ALGEBRA OF VECTORS

1.1 INTRODUCTION

In this chapter , we shall discuss the basic algebraic operations with vectors in three–dimensional space . Since forces , velocities , and various other quantities are vectors , therefore , the algebra of vectors is of great importance in mathematics , physics , and engineering .

1.2 SCALARS AND VECTORS

SCALAR

A quantity which has only magnitude but no direction is called a **scalar** . For example , mass , density , volume , length , time , temperature , speed and any real number , say π , $\sqrt{2}$, etc. The value of a scalar is a single real number with an appropriate unit of measurement such as cm , ft^3 , deg , or sec . Operations with scalars follow the same rules as in elementary algebra .

VECTOR

A quantity which has both magnitude and direction is called a **vector** . For example , displacement , velocity , acceleration , force , and momentum , etc .

A vector has two different natures , one geometric and the other algebraic (or analytic) . To study the applications of vectors , we need an understanding of both aspects .

1.3 GEOMETRIC REPRESENTATION OF A VECTOR

Geometrically , a vector is represented by a directed line segment (i.e. an arrow) OP whose direction is that of a vector and whose length represents the magnitude of the vector . The tail end O of the arrow is called the **origin** or **initial point** of the vector , and the head P is called the **terminal point** .

Figure (1.1)

Analytically , a vector is represented by a letter with an arrow over it , as $\vec{A}$. The magnitude of the vector $\vec{A}$ is denoted by $|\vec{A}|$ or simply A and is equal to the length OP as shown in figure (1.1) . The magnitude is also called the **length** or **norm** of the vector .

When the initial point of a vector is fixed , it is called **fixed** or **localized vector** . If the initial point is not fixed , it is called a **free** or **non–localized** vector .

1.4 FUNDAMENTAL DEFINITIONS USING GEOMETRIC REPRESENTATION

EQUAL VECTORS

Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal, written $\vec{A} = \vec{B}$, if they have the same magnitude and the same direction regardless of the position of their initial points as shown in figure (1.2). Thus a vector may be shifted from one location to another, provided neither the magnitude nor direction is changed.

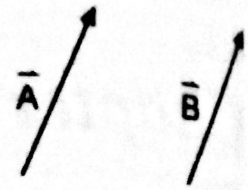

Figure (1.2)

NEGATIVE OF A VECTOR

A vector having the same magnitude as that of $\vec{A}$, but with direction opposite to that of $\vec{A}$, is defined as the negative of $\vec{A}$ and is denoted by $-\vec{A}$ as shown in figure (1.3).

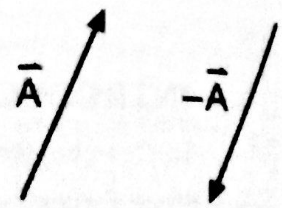

Figure (1.3)

UNIT VECTOR

A vector with unit magnitude is called the unit vector (or direction vector). The unit vector in the direction of a non–zero vector $\vec{A}$ denoted by $\hat{a}$ is defined as $\frac{\vec{A}}{A}$. Thus

$$\frac{\vec{A}}{A} = \hat{a} \quad \text{or} \quad \vec{A} = A\,\hat{a}$$

which shows that any vector $\vec{A}$ can be represented as the product of the magnitude of $\vec{A}$ and the unit vector $\hat{a}$ in the direction of $\vec{A}$. Note that to emphasize the fact that a particular vector is a unit vector, we put a **hat** or a **caret** over it like $\hat{a}$, $\hat{n}$, etc.

ZERO OR NULL VECTOR

A vector which has zero magnitude and no specific direction is called the **zero** or **null** vector and is denoted by $\vec{0}$. Geometrically, a null vector is represented by a point. It is the only vector we cannot represent as an arrow. A vector which is not null is called a **proper** vector.

ADDITION AND SUBTRACTION OF VECTORS

The operations of addition and subtraction familiar in the algebra of real numbers (scalars) are capable of extension to an algebra of vectors with suitable definitions.

ADDITION

The sum or resultant of two vectors $\vec{A}$ and $\vec{B}$ denoted by $\vec{A} + \vec{B}$ is a vector $\vec{C}$ formed by placing the initial point of $\vec{B}$ on the terminal point of $\vec{A}$ and then

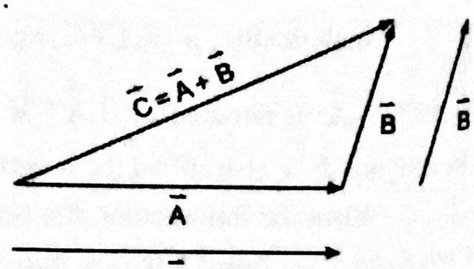

Figure (1.4)

joining the initial point of $\vec{A}$ to the terminal point of $\vec{B}$ as shown in figure (1.4) . This description of vector addition is called the **triangle law** of addition . It is also sometimes called the **parallelogram law** of addition because $\vec{C} = \vec{A} + \vec{B}$ is given by the diagonal of the parallelogram PQRS determined by $\vec{A}$ and $\vec{B}$ as shown in figure (1.5) .

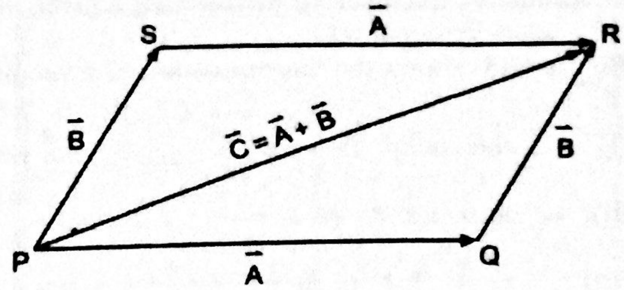

Figure (1.5)

SUBTRACTION

The difference of two vectors $\vec{A}$ and $\vec{B}$ denoted by $\vec{A} - \vec{B}$, is a vector $\vec{C}$ and is defined as the sum of $\vec{A}$ and $-\vec{B}$,

i.e. $\vec{C} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

Thus to subtract $\vec{B}$ from $\vec{A}$, we reverse the direction of $\vec{B}$ and add it to $\vec{A}$ as shown in figure (1.6) .

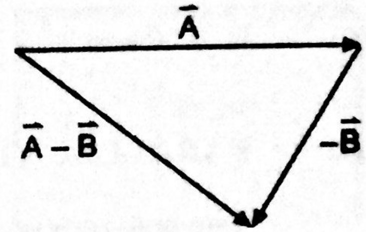

Figure (1.6)

POLYGON LAW OF VECTOR ADDITION

Let $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_n$ be n vectors . To find their sum , put all the vectors tip to nail i.e. initial point of $\vec{A}_{i+1}$ on the terminal point of $\vec{A}_i$ and complete a polygon of vectors . The sum is then the vector from initial point of $\vec{A}_1$ to the terminal point of $\vec{A}_n$ and is written as $\vec{A}_1 + \vec{A}_2 + \ldots\ldots + \vec{A}_n$. [See figure (1.7)] .

If this polygon is closed i.e. the terminal point of $\vec{A}_n$ coincides with the initial point of $\vec{A}_1$, then the sum is zero .

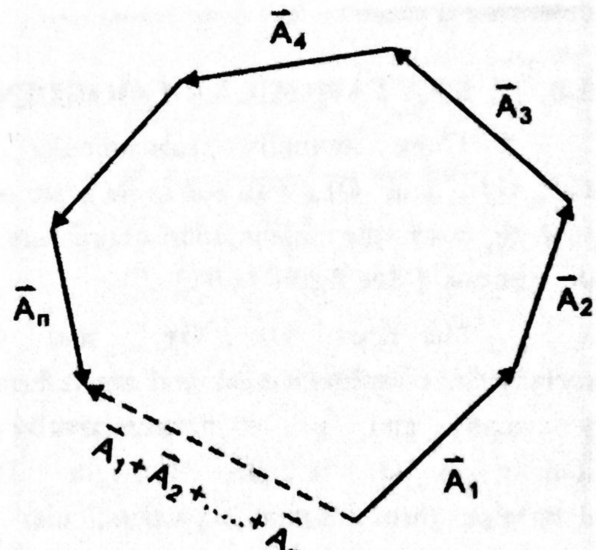

Figure (1.7)

SCALAR MULTIPLICATION

Let $\vec{A}$ be any vector and m any scalar . Then the vector $m\vec{A}$ called the scalar multiple of $\vec{A}$, is defined as follows :

The magnitude of $m\vec{A}$ is $|m|A$.

If $m > 0$ and $\vec{A} \neq \vec{0}$, then the direction of $m\vec{A}$ is that of $\vec{A}$.

If $m < 0$ and $\vec{A} \neq \vec{0}$, then the direction of $m\vec{A}$ is opposite to that of $\vec{A}$.

If $m = 0$ or $\vec{A} = \vec{0}$ (or both) , then $m\vec{A} = \vec{0}$.

Graphically , the result of multiplying a given vector by a scalar is a vector parallel to the given vector .

Figure (1.8) shows the multiplication of a vector $\vec{A}$ by scalars

(i) the vector $\vec{A}$

(ii) $m\vec{A}$ for $m = 2$

(iii) $m\vec{A}$ for $m = \frac{1}{2}$

(iv) $m\vec{A}$ for $m = -1$

(v) $m\vec{A}$ for $m = -\frac{3}{2}$

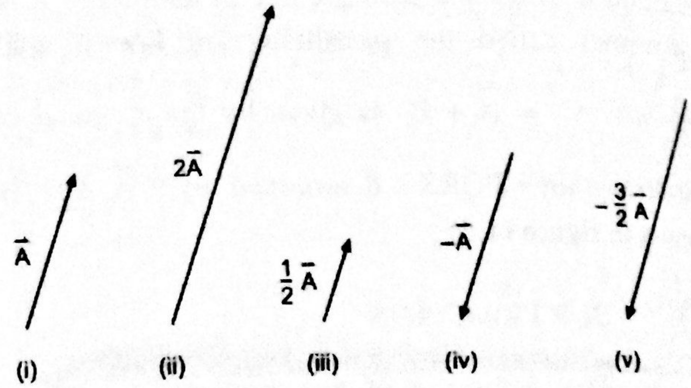

Figure (1.8)

1.5 PARALLEL VECTORS

Two non–zero vectors $\vec{A}$ and $\vec{B}$ are said to be parallel if and only if $\vec{B} = m\vec{A}$, where m is a scalar being positive or negative according as they have the same or opposite senses . In other words , two vectors are parallel if and only if they are scalar multiples of one another . Since $\vec{0} = 0\,\vec{A}$, where $\vec{A}$ is an arbitrary vector , therefore $\vec{0}$ is parallel to any vector $\vec{A}$. If $\vec{A}$ and $\vec{B}$ are not parallel , they determine a plane .

1.6 RECTANGULAR COORDINATE SYSTEM IN SPACE

Three mutually perpendicular intersecting lines Ox , Oy , and Oz with the same scale of measurement are said to constitute rectangular coordinate system in three–dimensions [See figure (1.9)] .

The lines Ox , Oy , and Oz are called the rectangular coordinate axes and are referred to as x – axis , y – axis , and z – axis respectively . Their point of intersection , O , is called the origin . The coordinate axes determine three mutually perpendicular planes called the xy , yz , and zx – planes . These planes are called the coordinate planes .

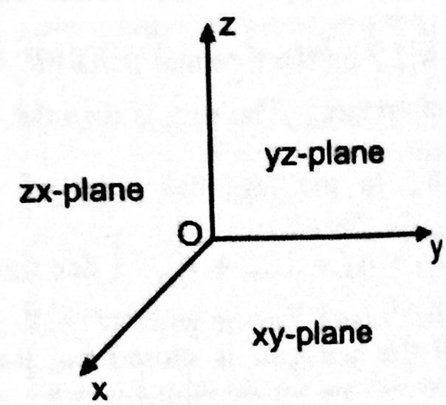

Figure (1.9)

1.7 UNIT VECTORS $\hat{i}, \hat{j}, \hat{k}$

Consider the rectangular coordinate system in three–dimensions . Take the points A (1 , 0 , 0) , B (0 , 1 , 0) and C (0 , 0 , 1) on the x , y , and z – axis respectively . We denote the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ by the symbols $\hat{i}, \hat{j}, \hat{k}$ and call them the unit vectors (or base vectors) in the directions of x , y , z – axis respectively as shown in figure (1.10) .

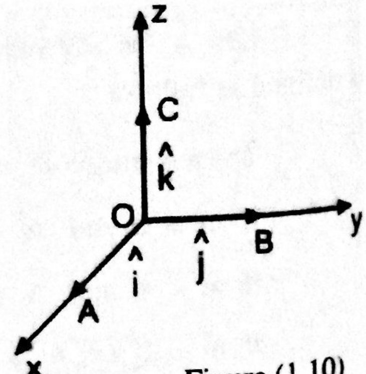

Figure (1.10)

1.8 COMPONENTS OF A VECTOR

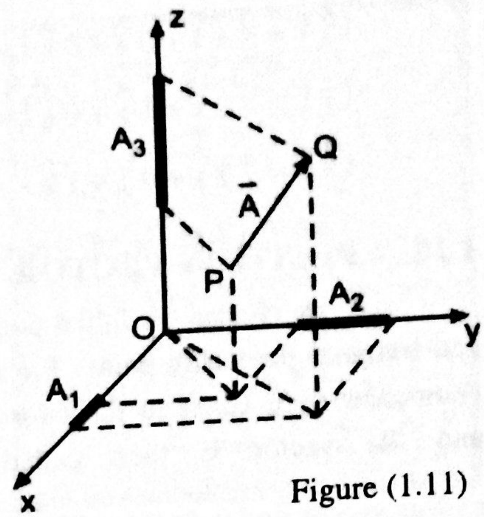

Figure (1.11)

Let $\vec{A}$ be a three-dimensional vector with $P(x_1, y_1, z_1)$ as the initial point and $Q(x_2, y_2, z_2)$ as the terminal point as shown in figure (1.11). Then the numbers (scalars) $A_1 = x_2 - x_1$, $A_2 = y_2 - y_1$, $A_3 = z_2 - z_1$ are called the rectangular components or simply components of $\vec{A}$ in the x, y, and z directions respectively.

The vectors $A_1\hat{i}, A_2\hat{j}, A_3\hat{k}$ are called the rectangular component vectors or simply component vectors of $\vec{A}$ in the x, y, and z directions respectively.

1.9 ANALYTIC REPRESENTATION OF A VECTOR

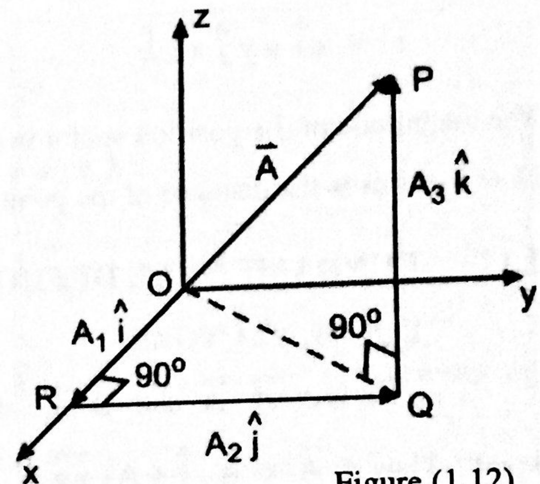

Figure (1.12)

Let the initial point of an arbitrary vector $\vec{A}$ in three-dimensions be at the origin of a rectangular coordinate system. Let (A_1, A_2, A_3) be the coordinates of the terminal point of vector $\vec{A}$ as shown in figure (1.12).

Then the vectors $A_1\hat{i}, A_2\hat{j}, A_3\hat{k}$ are the component vectors of $\vec{A}$ in the x, y, z directions respectively.

The sum or resultant of $A_1\hat{i}, A_2\hat{j}, A_3\hat{k}$ is the vector $\vec{A}$ using the polygon law of vector addition, so that we can write

$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k} \qquad (1)$$

Equation (1) is called the analytic representation of $\vec{A}$ in terms of components.

MAGNITUDE OF $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

From the above figure, we have by Pythagorean theorem

$$(OP)^2 = (OQ)^2 + (QP)^2 \qquad (2)$$

where OP denotes the magnitude of the vector $\overrightarrow{OP}$, etc. Similarly

$$(OQ)^2 = (OR)^2 + (RQ)^2 \qquad (3)$$

From equations (2) and (3) $\quad (OP)^2 = (OR)^2 + (RQ)^2 + (QP)^2$

or $A^2 = A_1^2 + A_2^2 + A_3^2$

or $A = |\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_2^2}$

MAGNITUDES OF UNIT VECTORS $\hat{i}$, $\hat{j}$, $\hat{k}$

$$|\hat{i}| = |1\hat{i}+0\hat{j}+0\hat{k}| = \sqrt{(1)^2+(0)^2+(0)^2} = 1$$

$$|\hat{j}| = |0\hat{i}+1\hat{j}+0\hat{k}| = \sqrt{(0)^2+(1)^2+(0)^2} = 1$$

$$|\hat{k}| = |0\hat{i}+0\hat{j}+1\hat{k}| = \sqrt{(0)^2+(0)^2+(1)^2} = 1$$

1.10 POSITION VECTOR

If we choose the initial point of a vector to be the origin and terminal point any point $P(x, y, z)$ in space, then its components are equal to the coordinates of the terminal point and the vector is then called the **position vector** or **radius vector** and is denoted by $\vec{r}$ as shown in figure (1.13).

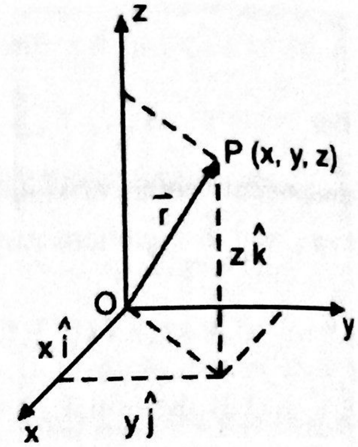

Figure (1.13)

Thus in terms of components, the position vector $\vec{r}$ is written as

$$\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$$

The magnitude of the position vector is given by $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$.

Note that this is the distance of the point P from the origin O.

1.11 FUNDAMENTAL DEFINITIONS USING ANALYTIC REPRESENTATION

EQUAL VECTORS

Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal, if and only if their corresponding components are equal. Thus if $\vec{A} = A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$ and $\vec{B} = B_1\hat{i}+B_2\hat{j}+B_3\hat{k}$, then $\vec{A} = \vec{B}$ implies that $A_1 = B_1$, $A_2 = B_2$, $A_3 = B_3$.

NEGATIVE OF A VECTOR

The negative of a vector $\vec{A}$ denoted by $-\vec{A}$ is obtained by multiplying each component of $\vec{A}$ by minus sign. Thus if $\vec{A} = A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$, then

$$-\vec{A} = -(A_1\hat{i}+A_2\hat{j}+A_3\hat{k}) = -A_1\hat{i}-A_2\hat{j}-A_3\hat{k}$$

UNIT VECTOR

A vector with unit magnitude is called the unit vector. The unit vector in the direction of a non-zero vector $\vec{A}$ denoted by $\hat{a}$ is obtained by dividing each component of $\vec{A}$ by its magnitude.

Thus, if $\vec{A} = A_1\hat{i}+A_2\hat{j}+A_3\hat{k}$, then the magnitude of $\vec{A}$ is $A = |\vec{A}| = \sqrt{A_1^2+A_2^2+A_3^2}$

and $$\hat{a} = \frac{\vec{A}}{A} = \frac{A_1\hat{i}+A_2\hat{j}+A_3\hat{k}}{A} = \frac{A_1}{A}\hat{i}+\frac{A_2}{A}\hat{j}+\frac{A_3}{A}\hat{k}$$

the direction ... where $r = |\vec{r}| = \sqrt{x^2 + y^2 ...}$

$$\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$$

ECTOR

... or null vector denoted by $\vec{0}$ is a vector whose components are all zero. Thus

$\hat{i} + 0\hat{j} + 0\hat{k}$.

ION AND SUBTRACTION OF VECTORS

ION

... or resultant of two vectors $\vec{A}$ and $\vec{B}$ denoted by $\vec{A} + \vec{B}$ is a vector $\vec{C}$ obtained ...

...sponding components of $\vec{A}$ and $\vec{B}$. Thus if

... $\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

$$\vec{A} + \vec{B} = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) + (B_1\hat{i} + B_2\hat{j} + B_3\hat{k})$$

$$= (A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}$$

TRACTION

... difference of two vectors $\vec{A}$ and $\vec{B}$ denoted by $\vec{A} - \vec{B}$ is a vector $\vec{C}$ obtain...

...e components of $\vec{B}$ from the corresponding components of $\vec{A}$.

$$= \vec{A} - \vec{B} = (\vec{A}\hat{i} + A_2\hat{j} + A_3\hat{k}) - (B_1\hat{i} + B_2\hat{j} + B_3\hat{k})$$

$$= (A_1 - B_1)\hat{i} + (A_2 - B_2)\hat{j} + (A_3 - B_3)\hat{k}.$$

ALAR MULTIPLICATION

...t $\vec{A}$ be any vector and m any scalar. Then the product $m\vec{A}$ called the scalar multipl...

...r obtained by multiplying each component of $\vec{A}$ by m. Thus if $\vec{A} = A_1\hat{i} + A_2\hat{j} + A$...

$$m\vec{A} = m(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) = (mA_1)\hat{i} + (mA_2)\hat{j} + (mA_3)\hat{k}$$

For example, if $m = -2$ and $\vec{A} = -3\hat{i} + 6\hat{j} + 2\hat{k}$, then

$$|\vec{A}| = \sqrt{(-3)^2 + (6)^2 + (2)^2} = \sqrt{9 + 36 + 4} = \sqrt{49} = 7$$

and $$|-2\vec{A}| = |(-2)(-3\hat{i} + 6\hat{j} + 2\hat{k})| = |6\hat{i} - 12\hat{j} - 4\hat{k}|$$

$$= \sqrt{(6)^2 + (-12)^2 + (-4)^2}$$

$$= \sqrt{36 + 144 + 16} = \sqrt{196}$$

$$= 14 = |-2|7 = |m||\vec{A}| = |m|A$$

EXAMPLE (1): **Determine the vector having initial point $P(x_1, y_1, z_1)$ and terminal point $Q(x_2, y_2, z_2)$ and find its magnitude.**

SOLUTION: Let $\vec{r}_1$ and $\vec{r}_2$ be the position vectors of the points P and Q, respectively as shown in figure (1.14). Then

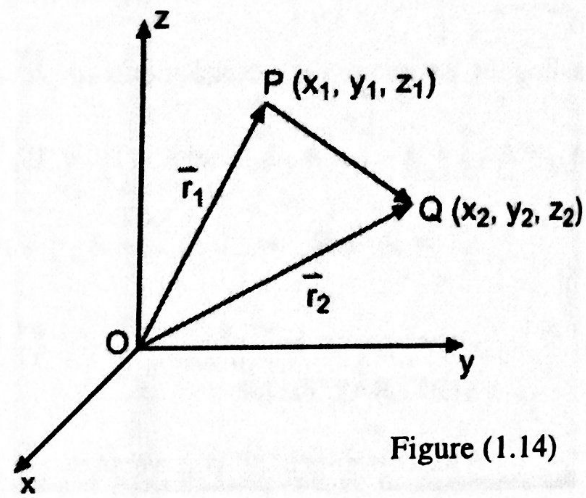

Figure (1.14)

$$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} \quad \text{and} \quad \vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

Now by the triangle law of vector addition, we have

$$\vec{r}_1 + \vec{PQ} = \vec{r}_2$$

or $$\vec{PQ} = \vec{r}_2 - \vec{r}_1$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

Magnitude of $\vec{PQ} = |\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Note that this is the distance between the points P and Q.

EXAMPLE (2): **Find a unit vector $\hat{a}$ in the direction of the vector from $P(1,0,1)$ to $Q(3,2,0)$.**

SOLUTION: The vector joining the points P and Q is

$$\vec{PQ} = (3-1)\hat{i} + (2-0)\hat{j} + (0-1)\hat{k} = 2\hat{i} + 2\hat{j} - \hat{k}$$

$$|\vec{PQ}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

Then the unit vector $\hat{a}$ in the direction of $\vec{PQ}$ is given by

$$\hat{a} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{2\hat{i} + 2\hat{j} - \hat{k}}{3} = \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$$

EXAMPLE (3): Let $\vec{A} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{B} = \hat{i} + 2\hat{j} + \hat{k}$, find

(i) $\vec{A} + \vec{B}$ (ii) $\vec{A} - \vec{B}$

SOLUTION: From the definitions of the vector sum and the vector difference, we have

(i) $$\vec{A} + \vec{B} = (2\hat{i} - 3\hat{j} + 4\hat{k}) + (\hat{i} + 2\hat{j} + \hat{k})$$
$$= (2+1)\hat{i} + (-3+2)\hat{j} + (4+1)\hat{k} = 3\hat{i} - \hat{j} + 5\hat{k}$$

(ii) $$\vec{A} - \vec{B} = (2\hat{i} - 3\hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + \hat{k})$$
$$= (2-1)\hat{i} + (-3-2)\hat{j} + (4-1)\hat{k} = \hat{i} - 5\hat{j} + 3\hat{k}$$

EXAMPLE (4): **Determine the condition under which two non–zero vectors**

$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ **and** $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$ **will be parallel.**

SOLUTION: If the vectors $\vec{A}$ and $\vec{B}$ are to be parallel, there must be a scalar m such that

$$\vec{B} = m\vec{A}$$

or $$B_1\hat{i} + B_2\hat{j} + B_3\hat{k} = m(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) = (mA_1)\hat{i} + (mA_2)\hat{j} + (mA_3)\hat{k}$$

Equating the corresponding components, we get

$B_1 = mA_1$, $B_2 = mA_2$, and $B_3 = mA_3$ for the same scalar m.

So the required condition is $$\frac{B_1}{A_1} = \frac{B_2}{A_2} = \frac{B_3}{A_3} = m$$

or $$A_1 : A_2 : A_3 = B_1 : B_2 : B_3$$

In other words, the corresponding components are proportional.

1.12 PROPERTIES OF VECTOR ADDITION

THEOREM (1.1): **If $\vec{A}$, $\vec{B}$, and $\vec{C}$ are any three vectors, then prove that**

(i) $\vec{A} + \vec{B} = \vec{B} + \vec{A}$ **(Commutative law for vector addition)**

(ii) $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$ **(Associative law for vector addition)**

(iii) $\vec{A} + \vec{0} = \vec{A}$ **($\vec{0}$ is the identity for vector addition)**

(iv) $\vec{A} + (-\vec{A}) = \vec{0}$ **($-\vec{A}$ is the inverse for $\vec{A}$)**

PROOF: (i) **GEOMETRICAL PROOF**

From the figure (1.15), we have $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$

or $\vec{A} + \vec{B} = \vec{C}$ (1)

Also $\overrightarrow{OR} + \overrightarrow{RQ} = \overrightarrow{OQ}$

or $\vec{B} + \vec{A} = \vec{C}$ (2)

From equations (1) and (2) we have

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

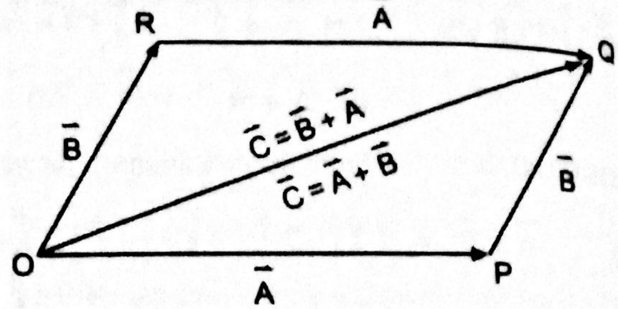

Figure (1.15)

ANALYTICAL PROOF

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$ then

$$\vec{A} + \vec{B} = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) + (B_1\hat{i} + B_2\hat{j} + B_3\hat{k})$$

$$= (A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}$$

$$= (B_1 + A_1)\hat{i} + (B_2 + A_2)\hat{j} + (B_3 + A_3)\hat{k}$$

$$= (B_1\hat{i} + B_2\hat{j} + B_3\hat{k}) + (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) = \vec{B} + \vec{A}.$$

(ii) GEOMETRICAL PROOF

From the figure (1.16), we have $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ} = (\vec{A} + \vec{B})$

and $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR} = (\vec{B} + \vec{C})$

Now $\overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR}$

i.e. $\vec{A} + (\vec{B} + \vec{C}) = \vec{D}$ (1)

Also $\overrightarrow{OQ} + \overrightarrow{QR} = \overrightarrow{OR}$

i.e. $(\vec{A} + \vec{B}) + \vec{C} = \vec{D}$ (2)

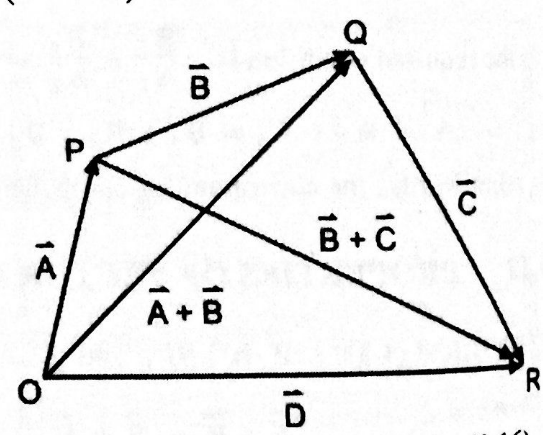

Figure (1.16)

From equations (1) and (2) $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

ANALYTICAL PROOF

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, $\vec{C} = C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$, then

$$\vec{A} + (\vec{B} + \vec{C}) = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) + [(B_1\hat{i} + B_2\hat{j} + B_3\hat{k}) + (C_1\hat{i} + C_2\hat{j} + C_3\hat{k})]$$

$$= (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) + [(B_1 + C_1)\hat{i} + (B_2 + C_2)\hat{j} + (B_3 + C_3)\hat{k}]$$

$$= [A_1 + (B_1 + C_1)]\hat{i} + [A_2 + (B_2 + C_2)]\hat{j} + [A_3 + (B_3 + C_3)]\hat{k}$$

$$= [(A_1+B_1)+C_1]\hat{i}+[(A_2+B_2)+C_2]\hat{j}+[(A_3+B_3)+C_3]\hat{k}$$

$$= [(A_1+B_1)\hat{i}+(A_2+B_2)\hat{j}+(A_3+B_3)\hat{k}]+[C_1\hat{i}+C_2\hat{j}+C_3\hat{k}]$$

$$= [(A_1\hat{i}+A_2\hat{j}+A_3\hat{k})+(B_1\hat{i}+B_2\hat{j}+B_3\hat{k})]+(C_1\hat{i}+C_2\hat{j}+C_3\hat{k})$$

$$= (\vec{A}+\vec{B})+\vec{C}$$

The proofs of (iii) and (iv) are obvious .

1.13 PROPERTIES OF SCALAR MULTIPLICATION

THEOREM (1.2): If $\vec{A}$ and $\vec{B}$ are vectors and m and n are scalars , then

(i) $m\vec{A} = \vec{A}m$ **(Commutative law for scalar multiplication)**

(ii) $m(n\vec{A}) = (mn)\vec{A}$ **(Associative law for scalar multiplication)**

(iii) $(m+n)\vec{A} = m\vec{A}+n\vec{A}$ **(Distributive law for scalar multiplication)**

(iv) $m(\vec{A}+\vec{B}) = m\vec{A}+m\vec{B}$ **(Distributive law for scalar multiplication)**

PROOF: (i) $m\vec{A} = m(A_1\hat{i}+A_2\hat{j}+A_3\hat{k})$

$$= (mA_1)\hat{i}+(mA_2)\hat{j}+(mA_3)\hat{k}$$

$$= (A_1m)\hat{i}+(A_2m)\hat{j}+(A_3m)\hat{k}$$

$$= (A_1\hat{i}+A_2\hat{j}+A_3\hat{k})m = \vec{A}m$$

(ii) $m(n\vec{A}) = m[n(A_1\hat{i}+A_2\hat{j}+A_3\hat{k})]$

$$= m(nA_1\hat{i}+nA_2\hat{j}+nA_3\hat{k})$$

$$= mnA_1\hat{i}+mnA_2\hat{j}+mnA_3\hat{k}$$

$$= mn(A_1\hat{i}+A_2\hat{j}+A_3\hat{k}) = (mn)\vec{A}$$

(iii) $(m+n)\vec{A} = (m+n)(A_1\hat{i}+A_2\hat{j}+A_3\hat{k})$

$$= (m+n)A_1\hat{i}+(m+n)A_2\hat{j}+(m+n)A_3\hat{k}$$

$$= (mA_1+nA_1)\hat{i}+(mA_2+nA_2)\hat{j}+(mA_3+nA_3)\hat{k}$$

$$= (mA_1\hat{i}+mA_2\hat{j}+mA_3\hat{k})+(nA_1\hat{i}+nA_2\hat{j}+nA_3\hat{k})$$

$$= m(A_1\hat{i}+A_2\hat{j}+A_3\hat{k})+n(A_1\hat{i}+A_2\hat{j}+A_3\hat{k}) = m\vec{A}+n\vec{A}$$

(iv) $m(\vec{A}+\vec{B}) = m[(A_1\hat{i}+A_2\hat{j}+A_3\hat{k})+(B_1\hat{i}+B_2\hat{j}+B_3\hat{k})]$