11. The Euler product for $\Gamma(z)$.

By defination of weierstrass product defination of $\Gamma(z)$ we obtain

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) \right].$$

Taking reciprocal.

$$\Gamma(z) = \frac{1}{z} e^{-\gamma z} \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right)^{-1} \exp\left(\frac{z}{n}\right) \right] \rightarrow ①$$

As we know that

$$\gamma = \lim_{n\to\infty} \left[H_n - \log n \right]$$

$$\log n = \log(n+1).$$

$$= \log n + \log(1)$$

$$= \lim_{n\to\infty} \left[H_n - \log(n+1) \right].$$

multiplying both sides by $-z$.

$$-\gamma z = \lim_{n\to\infty} \left[-zH_n + z\log(n+1) \right].$$

$$\because H_n = \sum_{m=1}^{n} \frac{1}{m}$$

$$= \lim_{n\to\infty} \left[-z \sum_{m=1}^{n} \frac{1}{m} + z \log(n+1) \right]$$

$$-\gamma z = \lim_{n\to\infty} \left[-z \sum_{m=1}^{n} \frac{1}{m} + z \sum_{m=1}^{n} \log\left(\frac{m+1}{m}\right) \right].$$

$$\sum_{m=1}^{n} \log\left(\frac{m+1}{m}\right) = \sum_{m=1}^{n} \log(m+1) - \log(m)$$

$$= \log(2) - \log(1) + \log(3) - \log(2)$$

$$+ \cdots + \log(n) - \log(n-1) + \log(n+1) - \log(n)$$

$$\sum_{m=1}^{n} \log\left(\frac{m+1}{m}\right) = \log(n+1).$$

Taking exponential on both sides

$$\exp(-\gamma z) = \lim_{n\to\infty}\left[\exp\left(\sum_{m=1}^{n} \frac{-z}{m}\right) \cdot \exp \sum_{m=1}^{n} \log\left(\frac{m+1}{m}\right)^{z}\right].$$

Put in eq ①.

$$\Gamma(z) = \frac{1}{z} e^{-\gamma z} \prod_{m=1}^{\infty}\left[\left(1+\frac{z}{n}\right)^{-1} \exp\left(\frac{z}{n}\right)\right].$$

$$z\Gamma(z) = \lim_{n\to\infty}\left[\exp\left(\sum_{m=1}^{n} \frac{-z}{m}\right) \exp \sum_{m=1}^{n} \log\left(\frac{m+1}{m}\right)^{z}\right]$$

$$\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)^{-1} \exp\left(\frac{z}{n}\right)\right].$$

$$= \lim_{n\to\infty}\left[\exp\left(\sum_{m=1}^{n} -\frac{z}{m}\right) \exp \sum_{m=1}^{n} \log\left(\frac{m+1}{m}\right)^{z}\right]$$

$$\lim_{n\to\infty} \prod_{m=1}^{n}\left[\left(1+\frac{z}{m}\right)^{-1} \exp\left(\frac{z}{m}\right)\right]$$

$$= \lim_{n\to\infty} \prod_{m=1}^{n}\left[\exp\left(-\frac{z}{m}\right)\left(1+\frac{1}{m}\right)^{z}\right]$$

$$\because \log \prod_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \log a_n.$$

$$\cdot \lim_{n\to\infty} \prod_{m=1}^{n}\left[\left(1+\frac{z}{m}\right)^{-1} \exp\left(\frac{z}{m}\right)\right].$$

$$= \lim_{n\to\infty} \prod_{m=1}^{n} \left[\exp\left(-\frac{z}{m}\right)\left(1+\frac{1}{m}\right)^{z}\left(1+\frac{z}{m}\right)^{-1}\exp\left(\frac{z}{m}\right)\right]$$

$$= \lim_{n\to\infty} \prod_{m=1}^{n} \left[\left(1+\frac{1}{m}\right)^{z}\left(1+\frac{z}{m}\right)^{-1}\right].$$

$$z\Gamma(z) = \prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right].$$

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right].$$

12. The difference equation

$$\Gamma(z+1) = z\Gamma(z).$$

proof:- from Euler's Product formula

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right] \to ①$$

put $z = z+1$.

$$\Gamma(z+1) = \frac{1}{z+1}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z+1}\left(1+\frac{z+1}{n}\right)^{-1}\right] \to ②$$

$\div$ing eq ② by eq ①

$$\frac{\Gamma(z+1)}{\Gamma(z)} = \frac{\frac{1}{z+1}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z+1} + \left(1+\frac{z+1}{n}\right)^{-1}\right]}{\frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right]}.$$

$$= \frac{z}{z+1}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z+1-z}\left(1+\frac{z}{n}\right)\left(1+\frac{z+1}{n}\right)^{-1}\right]$$

$$= \frac{z}{z+1} \prod_{n=1}^{\infty} \left[\frac{(1+1/n)(1+z/n)}{\left(1+\frac{z+1}{n}\right)} \right]$$

$$= \frac{z}{z+1} \prod_{n=1}^{\infty} \left[\frac{(1+1/n)(1+z/n)}{\left(1+\frac{z+1}{n}\right)} \right].$$

$$= \frac{z}{z+1} \lim_{n\to\infty} \prod_{m=1}^{n} \left[\frac{(1+1/m)(1+z/m)}{\left(1+\frac{z+1}{m}\right)} \right]$$

$$= \frac{z}{z+1} \lim_{n\to\infty} \prod_{m=1}^{n} \left[\frac{\left(\frac{m+1}{m}\right)\left(\frac{m+z}{m}\right)}{\left(\frac{m+z+1}{m}\right)} \right]$$

$$= \frac{z}{z+1} \lim_{n\to\infty} \prod_{m=1}^{n} \left[\left(\frac{m+1}{m}\right) \left(\frac{m+z}{m+z+1}\right) \right]$$

$$= \frac{z}{z+1} \lim_{n\to\infty} \left[\left((2) \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{(n+1)}{n}\right) \left(\frac{z+1}{z+2} \cdot \frac{z+2}{z+3} \cdot \frac{z+3}{z+4} \cdots \frac{n+z}{n+z+1} \right) \right]$$

$$= \frac{z}{z+1} \lim_{n\to\infty} \left[\frac{(n+1)(z+1)}{z+n+1} \right].$$

$$= \frac{z}{\cancel{z+1}} \cdot (\cancel{z+1}) \lim_{n\to\infty} \frac{(n+1)}{z+n+1}$$

$$= z \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)}{\left(\frac{z}{n}+1+1/n\right)} = z(1)$$

$$\frac{\Gamma(z+1)}{\Gamma(z)} = z \Rightarrow \Gamma(z+1) = z\Gamma(z)$$

is required result.