Presentation no 1

Chapter # 2
The Gamma and Beta Function

Topic 7.
The Euler or Mascheroni constant γ.

The Euler constant γ is defined by
$$\gamma = \lim_{n\to\infty} (H_n - \log n)$$
where $H_n = \sum_{m=1}^{n} \frac{1}{m}$

We shall prove that γ exist and that $0<\gamma<1$

For existance of γ

Let $A_n = H_n - \log n$ Now,

$$A_{n+1} - A_n = \{H_{n+1} - \log(n+1)\} - \{H_n - \log n\}$$
$$= H_{n+1} - H_n + \log n - \log(n+1)$$
$$= \sum_{m=1}^{n+1} \frac{1}{m} - \sum_{m=1}^{n} \frac{1}{m} + \log n - \log(n+1)$$
$$= \cancel{1} + \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{3}} + \cdots \cancel{\tfrac{1}{n}} + \frac{1}{n+1} - \left(\cancel{1} + \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{3}} \cdots \cancel{\tfrac{1}{n}}\right) + \log n - \log(n+1)$$
$$= \frac{1}{n+1} + \log n - \log(n+1)$$
$$= \frac{1}{n+1} + \log\left(\frac{n}{n+1}\right)$$
$$= \frac{1}{n+1} + \log\left(1 - \frac{1}{n+1}\right) \quad \text{--- (1)}$$

Expanding by Maclaurin series

then $\log(1-x) = -x - \frac{x^2}{2!} - \frac{2x^3}{3!} - \frac{3x^4}{4!} \cdots$

Now ① takes the form when we take $x = \frac{1}{n+1}$

$$\Rightarrow \cancel{\frac{1}{n+1}} + \left[-\cancel{\frac{1}{n+1}} - \frac{1}{2(n+1)^2} - \frac{1}{3(n+1)^3} + \cdots \infty \right]$$

$$= -\left[\frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \cdots \infty \right]$$

$$\Rightarrow A_{n+1} - A_n = -\sum_{m=1}^{\infty} \frac{1}{(m+1)(n+1)^{m+1}} < 0$$

Thus the sequence is decreasing

Furthermore, $\frac{1}{t}$ decreases as t increases so

$$\frac{1}{m} < \int_{m-1}^{m} \frac{dt}{t} < \frac{1}{m-1}$$

The sum of above inequality from $m=2$ to $m=n$

$$\sum_{m=2}^{n} \frac{1}{m} < \sum_{m=2}^{n} \int_{m-1}^{m} \frac{dt}{t} < \sum_{m=2}^{n} \frac{1}{m-1}$$

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \int_1^2 \frac{dt}{t} + \int_2^3 \frac{dt}{t} + \cdots \int_{n-1}^{n} \frac{dt}{t} < 1 + \frac{1}{2} + \frac{1}{3} \cdots \frac{1}{n-1}$$

Adding and subtracting 1 on left side of inequality and $\frac{1}{n}$ on right side

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - 1 < \log 2 - \log 1 + \log 3 - \log 2 + \cdots \log n - \log n-1$$

$$< 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n-1} + \frac{1}{n} - \frac{1}{n}$$

$\Rightarrow H_n - 1 < \log n < H_n - \frac{1}{n}$

Subtract H_n throughout

$-1 < -H_n + \log n < -\frac{1}{n}$

$1 > H_n - \log n > \frac{1}{n}$

or

$\frac{1}{n} < H_n - \log n < 1$

Taking limit $n \to \infty$ $\qquad \lim_{n\to\infty} \frac{1}{n} = 0$

$0 \le \lim_{n\to\infty} (H_n - \log n) < 1$

$\Rightarrow 0 \le \gamma < 1$ So proved

8 The Weierstrass Gamma function is defined as

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) \right]$$

9 A series for $\Gamma'(z)/\Gamma(z)$

As we know

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) \right]$$

Taking log on both sides

$$\log \frac{1}{\Gamma(z)} = \log \left[z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) \right] \right]$$

$$\Rightarrow \cancelto{0}{\log 1} - \log \Gamma(z) = \log z + \gamma z + \sum_{n=1}^{\infty} \left[\log\left(1 + \frac{z}{n}\right) - \frac{z}{n} \right] \quad \text{—(1)}$$

here above $\log \prod_{n=1}^{\infty} (a_n) = \log[a_1 a_2 a_3 \cdots]$

$= \log a_1 + \log a_2 + \log a_3 + \cdots$

$= \sum_{n=1}^{\infty} \log(a_n)$

Differentiate (1) w.r.t 'z'

$$-\frac{1}{\Gamma(z)} \Gamma'(z) = \frac{1}{z} + \gamma + \sum_{n=1}^{\infty} \left[\frac{1}{1 + z/n} \cdot \frac{1}{n} - \frac{1}{n} \right]$$

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$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{n=1}^{\infty}\left\{\frac{1}{z+n} - \frac{1}{n}\right\} \rightarrow (2)$$

10. This series is used to find $\Gamma'(1)$

Evaluation of $\Gamma(1)$ and $\Gamma'(1)$

In equation (2) put $z=1$ we get

$$\frac{\Gamma'(1)}{\Gamma(1)} = \frac{-1}{1} - \gamma - \sum_{n=1}^{\infty}\left\{\frac{1}{1+n} - \frac{1}{n}\right\}$$

as $\boxed{\Gamma(1)=1}$ we shall also prove it later

$$\Gamma'(1) = -1 - \gamma - \lim_{n\to\infty}\sum_{m=1}^{n}\left[\frac{1}{1+m} - \frac{1}{m}\right]$$

$$= -1 - \gamma - \lim_{n\to\infty}\left[\frac{1}{2} - 1 + \frac{1}{3} - \frac{1}{2} + \dots + \frac{1}{n+1} - \frac{1}{n}\right]$$

$$= -1 - \gamma + \lim_{n\to\infty}\left[1 - \frac{1}{n+1}\right]$$

$$= -1 - \gamma + 1 = -\gamma$$

So $\boxed{\Gamma'(1) = -\gamma}$

Now Find $\Gamma(1)$

As

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \cdot \prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)\exp\left(\frac{-z}{n}\right)\right]$$

put $z=1$

$$\frac{1}{\Gamma(1)} = 1 \cdot e^{\gamma} \prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right) \cdot e^{-\frac{1}{n}}\right]$$

$$= e^{\gamma} \prod_{n=1}^{\infty}\left[\frac{n+1}{n} \cdot e^{-\frac{1}{n}}\right]$$

$$= e^{\gamma} \lim_{n\to\infty} \prod_{m=1}^{n}\left[\frac{m+1}{m} e^{-\frac{1}{m}}\right].$$

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$$\frac{1}{\Gamma(1)} = e^{\gamma} \lim_{n\to\infty}\left[\frac{2}{1}e^{-1}\cdot\frac{3}{2}e^{-\frac{1}{2}}\cdot\frac{4}{3}e^{-\frac{1}{3}}\cdot\frac{5}{4}e^{-\frac{1}{4}} \cdots \frac{n+1}{n}e^{-\frac{1}{n}}\right]$$

$$= e^{\gamma} \lim_{n\to\infty}\left[\frac{n+1}{1} e^{-\left[1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}\right]}\right]$$

$$\Rightarrow \frac{1}{\Gamma(1)} = e^{\gamma} \lim_{n\to\infty}\left(n+1\, e^{-H_n}\right) \rightarrow (1)$$

$$\because H_n = \sum_{m=1}^{n} \frac{1}{n}$$

$\Rightarrow$ Since $\gamma = \lim_{n\to\infty}\left[H_n - \log n\right]$

$\Rightarrow$ $\gamma \approx \left[H_n - \log n\right]$

$H_n \approx \gamma + \log n$

$\Rightarrow$ $H_n = \gamma + \log n + \epsilon_n$

(1) becomes

$$\frac{1}{\Gamma(1)} = e^{\gamma} \lim_{n\to\infty}\left((n+1) e^{-\gamma - \log n - \epsilon_n}\right)$$

$$= e^{\gamma} \lim_{n\to\infty}\left((n+1)\, e^{-\gamma} e^{-\log n} e^{-\epsilon_n}\right)$$

$$= e^{\gamma} e^{-\gamma} \lim_{n\to\infty}(n+1)\frac{1}{n} \lim_{n\to\infty} e^{-\epsilon_n}$$

$$= 1\cdot 1 \lim_{n\to\infty} e^{-\epsilon_n} \quad \text{as } n\to\infty,\ \epsilon_n\to 0$$

$$\frac{1}{\Gamma(1)} = 1 \Rightarrow \Gamma(1) = 1 \quad \text{proved}$$