15. Euler Integral for $\Gamma(z)$

Theorem 6: If $Re(z) > 0$ then $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$

Proof:

We shall prove this theorem by using following four lemmas

Lemma 1: If $0 \leq \alpha < 1$, $1+\alpha \leq \exp(\alpha) \leq (1-\alpha)^{-1}$

Proof:

$$1+\alpha = 1+\alpha \quad \text{——— (i)}$$

$$\exp(\alpha) = 1 + \frac{\alpha}{\lfloor 1} + \frac{\alpha^2}{\lfloor 2} + \cdots \infty \quad \text{——— (ii)}$$

$$(1-\alpha)^{-1} = (1+(-\alpha))^{-1}$$

$$= 1 + \alpha + \alpha^2 + \alpha^3 + \cdots \quad \text{(by Binomial series expansion)}$$

——— (iii)

After compairing the above series, we find the required results.

Lemma 2: If $0 \leq \alpha < 1$, $(1-\alpha)^n \geq 1 - n\alpha$, for n, a +ve integer.

Proof: $(1-n\alpha) \leq (1-\alpha)^n$ ——— (A)

We shall prove the statement (A) by using principle of mathematical induction

Case I: for $n = 1$

$1-\alpha = 1-\alpha$ true.

Case II: Suppose that statement (A) is true for

$n = m$ i.e. $1 - m\alpha \leq (1-\alpha)^m$ ——— (B)

Case III: for $n = m+1$

multiplying eq (B) by $(1-\alpha)$

$$(1-\alpha)(1-m\alpha) \leq (1-\alpha)^{m+1}$$

$$1 - m\alpha - \alpha + m\alpha^2 \leq (1-\alpha)^{m+1}$$

$$1 - (m+1)\alpha + m\alpha^2 \leq (1-\alpha)^{m+1}$$

$$1 - (m+1)\alpha \leq (1-\alpha)^{m+1} \qquad \because m\alpha^2 \geq 0$$

Lemma 3 If $0 < t < n$, n is +ve integer

$$0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq \frac{t^2 e^{-t}}{n}$$

Proof From Lemma 1

$$1+\alpha \leq \exp(\alpha) \leq (1-\alpha)^{-1}$$

Put $\alpha = \frac{t}{n}$

$$1 + \frac{t}{n} \leq \exp\left(\frac{t}{n}\right) \leq \left(1 - \frac{t}{n}\right)^{-1}$$

taking power 'n' throughout

$$\left(1+\frac{t}{n}\right)^n \leq \left(e^{\frac{t}{n}}\right)^n \leq \left(1 - \frac{t}{n}\right)^{-n}$$

$$\left(1+\frac{t}{n}\right)^n \leq e^t \leq \left(1 - \frac{t}{n}\right)^{-n}$$

taking reciprocal

$$\left(1+\frac{t}{n}\right)^{-n} \geq e^{-t} \geq \left(1 - \frac{t}{n}\right)^n$$

or $\left(1-\frac{t}{n}\right)^n \leq e^{-t} \leq \left(1+\frac{t}{n}\right)^{-n}$ —— ①

From ①

$$\left(1-\frac{t}{n}\right)^n \leq e^{-t}$$

$$\Rightarrow \quad 0 \leq e^{-t} - \left(1-\frac{t}{n}\right)^n \quad \text{—— (i)}$$

Now consider

$$e^{-t} - \left(1-\frac{t}{n}\right)^n = e^{-t}\left(1 - e^{t}\left(1-\frac{t}{n}\right)^n\right) \quad \text{—— ②}$$

Also from ① $e^{-t} \leq \left(1+\frac{t}{n}\right)^{-n}$

$$\Rightarrow \left(1+\frac{t}{n}\right)^n \leq e^{t}$$

Putting in ②

$$e^{-t} - \left(1-\frac{t}{n}\right)^n \leq e^{-t}\left[1 - \left(1+\frac{t}{n}\right)^n\left(1-\frac{t}{n}\right)^n\right]$$

$$\Rightarrow e^{-t} - \left(1-\frac{t}{n}\right)^n \leq e^{-t}\left[1 - \left(1-\frac{t^2}{n^2}\right)^n\right] \quad \text{—— ③}$$

From lemma ②

$$(1-\alpha)^n \geq 1 - n\alpha$$

Put $\alpha = \frac{t^2}{n^2}$

$$1 - \frac{t^2}{n^2}(n) \leq \left(1-\frac{t^2}{n^2}\right)^n$$

$$1 - \frac{t^2}{n} \leq \left(1-\frac{t^2}{n^2}\right)^n$$

so ③ becomes

$$e^{-t} - \left(1-\frac{t}{n}\right)^n \leq e^{-t}\left[1-\left(1-\frac{t^2}{n}\right)\right] = e^{-t}\frac{t^2}{n}$$

$$\Rightarrow \quad e^{-t} - \left(1-\frac{t}{n}\right)^n \leq e^{-t}\frac{t^2}{n} \quad \text{--- (ii)}$$

from (i) & (ii)

$$0 \leq e^{-t}\left(1-\frac{t}{n}\right)^n \leq e^{-t}\frac{t^2}{n}$$

Lemm 4: If n is integral and $Re(z) > 0$ then

$$\Gamma(z) = \lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt$$

Proof Consider

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt$$

Put $\frac{t}{n} = y \Rightarrow t = ny$, $dt = n\,dy$; $t=0, y=0$; $t=n, y=1$

$$= \int_0^1 (1-y)^n n^{z-1} y^{z-1}\, n\,dy$$

$$= n^z \int_0^1 (1-y)^n y^{z-1}\,dy \quad \text{--- ①}$$

Now consider

$$\int_0^1 \underset{\text{(II)}}{(1-y)^n}\, \underset{\text{(I)}}{y^{z-1}}\,dy$$

An integration by parts gives us reduction formula

$$= (1-y)^n \frac{y^{z-1+1}}{z-1+1}\Big|_0^1 - \int_0^1 n(1-y)^{n-1}(-1)\frac{y^z}{z}\,dy$$

$$= \frac{n}{z}\int_0^1 (1-y)^{n-1} y^z\,dy$$

⋮ ⋮

(after n-1 times integration)

$$= \frac{n}{z}\,\frac{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(z+1)(z+2)\cdots(z+n-1)}\int_0^1 y^{z+n-1}\,dy$$

$$= \frac{n(n-1)(n-2)\cdots 321}{z(z+1)(z+2)\cdots(z+n-1)(z+n)}$$

$$= \frac{n!}{z(z+1)\cdots(z+n)}$$

Put in ①, ① becomes.

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt = n^z\,\frac{n!}{z(z+1)(z+2)\cdots(z+n)}$$

$$\lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt = \lim_{n\to\infty}\frac{n^z\,n!}{z(z+1)(z+2)\cdots(z+n)}$$

$$= \Gamma(z)$$ (Pg 11, eq 4)

Proof – Theorem 6

Consider

$$\int_0^\infty e^{-t} t^{z-1} dt - \Gamma(z)$$

$$= \int_0^\infty e^{-t} t^{z-1} dt - \lim_{n\to\infty} \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1} dt \quad \left(\text{by lemma } 4\right)$$

$$= \lim_{n\to\infty} \left[\int_0^\infty e^{-t} t^{z-1} dt - \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1} dt\right]$$

$$= \lim_{n\to\infty} \left[\int_0^n e^{-t} t^{z-1} dt + \int_n^\infty e^{-t} t^{z-1} dt - \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1} dt\right]$$

$$= \lim_{n\to\infty} \left[\int_0^n \left[e^{-t} - \left(1-\frac{t}{n}\right)\right]^n t^{z-1} dt\right] + \lim_{n\to\infty} \int_n^\infty e^{-t} t^{z-1} dt$$

$$= \lim_{n\to\infty} \int_0^n \left[e^{-t} - \left(1-\frac{t}{n}\right)^n\right] t^{z-1} dt + 0 \longrightarrow (1)$$

Consider

$$\int_0^n \left[e^{-t} - \left(1-\frac{t}{n}\right)^n\right] t^{z-1} dt \leq \int_0^n \frac{t^2}{n} e^{-t} t^{z-1} dt \quad \left(\text{by lemma } 3\right)$$

$\Rightarrow$

$$\left|\int_0^n \left[e^{-t} - \left(1-\frac{t}{n}\right)^n\right] t^{z-1} dt\right| \leq \int_0^n \frac{t^2}{n} e^{-t} t^{Re(z)-1} dt$$

$$\left(\text{since } |t^z| = t^{Re(z)}\right)$$

$$\Rightarrow \left|\int_0^n \left[e^{-t} - \left(1-\frac{t}{n}\right)^n\right] t^{z-1} dt\right| \leq \frac{1}{n} \int_0^n t^{2+Re(z)-1} e^{-t} dt$$

$$\left|\int_0^n \left[e^{-t} - \left(1-\frac{t}{n}\right)^n\right] t^{z-1} dt\right| \leq \frac{1}{n} \int_0^n t^{Re(z)+1} e^{-t} dt$$

As $\int_0^n e^{-t} t^{Re(z)+1} dt$ converges

so $\int_0^n e^{-t} t^{Re(z)+1} dt$ is bounded

Therefore

$$\lim_{n\to\infty} \int_0^n \left[e^{-t} - \left(1-\frac{t}{n}\right)^n\right] t^{z-1}\, dt = 0$$

Hence ① becomes

$$\int_0^\infty e^{-t} t^{z-1}\, dt - \Gamma(z) = 0$$

$$\Rightarrow \boxed{\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt}$$

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