Exercise Ch: 2 Solutions.

Problem 1: Start with.

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right), \quad (1)$$ Prove that

$$2\frac{\Gamma'(2z)}{\Gamma(2z)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)} = 2\log 2. \quad (2)$$

Solution: using (1) in (2) we have.

$$2\frac{\Gamma'(2z)}{\Gamma(2z)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)}$$

$$= -2\gamma - \frac{2}{2z} + \gamma + \frac{1}{z} + \gamma + \frac{1}{z+1/2} - \sum_{n=1}^{\infty}\left[\frac{2}{2z+n} - \frac{2}{n}\right]$$

$$+ \sum_{n=1}^{\infty}\left[\frac{1}{z+n} - \frac{1}{n}\right] + \sum_{n=1}^{\infty}\left[\frac{1}{z+1/2+n} - 1/n\right]$$

$$= \frac{2}{2z+1} - \lim_{n\to\infty}\sum_{m=1}^{2n}\left(\frac{2}{2z+m} - \frac{2}{m}\right) + \lim_{n\to\infty}\sum_{m=1}^{n}\left[\frac{1}{z+m} - \frac{1}{m}\right]$$

$$+ \lim_{n\to\infty}\sum_{m=1}^{n}\left[\frac{2}{2z+2m+1} - \frac{1}{m}\right]$$

$$= \frac{2}{2z+1} + \lim_{n\to\infty}\left(\sum_{m=1}^{2n}\frac{-2}{2z+m} + 2H_{2n} + \sum_{m=1}^{n}\frac{2}{2z+2m} - H_n\right.$$

$$\left. + \sum_{m=1}^{n}\frac{2}{2z+2m+1} - H_n\right)$$

$$= \frac{2}{2z+1} + \lim_{n\to\infty}\sum_{m=1}^{2n}\left(\frac{-2}{2z+m}\right) + \lim_{n\to\infty}(2H_{2n} - 2H_n)$$

$$+\left[\frac{2}{2z+2} + \frac{2}{2z+4} + \cdots + \frac{2}{2z+2n}\right] + \left[\frac{2}{2z+3} + \frac{2}{2z+5} + \cdots + \frac{2}{2z+2n}\right.$$

$$= \frac{2}{2z+1} + \lim_{n\to\infty}\sum_{m=1}^{2n}\frac{-2}{2z+m} + \lim_{n\to\infty}\sum_{m=2}^{2n+1}\frac{2}{2z+m} + \lim_{n\to\infty}(2H_{2n} - 2$$

$$= \frac{2}{2z+1} + \lim_{n\to\infty}\left[\frac{-2}{2z+1} + \frac{-2}{2z+2} + \cdots + \frac{-2}{2z+2n}\right]$$

$$+ \lim_{n\to\infty}\left[\frac{+2}{2z+2} + \frac{2}{2z+3} + \cdots + \frac{2}{2z+2n+1}\right]$$

$$+ \lim_{n\to\infty}(2H_{2n} - 2H_n)$$

$$= \frac{2}{2z+1} + \lim_{n\to\infty} \frac{-2}{2z+1} + \lim_{n\to\infty} \frac{2}{2z+2n+1} + \lim_{n\to\infty}\left(2H_{2n} - 2H_n\right)$$

$$= 0 + \lim_{n\to\infty} \left(2H_{2n} - 2H_n\right)$$

$$= \lim_{n\to\infty} 2\left[(H_{2n} - \log 2n) - (H_n - \log n) + \log 2n - \log n \right]$$

$$= 2\left[\lim_{n\to\infty} (H_{2n} - \log 2n) - \lim_{n\to\infty} (H_n - \log n) + \lim_{n\to\infty} (\log 2)\right]$$

$$= 2\left[\gamma - \gamma + \log 2\right] = 2\log 2$$

Problem (2): Show that

$$\Gamma'(1/2) = -(\gamma + 2\log 2)\sqrt{\pi}$$

Solution: $\dfrac{2\Gamma'(2z)}{\Gamma(2z)} - \dfrac{\Gamma'(z)}{\Gamma(z)} - \dfrac{\Gamma'(z+1/2)}{\Gamma(z+1/2)} = 2\log 2$

Now let $z = 1/2$ to get

$$2\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(1/2)}{\Gamma(1/2)} - \frac{\Gamma'(1)}{\Gamma(1)} = 2\log 2$$

$$\frac{\Gamma'(1/2)}{\Gamma(1/2)} = \frac{\Gamma'(1)}{\Gamma(1)} - 2\log 2$$

Since $\Gamma(1) = 1$, $\Gamma'(1) = -\gamma$, $\Gamma(1/2) = \sqrt{\pi}$

hence $\dfrac{\Gamma'(1/2)}{\sqrt{\pi}} = -\dfrac{\gamma}{1} - 2\log 2$

$$\Rightarrow \quad \Gamma'(1/2) = -(\gamma + 2\log 2)\sqrt{\pi}$$

Hence Proved.

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Problem (3): Use Euler's Integral form

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad — ①$$

to show that $\Gamma(z+1) = z\Gamma(z)$

Solution: From ①

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^{z} dt$$

$$= -t^z e^{-t}\Big|_0^\infty + z\int_0^\infty e^{-t} \cdot t^{z-1} dt$$

$$= 0 + z\,\Gamma(z)$$

$$\Gamma(z+1) = z\Gamma(z)$$

where $\lim_{t\to\infty} -e^{-t}t^z$ Converges for $Re(z) > 0$.

Problem (4): Show that

$$\Gamma(z) = \lim_{n\to\infty} n^z B(z,n)$$

Solution: since $\Gamma(z) = \lim_{n\to\infty} \frac{(n-1)!\,n^z}{(z)_n}$

and $B(z,n) = \dfrac{\Gamma(z)\Gamma(n)}{\Gamma(z+n)}$ — (A)

Since $(\alpha)_n = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$

i.e $(z)_n = \dfrac{\Gamma(z+n)}{\Gamma(z)}$ — ①

$\because \Gamma(n) = (n-1)\Gamma(n-1)$
$= (n-1)(n-2)\,\Gamma(n-2)$
$= (n-1)(n-2)\cdots\Gamma(1)$
$= (n-1)!$ — ②

using ① & ② in (A), we have.

$$B(z,n) = \frac{(n-1)!}{(z)_n}$$

So $\Gamma(z) = \lim_{n\to\infty} n^z B(z,n)$

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Problem (5): Derive the following Properties of the Beta function.

(a) $P\,B(P, q+1) = q\,B(P+1, q)$

Solution: we Know that

$$B(P, q) = \frac{\Gamma(P)\,\Gamma(q)}{\Gamma(P+q)}, \text{ so}$$

So

$$P\,B(P, q+1) = \frac{P\,\Gamma(P)\,\Gamma(q+1)}{\Gamma(P+q+1)} = \frac{\Gamma(P+1)\,q\,\Gamma(q)}{\Gamma(P+1+q)}$$

$$= q\,B(P+1, q).$$

(b) $B(P, q) = B(P+1, q) + B(P, q+1)$

since

$$B(P, q) = \frac{\Gamma(P)\,\Gamma(q)}{\Gamma(P+q)}$$

$\because \Gamma(z+1) = z\Gamma(z)$

$\Gamma(P+q+1) = (P+q)\,\Gamma(P+q)$

$$= \frac{\Gamma(P)\,\Gamma(q)}{\frac{\Gamma(P+q+1)}{P+q}}$$

$$= \frac{(P+q)\,\Gamma(P)\,\Gamma(q)}{\Gamma(P+q+1)}$$

$$= \frac{P\,\Gamma(P)\,\Gamma(q) + q\,\Gamma(P)\,\Gamma(q)}{\Gamma(P+q+1)}$$

$$= \frac{\Gamma(P+1)\,\Gamma(q)}{\Gamma(P+1+q)} + \frac{\Gamma(P)\,\Gamma(q+1)}{\Gamma(P+q+1)}$$

$$= B(P+1, q) + B(P, q+1)$$

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(c) $(p+q)\,B(p, q+1) = q\,B(p,q)$

$$(p+q)\,B(p,q+1) = (p+q)\frac{\Gamma(p)\,\Gamma(q+1)}{\Gamma(p+q+1)}$$

$$= \Gamma(p)\,\Gamma(q+1)\cdot\frac{p+q}{\Gamma(p+q+1)}$$

$$= \Gamma(p)\,\Gamma(q+1)\cdot\frac{1}{\Gamma(p+q)}$$

$$= \frac{\Gamma(p)\;q\,\Gamma(q)}{\Gamma(p+q)} \qquad \because \Gamma(p+q+ \; = \Gamma(p+q)$$

$$= q\,B(p,q)$$

(d) $B(p,q)\,B(p+q,n) = B(q,n)\,B(q+n$

$$B(p,q)\,B(p+q,n) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\cdot\frac{\Gamma(p+q)\,\Gamma}{\Gamma(p+q+n}$$

$$= \frac{\Gamma(p)\,\Gamma(q)\,\Gamma(n)}{\Gamma(p+q+n)}$$

$$= \frac{\Gamma(q+n)\,\Gamma(p)\,\Gamma(q)\,\Gamma(n)}{\Gamma(q+n)\,\Gamma(p+q+n)}$$

$$= \frac{\Gamma(q)\,\Gamma(n)}{\Gamma(q+n)}\cdot\frac{\Gamma(q+n)\,\Gamma(p)}{\Gamma(q+n+p)}$$

$$= B(q,n)\,,\; B(q+n,p)$$

Problem (6): Show that for Positiv
integer n

$$B(p, n+1) = \frac{n!}{(p)_{n+1}}$$

Solution :

$$B(p,n+1) = \frac{\Gamma(p)\,\Gamma(n+1)}{\Gamma(p+n+1)}$$

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$$= \frac{\Gamma(P)\Gamma(n+1)}{\Gamma(P+n+1)}$$

$(B)_k = B(B+1)(B+2)\cdots(B+k-1)$

$$\therefore \Gamma(B+k) = (B)_k \Gamma(B)$$

So

$$\Gamma(P+1+n) = (P+1)_n \Gamma(P+1)$$

$= (B+k-1)(B+k-2)\cdots(B+2)(B+1)B\cdot\frac{\Gamma(B)}{\Gamma(B)}$

$$= \frac{\Gamma(P)\Gamma(n+1)}{(P+1)_n \Gamma(P+1)}$$

$\because \Gamma(n+1) = n!$

$= \frac{\Gamma(B+k)}{\Gamma(B)}$

$$= \frac{\Gamma(P)\, n!}{(P+1)_n P\Gamma(P)}$$

$$= \frac{n!}{P(P+1)_n}$$

$\because (P+1)_n P = (P)_{n+1}$

$$= \frac{n!}{(P)_{n+1}}$$

Problem(7): Evaluate $\int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1}\,dx$

Solution: Let $A = \int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1}\,dx$

now

Let $y = \frac{1+x}{2}$ when $x=1 \Rightarrow y=1$, $x=-1 \Rightarrow y=0$

$\Rightarrow 2y = 1+x$

$\Rightarrow 2y-1 = x$

$\Rightarrow 2-2y = 1-x$

$\Rightarrow 2(1-y) = 1-x$

Hence

$$A = \int_0^1 2^{p-1} y^{p-1} \cdot 2^{q-1}(1-y)^{q-1}\, 2\,dy$$

$$= 2^{p+q-1}\int_0^1 y^{p-1}(1-y)^{q-1}\,dy$$

$$= 2^{p+q-1} B(p,q) \quad \text{by definition.}$$

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Problem (08): Show that for $0 \le k \le n$

$$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k}$$

Note Particularly the special case $\alpha = 1$.

Solution: Consider $(\alpha)_{n-k}$ for $0 \le k \le n$, then

$$(\alpha)_{n-k} = \alpha \cdot (\alpha+1) \cdot \ldots \cdot (\alpha+n-k-1)$$

$$= \frac{\alpha(\alpha+1)\ldots(\alpha+n-k-1)\left[(\alpha+n-k)(\alpha+n-k+1)\ldots(\alpha+n-1)\right]}{\left[(\alpha+n-1)(\alpha+n-2)\ldots(\alpha+n-k)\right]}$$

$$= \cancel{\frac{(\alpha)_n}{(\alpha+n-k)_k}}$$

$$= \frac{(\alpha)_n}{(-1)^k \left[(1-\alpha-n)(2-\alpha-n)\ldots(k-\alpha-n)\right]}$$

$$= \frac{(\alpha)_n}{(-1)^k (1-\alpha-n)_k} = \frac{(\alpha)_n}{(-1)^n\left[(1-\alpha-n)(2-\alpha-n)\ldots(k-\alpha-n+1-1)\right]}$$

$$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k}.$$

Corrolary
~~Note~~ for $\alpha = 1$, we have

$$(1)_{n-k} = \frac{(-1)^k (1)_n}{(-n)_k}$$

$$\Rightarrow \boxed{(n-k)! = \frac{(-1)^k n!}{(-n)_k}}$$

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Problem (9) Show that if α is not an integer

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{(-1)^n}{(\alpha)_n}.$$

Solution: Consider an α which is not an integer

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{\Gamma(1-\alpha-n)}{-\alpha\,\Gamma(-\alpha)}$$

$$\boxed{\Gamma(z+1) = z\Gamma(z)}$$

$$= \frac{\Gamma(1-\alpha-n)}{(-\alpha)(-\alpha-1)\,\Gamma(-\alpha-1)}$$

$$= \frac{\Gamma(1-\alpha-n)}{(-\alpha)(-\alpha-1)\cdots\cdots(-\alpha-n+1)\,\Gamma(1-\alpha-n)}$$

$$= \frac{1}{(-1)^n\,\alpha\,(\alpha+1)\ldots(\alpha+n-1)}$$

$$= \frac{1}{(-1)^n(\alpha)_n}$$

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