2nd chapter

Topic NO 13: The order Symbols o and O

Let R be a region in a complex z-plane
If and only if

$$\lim_{z \to c \text{ in } R} \frac{f(z)}{g(z)} = 0$$

we write

$$f(z) = o[g(z)] \quad \text{as } z \to c \text{ in } R$$

If and only if $\left|\frac{f(z)}{g(z)}\right|$ is bounded as $z \to c$ in R.
we write

$$f(z) = O\,g(z) \quad \text{as } z \to c \text{ in } R$$

Example (a)

Since $\lim_{z \to 0} \frac{\sin^2 z}{z} = 0$, we may write

$$\sin^2 z = o(z) \quad \text{as } z \to 0$$

e.g If $\left|\frac{\cos x - 4x}{x}\right|$ is bounded as $x \to 0$ (x is real)

then $\cos x - 4x = O(x)$ as $x \to 0$

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Topic 16: The Beta Function:

we define $B(P,Q)$ as:

$$B(P,Q) = \int_0^1 t^{P-1}(1-t)^{Q-1}dt \quad , (i) Re(P) > 0 \quad , Re(Q) > 0$$

Lemma 1:

If we put $t = \sin^2\varphi$, $\Rightarrow t=0$, $\varphi = 0$

$t=1$, $\varphi = \pi/2$

$dt = 2\sin\varphi\cos\varphi$ in (i)

we write $B(P,Q)$ as

$$B(P,Q) = \int_0^{\pi/2} \sin^{2P-2}\varphi \,(1-\sin^2\varphi)^{Q-1}\, 2\sin\varphi\cos\varphi d\varphi$$

$$B(P,Q) = 2\int_0^{\pi/2} \sin^{2P-1}\varphi \cos^{2Q-1}\varphi \, d\varphi$$

Theorem:

If $Re(P) > 0$, $Re(Q) > 0$. Then

$$B(P,Q) = \frac{\Gamma(P)\Gamma(Q)}{\Gamma(P+Q)}$$

Proof:

Consider

$$\Gamma(P)\Gamma(Q) = \int_0^\infty u^{P-1}e^{-u}du \cdot \int_0^\infty v^{Q-1}e^{-v}dv$$

$$= \int_0^\infty \int_0^\infty u^{P-1} v^{Q-1} e^{-(u+v)} du dv$$

Let

$u = x^2$, $v = y^2$

$du = 2x dx$, $dv = 2y dy$

$u = 0 \Rightarrow x = 0$ $\quad$ $v=0 \Rightarrow y = 0$

$u \to \infty \Rightarrow x \to \infty$ $\quad$ $v \to \infty \Rightarrow y \to \infty$

$$\Gamma(P)\Gamma(q) = \int_0^\infty\int_0^\infty x^{2p-2}\, y^{2q-2}\, e^{-(x^2+y^2)} (2x\,dx)(2y\,dy)$$

$$\Gamma(P)\Gamma(q) = 4\int_0^\infty\int_0^\infty x^{2p-1}\, y^{2q-1}\, e^{-(x^2+y^2)}\, dx\,dy \quad -(ii)$$

$x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$, $0 < r < \infty$

$r^2 = x^2 + y^2$, $0 \le \theta \le \pi/2$

$$\because dx\,dy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta$$

$$dx\,dy = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta$$

$$dx\,dy = (r\cos^2\theta + r\sin^2\theta)\, dr\,d\theta$$

$$dx\,dy = r\,dr\,d\theta$$

(ii) =>

$$\Gamma(P)\Gamma(q) = 4\int_0^\infty\int_0^{\pi/2} e^{-r^2} (r\cos\theta)^{2p-1} (r\sin\theta)^{2q-1}\, r\,dr\,d\theta$$

$$= 4\int_0^\infty\int_0^{\pi/2} e^{-r^2} r^{2p-1} \cos^{2p-1}\theta\; r^{2q-1} \sin^{2q-1}\theta\; r\,dr\,d\theta$$

$$= 4\int_0^\infty\int_0^{\pi/2} e^{-r^2} r^{2p+2q-1} \sin^{2q-1}\theta \cos^{2p-1}\theta\, dr\,d\theta$$

$$\Gamma(P)\Gamma(q) = 2\int_0^\infty e^{-r^2} r^{2p+2q-1}\, dr \cdot 2\int_0^{\pi/2} (\sin\theta)^{2q-1} (\cos\theta)^{2p-1}\, d\theta \quad -(iii)$$

put $r^2 = t$, $\theta = \pi/2 - \varphi$

$r = \sqrt{t}$, $d\theta = -d\varphi$

$dr = \frac{1}{2\sqrt{t}}\, dt$ in (iii)

$r = 0$, $t = 0$, $\theta = 0$, $\varphi = \pi/2$

$r \to \infty$, $t \to \infty$, $\theta = \pi/2$, $\varphi = 0$

$$\text{(iii)} \Rightarrow \Gamma(P)\Gamma(q) = 2\int_0^\infty e^{-t}\, t^{p+q-1/2} \frac{1}{2\sqrt{t}}\, dt \cdot 2\int_{\pi/2}^{0} (\cos(\pi/2-\varphi))^{2p-1} (\sin(\pi/2-\varphi))^{2q-1} (-d\varphi)$$

$$\Gamma(P)\Gamma(q) = \int_0^\infty e^{-t}\, t^{p+q-\frac{1}{2}-\frac{1}{2}}\, dt \cdot 2\int_0^{\pi/2} \sin^{2p-1}\varphi \cos^{2q-1}\varphi\, d\varphi \quad -(iv)$$

using $B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}\varphi \cos^{2q-1}\theta \, d\varphi$

$\because \Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$

so $\Gamma(p+q) = \int_0^{\infty} e^{-t} t^{p+q-1} dt$

using $B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}\varphi \cos^{2q-1}\theta \, d\varphi$ and $\Gamma(p+q) = \int_0^{\infty} e^{-t} t^{p+q-1} dt$

in (iv)

(iv) $\Rightarrow$ $\Gamma(p)\Gamma(q) = \Gamma(p+q)\, B(p,q)$

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Hence proved.