),\ y = y(s),\ z = z(s)$.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of the point P, then $\hat{T} = \frac{d\vec{r}}{ds}$ is the unit vector tangent to the curve Γ at P as shown in figure (4.4).

Since the curve Γ lies on the level surface, therefore, the coordinates of any point on the curve must satisfy equation (1), and so $\phi[x(s), y(s), z(s)] = C$.

Differentiating this equation w.r.t. s using the chain rule,

$$\frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} = 0$$

or $$\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right) = 0$$

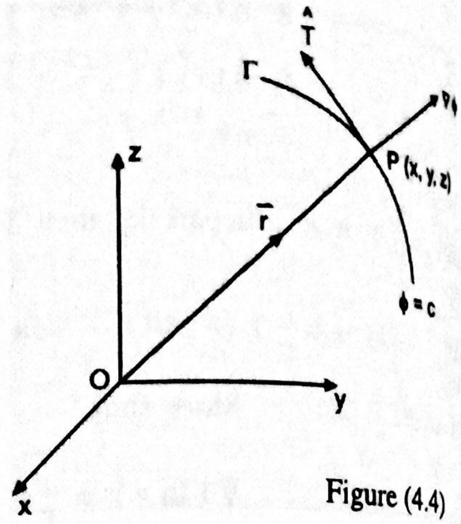

Figure (4.4)

or $$\nabla\phi\cdot\frac{d\vec{r}}{ds} = 0 \text{ or } \nabla\phi\cdot\hat{T} = 0$$

which implies that $\nabla\phi$ is a vector perpendicular to the unit tangent vector $\hat{T}$ and therefore to the surface $\phi(x, y, z) = C$.

EXAMPLE (3): **Find a unit normal vector $\hat{n}$ to the surface given by $z = x^2 + y^2$ at the point (1, 2, 5).**

SOLUTION: Since $z = x^2 + y^2$, the surface is defined as $\phi(x, y, z) = x^2 + y^2 - z = 0$.

Then $$\nabla\phi = \hat{i}\frac{\partial}{\partial x}(x^2 + y^2 - z) + \hat{j}\frac{\partial}{\partial y}(x^2 + y^2 - z) + \hat{k}\frac{\partial}{\partial z}(x^2 + y^2 - z)$$

$$= 2x\hat{i} + 2y\hat{j} - \hat{k}$$

Thus at the point (1, 2, 5) the value of the gradient is

$$\nabla\phi = 2\hat{i} + 4\hat{j} - \hat{k}$$

Hence a unit normal vector $\hat{n}$ to the surface at the given point is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{(2)^2 + (4)^2 + (-1)^2}}$$

$$= \frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{21}}$$

Another unit normal vector to the surface at the given point in the opposite direction is $-\frac{2\hat{i} + 4\hat{j} - \hat{k}}{\sqrt{21}}$

4.8 DIRECTIONAL DERIVATIVE

We know that the first partial derivatives of $\phi(x, y, z)$ are the rates of change of ϕ in the directions of the coordinate axes. It seems unnatural to restrict attention to these three directions, and we may ask for the rate of change of ϕ in any direction. This leads us to the idea of a directional derivative

To define the directional derivative we choose a point $P(x, y, z)$ in space and a direction at P, given by a unit vector $\hat{T}$. Let C be the ray drawn from P in the direction of $\hat{T}$, and let $P'(x+\Delta x, y+\Delta y, z+\Delta z)$ be a neighbouring point on C, whose distance from P is Δs as shown in figure (4.5). Let ϕ be a differentiable scalar point function and $\phi(x, y, z)$ and $\phi(x+\Delta x, y+\Delta y, z+\Delta z)$ be the value of this function at P and P′ respectively. Then the limit

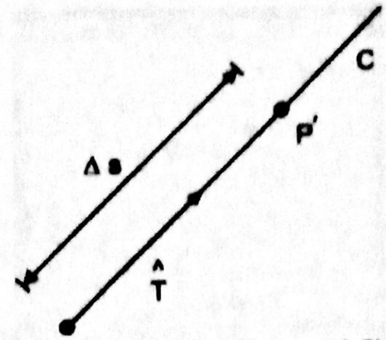

Figure (4.5)

$$\underset{\Delta s \to 0}{\mathrm{Lt}} \frac{\Delta \phi}{\Delta s} = \underset{\Delta s \to 0}{\mathrm{Lt}} \frac{\phi(P') - \phi(P)}{\Delta s} \qquad (1)$$

if it exists, is called the directional derivative of ϕ at P in the direction of $\hat{T}$ and is denoted by $\frac{\partial \phi}{\partial s}$.

Obviously, $\frac{\partial \phi}{\partial s}$ is the rate of change of ϕ at P w.r.t. the distance s measured in the direction of $\hat{T}$.

Equation (1) can be written as

$$\frac{\partial \phi}{\partial s} = \underset{\Delta s \to 0}{\mathrm{Lt}} \frac{\phi(x+\Delta x, y+\Delta y, z+\Delta z) - \phi(x, y, z)}{\Delta s} \qquad (2)$$

In this way there are now infinitely many directional derivatives of ϕ at P, each corresponding to a certain direction. Now from elementary calculus, equation (2) can be written as

$$\frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds}$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right) \cdot \left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right)$$

$$= \nabla \phi \cdot \frac{d\vec{r}}{ds} = \nabla \phi \cdot \hat{T} \qquad (3)$$

Since $\hat{T}$ is a unit vector, directional derivative of ϕ $\left(\text{i.e. } \frac{\partial \phi}{\partial s}\right)$ is the component of $\nabla \phi$ in the direction of this unit vector. From equation (3) we have the operator equivalence.

$$\frac{\partial}{\partial s} = \hat{T} \cdot \nabla \qquad (4)$$

This means that the operator $\hat{T} \cdot \nabla$ applied to the scalar function ϕ differentiates it w.r.t. the distance s in the direction of $\hat{T}$.

PARTICULAR CASES

(i) If $\hat{r}$ is a unit vector in the direction of the position vector $\vec{r}$, then the direction derivative of ϕ in this direction is

$$\frac{\partial \phi}{\partial r} = \nabla \phi \cdot \hat{r}$$

(ii) If, in particular, $\hat{T}$ has the direction of the positive x–axis, then $\hat{T} = \hat{i}$ and from equation (...) we have

$$\frac{\partial \phi}{\partial s} = \nabla \phi \cdot \hat{i}$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right) \cdot \hat{i} = \frac{\partial \phi}{\partial x}$$

Similarly, $\frac{\partial \phi}{\partial y} = \nabla \phi \cdot \hat{j}$ and $\frac{\partial \phi}{\partial z} = \nabla \phi \cdot \hat{k}$

It therefore, follows that the first partial derivatives $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$, and $\frac{\partial \phi}{\partial z}$ are the directional derivatives in the directions of the coordinates axes.

THEOREM (4.4): Show that the maximum value of the directional derivative of $\phi(x, y, z)$ is equal to the magnitude of $\nabla \phi$ (i.e. $|\nabla \phi|$) and it takes place in the direction of $\nabla \phi$.

PROOF: We know that

$$\frac{\partial \phi}{\partial s} = \nabla \phi \cdot \hat{T}$$

$$= |\nabla \phi| |\hat{T}| \cos \theta = |\nabla \phi| \cos \theta$$

where θ is the angle between $\nabla \phi$ and $\hat{T}$. Since $-1 \leq \cos \theta \leq 1$, therefore $\frac{\partial \phi}{\partial s}$ is maximum when $\cos \theta = 1$ or $\theta = 0^\circ$ i.e. when the direction of $\hat{T}$ is the direction of $\nabla \phi$ and $\left.\frac{\partial \phi}{\partial s}\right|_{max} = |\nabla \phi|$.

Thus the maximum value of the directional derivative takes place in the direction of $\nabla \phi$ and has the magnitude as $|\nabla \phi|$.

NOTE: The directional derivative $\frac{\partial \phi}{\partial s}$ is zero, when $\theta = \frac{\pi}{2}$ i.e. when $\nabla \phi$ and $\hat{T}$ are orthogonal to each other.

EXAMPLE (4): Find the directional derivative of $\phi(x, y, z) = x^2 + y^2 + z^2$ at the point $(1, 1, 1)$ in the direction of the vector $\hat{i} + \hat{j} + \hat{k}$. Find its maximum value and the direction in which it takes place.

SOLUTION: Since $\phi(x, y, z) = x^2 + y^2 + z^2$, therefore

$$\nabla \phi = \hat{i}\frac{\partial}{\partial x}(x^2 + y^2 + z^2) + \hat{j}\frac{\partial}{\partial y}(x^2 + y^2 + z^2) + \hat{k}\frac{\partial}{\partial z}(x^2 + y^2 + z^2)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Then at the point $(1, 1, 1)$ the value of this gradient is $\nabla \phi = 2\hat{i} + 2\hat{j} + 2\hat{k}$

The unit vector in the direction of $\hat{i} + \hat{j} + \hat{k}$ is $\hat{T} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{1 + 1 + 1}} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

Thus the required directional derivative of ϕ at (1 , 1 , 1) is

$$\frac{\partial \phi}{\partial s} = \nabla \phi . \hat{T} = (2\hat{i} + 2\hat{j} + 2\hat{k}) . \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)$$

$$= \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

Now $|\nabla \phi| = \sqrt{(2)^2 + (2)^2 + (2)^2} = \sqrt{12} = 2\sqrt{3}$

The maximum value of the directional derivative $\frac{\partial \phi}{\partial s}$ at the point (1 , 1 , 1) is $|\nabla \phi| = 2\sqrt{3}$, and its direction is that of $\nabla \phi = 2\hat{i} + 2\hat{j} + 2\hat{k}$.

DIRECTIONAL DERIVATIVE ALONG A CURVE

So far , we have defined the directional derivative along a straight line . Instead of a straight line , we can consider an arbitrary smooth curve passing through the point P (x , y , z) . If s measures the distance along the curve as shown in figure (4.6) , then we know that the unit tangent vector $\hat{T}$ to the curve at P is given by

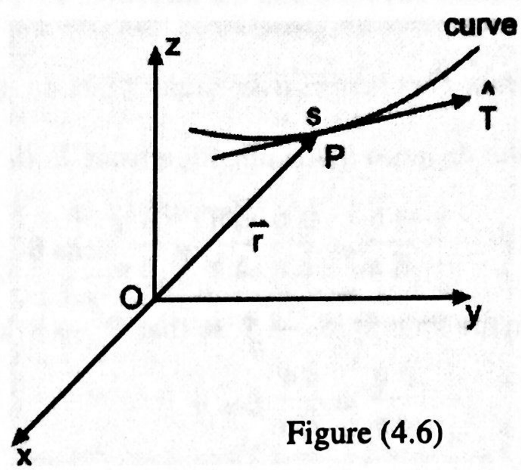

Figure (4.6)

$$\hat{T} = \frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}$$

and again , in this case , we can show that

$$\frac{\partial \phi}{\partial s} = \nabla \phi . \hat{T} = |\nabla \phi| \cos \theta$$

where θ is the angle between the unit tangent vector $\hat{T}$ to the curve at P and $\nabla \phi$.

Thus we can say that the directional derivative along an arbitrary smooth curve is the same as the directional derivative along a straight line tangent to the curve .

4.9 NORMAL DERIVATIVE

In various applications , the direction along which a directional derivative is formed is that of a specified unit normal vector $\hat{n}$ to a curve in the plane or to a surface in space . The directional derivative is then denoted by $\frac{\partial \phi}{\partial n}$ and is given by

$$\frac{\partial \phi}{\partial n} = \nabla \phi . \hat{n} \qquad (1)$$

i.e. the normal derivative of ϕ is the component of $\nabla \phi$ in the direction of the unit normal vector $\hat{n}$.

From equation (1) we have the operator equivalence $\frac{\partial}{\partial n} = \hat{n} . \nabla$

which when applied to any scalar function always gives the normal derivative .

4.10 ALTERNATIVE DEFINITION OF GRADIENT

An alternative definition of the gradient can be given as follows :

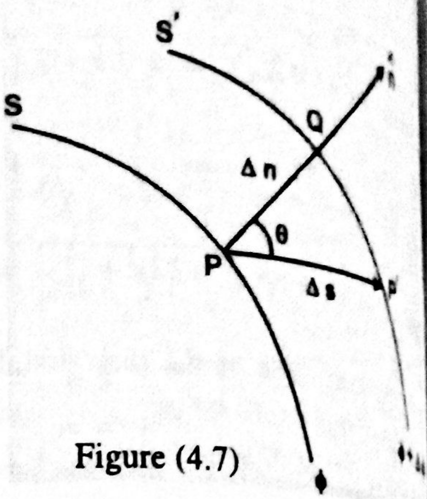

Figure (4.7)

Let S and S′ be the level surfaces of the function ϕ through P (x , y , z) and P′ (x + Δx , y + Δy , z + Δz) with values ϕ and $\phi + \Delta\phi$ respectively , and let Δs be the distance between the points P and P′. Draw PQ normal to S at P , to meet S′ in Q and let Δn be the length PQ . Accordingly , QP′ is approximately orthogonal to PQ and if θ is the angle between the normal direction and the direction PP′ as shown in figure (4.7) , then $\frac{\Delta n}{\Delta s} = \cos\theta$

Since the change in the scalar function ϕ in passing from P to P′ is $\Delta\phi$, its average rate of change per unit distance in the direction from P to P′ is $\frac{\Delta\phi}{\Delta s}$. Now

$$\frac{\Delta\phi}{\Delta s} = \frac{\Delta\phi}{\Delta n}\frac{\Delta n}{\Delta s} = \frac{\Delta\phi}{\Delta n}\cos\theta$$

In the limit as $S' \to S$ so that $P' \to P$ keeping θ constant , this gives

$$\frac{\partial\phi}{\partial s} = \frac{\partial\phi}{\partial n}\cos\theta \qquad (1)$$

If $\hat{n}$ is the unit vector normal to the level surface S at P and having the direction PQ , and $\hat{T}$ is the unit vector in the direction PP′ , then $\cos\theta = \hat{n}\cdot\hat{T}$ and equation (1) becomes

$$\frac{\partial\phi}{\partial s} = \frac{\partial\phi}{\partial n}\,\hat{n}\cdot\hat{T} \qquad (2)$$

Also we know that $\frac{\partial\phi}{\partial s} = \nabla\phi\cdot\hat{T}$ (3)

Comparison of equations (2) and (3) shows that

$$\nabla\phi = \frac{\partial\phi}{\partial n}\hat{n} \qquad (4)$$

Thus $\nabla\phi$ is a vector in the direction of the normal to the surface and has magnitude equal to the rate of change of ϕ along this normal i.e. $|\nabla\phi| = \frac{\partial\phi}{\partial n}$ (5)

Equation (5) states that the maximum value of the directional derivative takes place along the normal direction .

NOTE: The definition of gradient given in equation (4) is invariant in the sense that $\nabla\phi$ is independent of the choice of the coordinate system .

Also $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$ (6)