Lemma (10) . Page(56)

Prove that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m,n-m)$$

Proof Let

$m = j$ | $n = i - j$
$m \geq 0$ | $n \geq 0$
$\Rightarrow j \geq 0$ | $i - j \geq 0$
$0 \leq j$ | $\Rightarrow j \leq i$

$\Rightarrow 0 \leq j \leq i$

$0 \leq j \leq i < \infty \ , \ 0 \leq i < \infty$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m,n) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} A(j, i-j)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m, n-m).$$

Prove that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m, n+m)$$

Proof: Consider

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m, n+m)$$

Let $m = j$, $n = i - j$, $i - j \geq 0$

$i \geq j \Rightarrow 0 \leq j \leq i$

$0 \leq j \leq i < \infty \ , 0 \leq i < \infty$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{i} A(j, i)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m, n)$$

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$$d_2 = \frac{(a)_2 (b)_2}{(c)_2 \cdot 2!} d_0$$

$$d_3 = \frac{(a)_3 (b)_3}{(c)_3 \, 3!} d_0$$

After continuing the Process, we have

$$d_n = \frac{(a)_n (b)_n}{(c)_n \, n!} d_0$$

So
eq. 2 becomes.

$$\cancel{y = \sum_{n=0}^{\infty} \frac{(a_n)(b_n)}{(c)_n \, n!} d_0 x^n} \quad y = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} d_0 \frac{x^n}{n!}$$

For convience Take $d_0 = 1$ $\quad$ $\boxed{\because (-m)_n = 0 \text{ if } m < n}$

$$y = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{x^n}{n!}$$

Theorem: If $|z| < 1$ and $\left|\frac{z}{1-z}\right| < 1$, then.

$$F(a, b; c; z) = (1-z)^{-a} F\left(a, c-b; c; \frac{-z}{1-z}\right)$$

Proof: $(1-z)^{-a} F\left(a, c-b; c, \frac{-z}{1-z}\right)$

$$= (1-z)^{-a} \sum_{m=0}^{\infty} \frac{(a)_m (c-b)_m}{(c)_m} \frac{(-1)^m z^m}{(1-z)^m \, m!}$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m (c-b)_m}{(c)_m} \cdot \frac{(-1)^m z^m (1-z)^{-(a+m)}}{m!}$$

$$\boxed{\because (1-t)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n t^n}{n!}} \qquad \because (a+m)_n (a)_m = (a)_{m+n}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a+m)_n (a)_m (c-b)_m (-1)^m z^{n+m}}{(c)_m \, m! \, n!}$$

Now using Lemma.

$$\boxed{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A(m,n-m)} \quad -(1)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{n+m}\,(c-b)_m\,(-1)^m z^{n+m}}{(c)_m\, m!\, n!}$$

using Lemma (1)

$(1-z)^{-a} F(a, c-b; c, -z/1-z)$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(a)_n\,((c-b)_m(-1)^m z^n}{(c)_m\; m!\,(n-m)!}$$

Since

$$(\alpha)_{n-m} = \frac{(-1)^m(\alpha)_n}{(1-\alpha-n)_m}$$

Put $\alpha = 1$

$$(1)_{n-m} = \frac{(-1)^m(1)_n}{(-n)_m}$$

$\Rightarrow$

$$(n-m)! = \frac{(-1)^m\, n!}{(-n)_m}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{(a)_n\,(c-b)_m}{(c)_m \cdot m!} \cdot \frac{(-n)_m\,(-1)^m z^n}{(-1)^m\, n!}$$

$$= \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n} \frac{(-n)_m (c-b)_m}{(c)_m} \cdot \frac{1}{m!}\right] \frac{(a)_n z^n}{n!}$$

$$= \sum_{n=0}^{\infty} F(-n, c-b; c; 1)\, \frac{(a)_n z^n}{n!}$$

$$\boxed{\text{Since } F(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n}}$$

$$= \sum_{n=0}^{\infty} \frac{(c-(c-b))_n}{(c)_n}\, \frac{(a)_n z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n\, z^n}{(c)_n\, n!} = F(a, b; c; z)$$

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Theorem (21): Page (60)

If $|z|<1$, then

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; z)$$

Proof: Since

$$F(a;b;c;z) = (1-z)^{-a} F\left(a, c-b; c; \frac{-z}{1-z}\right)$$

Let

$$y = \frac{-z}{1-z} \Rightarrow y(1-z) = -z$$

$$\Rightarrow y - yz = -z \Rightarrow y = z(y-1)$$

$$\Rightarrow z = \frac{-y}{1-y}$$

So consider

$$F\left(a; \overset{c-b}{\cancel{b}}; c; \frac{-z}{1-z}\right) = F(a; c-b; c; y)$$

$$(1-z)^a F(a,b;c;z) = F(c-b, a; c; y)$$

$$\Rightarrow (1-z)^a F(a,b;c;z) = (1-y)^{-(c-b)} F\left(c-b, c-a; c; \frac{-y}{1-y}\right)$$

Put $y = \dfrac{-z}{1-z}$ and $z = \dfrac{-y}{1-y}$

$$(1-z)^a F(a,b;c;z) = \left(1 + \frac{z}{1-z}\right)^{-c+b} F(c-a, c-b; c; z)$$

$$\Rightarrow F(a,b;c;z) = (1-z)^{-a} (1-z)^{c-b} F(c-a, c-b; c; z)$$

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b, c; z)$$

$$\boxed{(1-z)^a F(a,b;c;z) = F\left(a, c-b; c; \frac{-z}{1-z}\right)}$$

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Contigrous Functions:

A function $F(a, b; c; z)$ is said to be Contiguous ~~functio~~ to the six of the functions if we increase or decrease the three parameters a, b, c by unity.
The six functions are $F(\mp a)$, $F(\mp b)$, $F(\mp c)$

$$F(a+) = F(a+1, b; c; z)$$

$${}_2F_1(a; b; c; z) = F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \delta_n, \quad \delta_n = \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

$$F(a+) = F(a+1, b; c; z)$$

$$= \sum_{n=0}^{\infty} \frac{(a+1)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

Since $(a+1)_n = (a+1)(a+2)\cdots(a+\overline{n+1}-2)(a+\overline{n+1}-1)$

$$= \frac{(a)_n (a+n)}{a}$$

$$= \sum_{n=0}^{\infty} \frac{a+n}{a} \cdot \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

$$F(a+) = \sum_{n=0}^{\infty} \frac{a+n}{a} \delta_n$$

Similarly

$$F(b+) = \sum_{n=0}^{\infty} \frac{b+n}{b} \delta_n$$

$$F(c+) = \sum_{n=0}^{\infty} \frac{c+n}{c} \delta_n$$

and

$$F(a-) = F(a-1, b; c; z)$$

$$= \sum_{n=0}^{\infty} \frac{(a-1)_n (b)_n}{c_n} \frac{z^n}{n!}$$

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Since $(a-1)_n = \frac{(a-1)a(a+1)\cdots(a-1+n-1)(a+n-1)}{a+n-1}$

$= \frac{(a-1)}{a+n-1}(a)_n.$

$F(a-) = \sum_{n=0}^{\infty} \frac{a-1}{a+n-1}\delta_n.$

$F(b-) = \sum_{n=0}^{\infty} \frac{b-1}{b+n-1}\delta_n$

$F(c-) = \sum_{n=0}^{\infty} \frac{c+n-1}{c-1}\delta_n$

The six of the functions are said to be Contiguous to the function $F(a, b; c; z)$ if we increase or decrease the three parameters by a, b, c by unity. The six functions are $F(a\pm)$, $F(b\pm)$, $F(c\pm)$

Contiguous Functions Relations:

1) Prove $(a-b)F = aF(a+) - bF(b+)$

Here $\theta = z\frac{d}{dz}$

$(\theta+a)F = \sum_{n=0}^{\infty}(a+n)\delta_n$

$\Rightarrow (\theta+a)F = \sum_{n=0}^{\infty} a\frac{(a+n)}{a}\delta_n = aF(a+)$ —(i)

$(\theta+b)F = \sum_{n=0}^{\infty}(b+n)\delta_n = bF(b+)$ —(ii)

(i) − (ii) $\Rightarrow (a+\theta)F - (\theta+b)F = aF(a+) - bF(b+)$

$\Rightarrow (a-b)F = aF(a+) - bF(b+)$

2) $(a-c+1)F = aF(a+) - (c-1)F(c-)$

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since $(\theta+c-1)F = \sum_{n=0}^{\infty} (n+c-1)\delta_n = (c-1)F(c-)$ (iii)

(i) − (iii) ⇒

$(a-c+1)F = aF(a+) - (c-1)F(c-)$.

3) $[a+(b-c)z]F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)\, zF(c+)$

$\theta = z\frac{d}{dz}$

Since $\theta F = \sum_{n=0}^{\infty} n\,\delta_n$

$= \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (n-1)!} \cdot \frac{z^n}{(n-1)!}$

$= \sum_{n=0}^{\infty} \frac{(a)_{n+1} (b)_{n+1}}{(c)_{n+1}} \cdot \frac{z^{n+1}}{n!}$

$= z\sum_{n=0}^{\infty} \frac{(a+n)(b+n)}{(c+n)}\delta_n$

$= z\sum_{n=0}^{\infty} \left[n + (a+b-c) + \frac{(c-a)(c-b)}{c+n}\right]\delta_n$

$= z\sum_{n=0}^{\infty} n\,\delta_n + z(a+b-c)\sum_{n=0}^{\infty}\delta_n + \frac{z(c-a)(c-b)}{c+n}\sum_{n=0}^{\infty}\delta_n$

$= z\sum_{n=0}^{\infty} n\,\delta_n + z(a+b-c)\sum_{n=0}^{\infty}\delta_n + z(c-a)(c-b)\sum_{n=0}^{\infty}\frac{1}{c+n}\delta_n$

$\theta F = z\,\theta F + z(a+b-c)F + z(c-a)(c-b)\frac{1}{c}F(c+)$

$(1-z)\theta F = z(a+b-c)F + (c-a)(c-b)\frac{z}{c}F(c+)$ (iv)

From (1) $\theta F = aF(a+) - aF$

put in (iv)

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$$(1-z)(aF(a+)-aF) = z(a+b-c)F + \frac{(c-a)(c-b)}{c} zF(c+)$$

$$\Rightarrow (1-z)aF(a+) - (1-z)aF = z(a+b-c)F + (c-a)(c-b)\frac{z}{c}F(c+)$$

$$\Rightarrow (1-z)aF(a+) - aF + zaF = z(a+b-c)F + \text{''}$$

$$\Rightarrow aF - zaF + (a+(b-c)z)F = (1-z)aF(a+) + \text{''}$$

$$\Rightarrow (a+(b-c)z)F = (1-z)aF(a+) - c^{-1}(c-a)(c-b)zF(c+)$$

4) $(1-z)F = F(a-) - c^{-1}(c-b)zF(c+)$

Consider $\theta F(a-) = \theta \sum_{n=0}^{\infty} \frac{a-1}{a+n-1} \cdot \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$

$$= \theta \sum_{n=0}^{\infty} \frac{(a-1)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad \because \frac{(a-1)(a)_n}{a+n-1} = (a-1)_n$$

$$= \sum_{n=1}^{\infty} \frac{(a-1)_n (b)_n}{(c)_n} \frac{z^n}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(a-1)_{n+1} (b)_{n+1}}{(c)_{n+1}} \frac{z^{n+1}}{n!} \qquad \text{Replace } n \text{ by } n+1.$$

$$= \sum_{n=0}^{\infty} \frac{(a-1)(a)_n (b+n)(b)_n}{(c+n)(c)_n} \frac{z^{n+1}}{n!} \qquad \because (a-1)_{n+1} = (a-1)(a)_n \;\&\; (b)_{n+1} = (b+n)(b)_n,\; (c)_{n+1} = (c+n)(c)_n$$

$$= (a-1)z \sum_{n=0}^{\infty} \frac{b+n}{c+n} \delta_n$$

Since $\frac{b+n}{c+n} = 1 - \frac{c-b}{c+n}$

So $= (a-1)z \sum_{n=0}^{\infty} \delta_n \left[1 - \frac{c-b}{c+n}\right]$

$$\theta F(-a) = (a-1)z \sum_{n=0}^{\infty} \delta_n - \frac{(a-1)(c-b)}{c} z \sum_{n=0}^{\infty} \frac{c}{c+n} \delta_n \quad -(V)$$

$$\theta F(-a) = (a-1)zF - \frac{(a-1)(c-b)}{c} z F(c+)$$

Replace a by a-1 in (i)
Since ① is

$$(\theta + a)F = aF(a+)$$

$$\Rightarrow \theta F + aF = aF(a+)$$

$$\theta F = aF(a+) - aF$$

$$\theta F(a-) = (a-1)F - (a-1)F(a-)$$

using in (4) we have

$$(1-z)(a-1)F - (a-1)F(a-)$$
$$= (a-1)zF - \frac{(a-1)(c-b)}{c} zF(c+)$$

$$\Rightarrow F - F(a-) = zF - c^{-1}(c-b)zF(c+)$$

$$\Rightarrow (1-z)F = F(a-) - c^{-1}(c-b)zF(c+)$$

Since a & b may be interchanged without affecting the hyper geometric series
So we have

5) $(1-z)F = F(b-) - c^{-1}(c-a)zF(c+)$

6) $[2a - c + (b-a)z]F = a(1-z)F(a+) - (c-a)F(a-)$

From ③ & ④

③ $-(c-a) \times 4$

$$[a + (b-c)z]F - (c-a)(1-z)F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)zF(c+) - (c-a)F(a-) + c^{-1}(c-a)c^{-1}(c-b)zF(c+)$$

$$\Rightarrow [a + bz - cz - (c - cz - a + az)]F$$
$$= a(1-z)F(a+) - (c-a)F(a-)$$

$$\Rightarrow [a + bz - \cancel{cz} - c + \cancel{cz} + a - az]F = a(1-z)F(a+) - (c-a)F(a-)$$

$$\Rightarrow [2a - c + (b-a)z]F = a(1-z)F(a+) - (c-a)F(a-)$$

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(7) $(a+b-c)F = a(1-z)F(a+) - (c-b)F(b-)$

from ③ & ⑤

③ $-$ $(c-b)$ ⑤

$[a+(b-c)z]F - (c-b)(1-z)F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)zF(c+) - (c-b)F(b-) + c^{-1}(c-b)(c-a)zF(c+)$

$\Rightarrow$

$[a+bz-cz-c+cz+b-bz]F = a(1-z)F(a+) - (c-b)F(b-)$

$(a+b-c)F = a(1-z)F(a+) - (c-b)F(b-)$

(8) $(c-a-b)F = (c-a)F(a-) - b(1-z)F(b+)$

from ① & ⑥

$(1-z)$ ① $-$ ⑥

$[(a-b)(1-z)F] - [2a-c+(b-a)z]F = (1-z)aF(a+) - (1-z)b(Fb+) - a(1-z)F(a+) + (c-a)F(a-)$

$\Rightarrow [a-a\cancel{z}-b+b\cancel{z}-2a+c-\cancel{b}z+\cancel{az}]F = (c-a)F(a-) - b(1-z)F(b+)$

$\Rightarrow [c-a-b]F = (c-a)F(a-) - b(1-z)F(b+)$

(9) $(b-a)(1-z)F = (c-a)F(a-) - b(1-z)F(b+)$

From ⑦ $-$ ⑥ we have

$(a+b-c)F - [2a-c+(b-a)z]F = a(1-z)F(a+) - (c-b)F(b-) - a(1-z)F(a+) + (c-a)F(a-)$

$[aF+bF-\cancel{cF}-2aF+\cancel{cF}-bz+az]F = (c-a)F(a-) - (c-b)F(b-)$

$[-a(1-z)+b(1-z)]F = (c-a)F(a-) - (c-b)F(b-)$

$\Rightarrow (b-a)(1-z)F = $ "

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(10) $[1 - a + (c-b-1)z]F = (c-a)F(a-) - (c-1)(1-z)F(c-)$

$(1-z) \times$ (2) $-$ (6) implies that

$$(1-z)(a-c+1)F - (2a-c+(b-a)z)F = (1-z)aF(a+) - (1-z)(c-1)F(c-) - a(1-z)F(a+) + (c-a)F(a-)$$

$$\Rightarrow [(1-z)(a-c+1) - 2a + c - bz + az]F = (c-a)F(a-) - (c-1)(1-z)F(c-)$$

$$\Rightarrow [1+\cancel{a} - \cancel{az} - \cancel{c} + cz - z - 2\cancel{a} + \cancel{c} - bz + \cancel{az}]F = \text{''}$$

$$\Rightarrow [1 - a + cz - bz - z]F = \text{''}$$

$$\Rightarrow [1 - a + (c-b-1)z]F = (c-a)F(a-) - (c-a)(1-z)F(c-)$$

(11) $[2b - c + (a-b)z]F = b(1-z)F(b+) - (c-b)F(b-)$

Interchanging a & b in (6) we get (11).

(12) $[b + (a-c)z]F = b(1-z)F(b+) - c^{-1}(c-a)(c-b)zF(c+)$

Interchanging b to a and a to b in (3) we get (12)

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(13) $[b-c+1]F = bF(b+) - (c-1)F(c-)$

Interchanging a & b in (2) we get (13)

(14) $[1-b+(c-a-1)z]F = (c-b)F(b-) - (c-1)(1-z)F(c-)$

Interchanging a & b in (10) we get (14).

(15) $[c-1+(a+b+1-2c)z]F$

$$= (c-1)(1-z)F(c-) - c^{-1}(c-a)(c-b)zF(c+)$$

From

(3) $-(1-z)(2)$

$$\Rightarrow [a+(b-c)z]F - (1-z)(a-c+1)F$$

$$= a(1-z)F(\cancel{a+}) - c^{-1}(c-a)(c-b)zF(c+)$$

$$- a(1-\cancel{z})F(a+) + (1-z)(c-1)F(c-)$$

$$\Rightarrow [\cancel{a}+bz-cz-\cancel{a}+c-1+az-cz+z]F$$

$$= (1-z)(c-1)F(c-) - c^{-1}(c-a)(c-b)zF(c+)$$

$$\Rightarrow [c-1+(a+b+1-2c)z]F$$

$$= (1-z)(c-1)F(c-) - c^{-1}(c-a)(c-b)zF(c+$$

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