SIX

VECTOR SPACE STRUCTURE

6.1 INTRODUCTION

Vector space is an abstract mathematical model which is built on the foundation of an abelian group, via actions on the elements of the abelian group by the elements of the structure of a field. It thus, combines a model of a field with that of an abelian group showing colours of the actions of field elements on the elements of an abelian group. The readers have already been made familiar to the structures of abelian group in chapter 4 and that of a field in the earlier chapter 5. In this chapter, model of vector space is formed with the use of the concept of a function.

Let A be an abelian group and F be the structure of a field. A function f from the set F × A into A, is denoted by, $f: F \times A \longrightarrow A$ and defined by

$$f(\alpha, a) = \alpha \cdot a \in A, \text{ where } \alpha \in F \text{ and } a \in A$$

In this way, f causes an operation '·', by the elements of F on the elements of the group A, for all $\alpha \in F$ and $\forall a \in A$. $\alpha \cdot a$ is an element of the abelian group A, which is conveniently denoted by the juxtaposition $\alpha a = (\alpha a) = \alpha \cdot a$.

If A is the abelian group of all vectors of R^2 (or R^3) and F = R the field of real numbers then the operation '·' on R^2 by the elements of R is understood to be the operation of scalar multiplication by the elements of R to the vectors of R^2 (mentioned in chapter 2 on vector algebra). In order to form A to be called a vector space, we specialize the action of the field elements on A axiomatically as follows:

6.1.1 DEFINITION

Let V be an arbitrary abelian group and F, the structure of a field. A function

$f: F \times V \to V$, be from $(F \times V)$ into V. Let the action '·' which corresponds to f on V, by the field elements, be observed by the following **four** axioms:

1. $\alpha(a_1 + a_2) = \alpha a_1 + \alpha a_2$
2. $(\alpha+\beta)\, a = \alpha a + \beta a$
3. $(\alpha\beta)\, a = \alpha\, (\beta a) = \beta(\alpha a) = (\beta\alpha)\, a$
4. $1_F\, a = a$,

for all $a, a_1, a_2 \in A$ and $\forall \alpha, \beta \in F$, with multiplicative identity element 1_F of F.

Then V is called a **vector space** over the field F and is denoted by V(F).

In place of a group structure of an abelian group vector space can be built independently on a set as it follows by definition:

6.1.2. DEFINITION

Let V be a non-empty set with a commutative binary operation (+) on V. For a field F and a functional field-operation '.' from F × V into V, defined by $\alpha v \in V$, for $\alpha \in F$ and $v \in V$, V is said to form a vector space over the field F and denoted by V(F) if it observes the following axioms;

(a) (V, +) is an abelian group.

(b) The operation '.' on V observes that

(1) $\alpha\,(v_1 + v_2) = \alpha v_1 + \alpha v_2$, is distributive on addition (+)

(2) $(\alpha+\beta).v = \alpha v+\beta v$ (Field addition is distributive over group multiplication)

(3) $(\alpha\, \beta)v = \alpha(\beta\, v) = \beta(\alpha\, v) = (\beta\alpha)\, v$ (multiplication of field is association and commutation on V)

(4) $1_F v = v$,

for all $v, v_1, v_2 \in V$, $\forall\, \alpha\, , \beta \in F$ and 1_F, the multiplicative identity of the field F.

6.1.3 IMMEDIATE OBSERVATIONS FROM THE DEFINITION

(i) Every vector space falls in the category of an abelian group.

(ii) No Abelian group A forms a vector space unless field multiplication on A observes the axioms (1–4) of the definitions.

(iii) If, instead of field, a Ring R with identity is used to form V, a vector space is called a R-module.

(iv) All the elements of a vector space are called vectors.

(v) Same abelian group (V, +) forms a different vector space with different field. i.e., $V(F) \neq V(\overline{F})$, if $F \neq \overline{F}$.

(vi) If, either the abelian group V or the field F, is infinite, then the vector space V(F) is infinite.

(vii) V(F) is a finite vector space if V and F both are finite.

(viii) An abelian group V may form a vector space over one field and may not be a vector space over another field.

(ix) Each vector space V(F) must be a system closed under the group operation and the field operation, called **scalar multiplication.**

(x) If $S \subseteq V(F)$ and $T \subseteq V(F)$, then the sum (difference) $S \pm T = \{s \pm t : s \in S, t \in T\}$ is another subset of V(F).

(xi) If v is a non-zero vector of V(F) and $\alpha \in F$, then αv is a vector collinear with v, for all non-zero $\alpha \in F$.

(xii) Every field F is a vector space over itself.

6.1.4. PROPOSITION

Let V(F) be a vector space. If 0_V and 0_F be the additive identities of the group V and the field F respectively. Then,

(i) $\alpha 0_V = 0_V, \forall \alpha \in F$

(ii) ~~$0_F 0_V = 0_V$~~ (ii) $0_F v = 0_V$

(iii) $\alpha v = 0_V \Rightarrow$ either $\alpha = 0_F$ or $v = 0_V$

(iv) $(-\alpha) v = \alpha(-v) = -(\alpha v), \forall \alpha \in F, \forall v \in V$

(v) $\alpha (v_1 - v_2) = \alpha v_1 - \alpha v_2, \forall \alpha \in F, \forall v_1, v_2 \in V$

(vi) $\alpha v = v \Leftrightarrow \alpha = 1_F \in F$.

PROOF

(i) For $\alpha \in F$ and $v \in V$, $\alpha v \in V$ and $\alpha v = \alpha(0_V + v)$

i.e., $\alpha v = \alpha 0_V + \alpha v$, by def. 6.1.2 b(1)

$\Rightarrow \alpha v + 0_V = \alpha v + \alpha 0_V$, by definition

$\Rightarrow 0_V = \alpha 0_V, \forall \alpha \in F$, by the additive group cancellation law.

(ii) $\alpha v = (\alpha + 0_F) v$

$= \alpha v + 0_F v$, by b (1)

$\Rightarrow \quad \alpha v = \alpha v + 0_V = \alpha v + 0_F v$

$\Rightarrow \quad 0_V = 0_F v$, by the additive group cancellation law.

(iii) Suppose that $\alpha \neq 0_F$ and $v \neq 0_V$. Let $\alpha v = 0_V$. Then $\alpha^{-1} \in F$ and

hence $\alpha^{-1}(\alpha v) = \alpha^{-1} 0_V = 0_V$, by prop. 6.1.4 (i)

$\Rightarrow \quad (\alpha^{-1}\alpha) v = 0_V$, by def. 6.1.2 b(3)

$\Rightarrow \quad 1_F v = 0_V$

$\Rightarrow \quad v = 0_V$, by b(4)

which provides contradiction to the assumption that $v \neq 0_V$. Hence $\alpha \neq 0_F$ is wrong, which gives that $\alpha = 0_F$

The proof (iii) can be completed by the argument supposing, that $0_V \neq v$ and

$\alpha v = 0_V$

Then, $0_V = \alpha v = (\alpha + 1_F) v = \alpha v + 1_F v$

$= 0_V + v$, by def. 6.1.2 b(4)

$= v$

$\Rightarrow v = 0_V$

which contradicts to the supposition that $v \neq 0_V$.

Hence $\alpha = 0_F$.

(iv) $0_F v = 0_V = [\alpha + (-\alpha)] v = \alpha v + (-\alpha) v$, by def. 6.1.2 b(2)

$\Rightarrow \quad (-\alpha) v = -(\alpha v)$

Similarly, $0_V = \alpha 0_V = \alpha [v + (-v)]$

$= \alpha v + \alpha(-v)$, by def. 6.1.2 b(1)

$\Rightarrow \alpha(-v) = -(\alpha v)$,

which proves the equality (iv).

(v) $\alpha(v_1 - v_2) = \alpha\,[v_1 + (-v_2)] = \alpha v_1 + \alpha(-v_2)$

$= \alpha v_1 - (\alpha v_2),$ by prop. 6.1.4 (iv)

(vi) Suppose that $\alpha \neq 0_F$ and $\alpha v = v$, for $0_V \neq v$

Then $\alpha v = v \Rightarrow \alpha v - v = 0_V$

$\Rightarrow \alpha v + (-1_F)v = 0_V,$ by prop. 6.1.4 (iv)

$\Rightarrow (\alpha - 1_F)\, v = 0_V$

$\Rightarrow \alpha - 1_F = 0_F$, since $v \neq 0_V$ by prop. 6.1.4

$\Rightarrow \alpha = 1_F,$

Thus $\alpha v = v$ if, and only if $\alpha = 1_F$, whenever $v \neq 0_V$

6.1.5 PROPOSITION

Let V(F) be a vector space. Then the following cancellation laws h... within V(F);

(l_1) $\alpha v = \beta v \Rightarrow \alpha = \beta,\ \forall\, \alpha, \beta \in F,\ v\,(\neq 0_V) \in V$

(l_2) $\alpha v_1 = \alpha v_2 \Rightarrow v_1 = v_2,\ \forall \alpha \in F,\ \forall v_1, v_2 \neq 0_V$

PROOF

(l_1) $\alpha v = \beta v \Rightarrow \alpha v - \beta v = 0_V$

$\Rightarrow \alpha v + (-\beta)\, v = 0_V$

$\Rightarrow [\alpha + (-\beta)]\, v = 0_V$

$\Rightarrow \alpha + (-\beta) = 0_F$, v being non-zero by proposition 5.1.4

$\Rightarrow 0_F = \alpha - \beta \Rightarrow \alpha = \beta$, in F.

(l_2) $0 \neq \alpha$ and $\alpha v_1 = \alpha v_2 \Rightarrow \alpha v_1 - \alpha v_2 = 0_V$

$\Rightarrow \alpha v_1 + \alpha(-v_2) = 0_V,$ by def. 6.1

$\Rightarrow \alpha\,[v_1 + (-v_2)] = 0_V,$ by def. 6.1

$\Rightarrow \alpha\,(v_1 - v_2) = 0_V,$

$\Rightarrow v_1 - v_2 = 0_V \Rightarrow v_1 = v_2,$

6.2 IMPORTANT CONCEPTS OF VECTOR SPACE V(F)

The concepts which play an important role in the development of the structural theory of vector spaces are particularly explained to communicate them to the readers in a neat and explicit way. Special attention is required to be taken by the readers to learn the following concepts:

6.2.1 LINEAR COMBINATION OF A SET OF VECTORS OF V(F)

Let $\{v_1, v_2, v_3, \ldots, v_n\} = S$ be a finite subset of vectors of V(F). For each choice of $\alpha_1, \alpha_2, \ldots, \alpha_n$ elements, called scalars, of the field F, a vector $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = v$ is called a **linear combination** of the set S of vectors $v_1, v_2, \ldots, v_n$ of V(F). If we change another set of elements α_i $(i = 1, 2, \ldots, n)$ of the field F, then another vector obtained will also be the linear combination of the same set S of vectors of V(F). Infact, the set of all the linear combinations of the vectors of S form a huge set and is denoted by L(S) as defined below:

6.2.2 DEFINITION

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a finite subset of vector space V(F). For scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$, the set $\{v \in V: v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n; \alpha_i \in F, v_i \in S\}$ is called **linear span** of a subset S of V(F) and is denoted by L(S)

i.e., $L(S) = \{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n : \alpha_i \in F, v_i \in S\}$ is subset of V(F).

It is important to note that,

$S \subseteq L(S)$

$L(S) = 0_v$, if $S = \{0_v\}$

$L(S) = \{\alpha v : \text{for all } \alpha \in F\}$, if $S = \{v\}$ is a singleton set.

$L(v_i) \subseteq L(S)$, for each $v_i \in S$ where $i = 1, 2, \ldots, k$ $(k \leq n)$

$L(S) = L(v_1) + L(v_2) + \ldots + L(v_n)$, if $S = \{v_1, v_2, \ldots, v_n\}$

If S and T are subsets of V(F) then $S + T = \{s + t : s \in S, t \in T\}$ is a subset of V(F).

6.2.3 LEMMA

Let V(F) be a vector space and S, T be two subsets of V(F). Then

(i) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

(ii) $L(S \cup T) = L(S) + L(T)$

(iii) $L(L(S)) = L(S)$

PROOF

(i) Let $S \subseteq T$. If $T = \{v_1, v_2, \ldots, v_n\}$ and $S = \{v_1, v_2, \ldots, v_s\}$, where $s < n$.

Take $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_s v_s \in L(S)$. Then,

$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_s v_s = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_s v_s + 0_F v_{s+1} + \ldots + 0_F v_n \in L(T)$,
for all $v \in L(S) \Rightarrow v \in L(T) \Rightarrow L(S) \subseteq L(T)$.

(ii). If $S = \{v_1, v_2, \ldots, v_m\}$ and $T = \{w_1, w_2, \ldots w_s\}$,

then, $S \cup T = \{v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_s\}$

Take, $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m + \beta_1 w_1 + \beta_2 w_2 + \ldots + \beta_s w_s \in L(S \cup T)$,

where $\alpha_i, \beta_i \in F$. Then

$$v = (\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m) + (\beta_1 w_1 + \beta_2 w_2 + \ldots + \beta_n w_n) \in L(S) + L(T)$$

$$\Rightarrow L(S \cup T) \subseteq L(S) + L(T) \qquad \ldots\ldots\ldots\ldots (a)$$

If $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m \in L(S)$ and $\beta_1 w_1 + \beta_2 w_2 + \ldots + \beta_s w_s) \in L(T)$

then $v = (\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m) + (\beta_1 w_1 + \beta_2 w_2 + \ldots + \beta_s w_s) \in L(S) + L(T)$

$$\Rightarrow v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m + \beta_1 w_1 + \beta_2 w_2 + \ldots + \beta_s w_s \in L(S \cup T)$$

$$\Rightarrow L(S) + L(T) \subseteq L(S \cup T) \qquad \ldots\ldots\ldots\ldots (b)$$

(a) and (b) consequently imply that

$$L(S \cup T) = L(S) + L(T)$$

(iii) Let $v = \sum_{i=1}^{m} \alpha_i v_i \in L(S)$ and $\bar{v} = \sum_{i=1}^{m} \beta_i v_i \in L(S) \ldots$ etc

$$\bar{\bar{v}} = \sum_{i=1}^{m} \gamma_i v_i \in L(S)$$

Then, $\mu v + \lambda \bar{v} + \xi \bar{\bar{v}} + \ldots$

$$= \sum_{i=1}^{m} (\mu\alpha_i) v_i + \sum_{i=1}^{m} (\lambda\beta_i) v_i + \sum_{i=1}^{m} (\xi\gamma_i) v_i \ldots \in L(L(S))$$

$$= \sum_{i=1}^{m} (\mu\alpha_i + \lambda \beta_i + \xi\gamma_i) v_i \in L(S)$$

$\Rightarrow L(L(S)) \subseteq L(S)$ .... (a)

Since $S \subseteq L(S)$, $L(S) \subseteq L(L(S)) \subseteq L(S)$ .... (b)

therefore (a) and (b) $\Rightarrow L(L(S)) = L(S)$

6.2.4 LINEARLY DEPENDENT (L.D) SUBSET OF A VECTOR SPACE V(F)

Let v_1 be a vector of V(F). For a scalar $\alpha \in F$, if $\alpha v_1 = v_2$ then the vector v_2 of V(F) is called linearly dependent on v_1. If $\alpha = 0_F$, then $0_F v_1 = 0_v$, which means that 0_v is a vector dependent on any vector $v_1 \in V(F)$. Similarly a vector $v_3 = v_1 + \alpha v_1 = v_1 + v_2$, is L.D on v_1 and v_2 both. In fact, the concept of a linearly dependent vector is understood as a vector included in the linear span of those vectors on which it is linearly dependent. Concept of a linearly dependent subset of a vector space is defined as a subset of V(F) which contains a vector linearly dependent on the remaining vectors or is in linear Span of the remaining vectors.

6.2.5 DEFINITION

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a subset of vector space V(F). S is said to be **linearly dependent** if, there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$, not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0_v \qquad \ldots\ldots\ldots\ldots (1)$$

It may be understood that there exists a non-zero solution $(\alpha_1, \alpha_2, \ldots, \alpha_n) \neq (0, 0, \ldots 0)$ in F^n of the equation (1) i.e., if $\alpha_i \neq 0$ for some i, than

$$\alpha_i v_i = -\alpha_1 v_1 - \alpha_2 v_2 - \ldots \alpha_{i-1} v_{i-1} - \alpha_{i+1} v_{i+1} \ldots - \alpha_n v_n$$

giving $v_i = -\left(\frac{\alpha_1}{\alpha_i}\right) v_1 - \left(\frac{\alpha_2}{\alpha_i}\right) v_2 - \ldots - \left(\frac{\alpha_n}{\alpha_i}\right) v_n$ as a linear combination of the *remaining* vectors $\{v_1, v_2, \ldots v_{i-1}, v_{i+1}, \ldots v_n\}$ of S.

6.2.6 IMPORTANT OBSERVATIONS

It is important to note that,

(1) The zero vector 0_V of V(F) is always linearly dependent because $\alpha 0_V =$ for non-zero $\alpha (\neq 0_F)$.

(2) Any finite subset S of V(F), which contains zero vector is L.D.

(3) Let $S = \{v_1, v_2, \ldots, v_n\}$ be a finite subset of V(F). S is L.D in V(F) there exists at least one vector $v_i \in S$ such that $v_i \in L(S - \{v_i\})$.

(4) A singleton subset $\{v \neq 0_V\}$ of V(F) is not L.D, because there exists no $0_F \neq \alpha \in F$ such that $\alpha v = 0_V$.

(5) A subset $= \{v_1, v_2\}$ containing two non-zero vectors of vector space V(F) is L.D. if and only if, there exist two non-zero scalars $\alpha_1, \alpha_2 \in F$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 = 0_V$$

(6) Let $S = \{v_1, v_2, \ldots, v_n\}$ be a finite subset of a vector space V(F), which is L.D. Then the extended set $S' = \{v_1, v_2, \ldots, v_n, v\}, v \neq 0_V$, within V remains L.D.

(7) Any subset S_1 of a linearly dependent subset S of V(F) is not essentially L.D in V(F).

(8) If $v_1, v_2, \ldots, v_m \in V(F)$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in F$ such that the $S = \{v_2 + \alpha_2 v_1, v_3 + \alpha_4 v_1, \ldots + v_n + \alpha_n v_1\}$ is L.D in V(F), then the $S_1 = \{v_1, v_2, \ldots, v_n\}$ is also L.D in V(F).

PROOF

Since the set S is L.D, therefore, by definition, there exist scalars λ_i (i = 2, 3, ..., n) $\in F$, not all zero, such that $\lambda_2 (v_2 + \alpha_2 v_1) + \lambda_3 (v_3 + \alpha_3 v_1) + \ldots (v_n + \alpha_n v_1) = 0_V$

$$\Rightarrow \quad \lambda_2 v_2 + \lambda_3 v_3 + \ldots + \lambda_n v_n + (\lambda_2\alpha_2 + \lambda_3\alpha_3 + \ldots + \lambda_n\alpha_n) v_1 = 0_V$$

Since λ_i's (i = 2, 3, ..., n) are not all zero's of F, therefore $S_1 = \{v_1, v_2, \ldots, v_n\}$ is L.D.

6.2.7 THEOREM

In a vector space V(F), if the subset $S = \{v_1, v_2, \ldots, v_m\}$ of vectors (non-zero) of V(F) $(m \geq 2)$ is L.D then, there exists an integer k, $2 \leq k \leq m$ such that v_k is a linear combination of the preceding vectors $\{v_1, v_2, \ldots v_{k-1}\}$ and conversely.

PROOF

Let the subset $S = \{v_1, v_2, \ldots, v_m\}$ $(m \geq 2)$ of vector space V(F) be L.D. Suppose that v_k, $2 \leq k \leq m$ is the first vector of S from the right for which the set $\{v_1, v_2, \ldots, v_k\}$ is L.D. Then there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in F$ such that $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k = 0_V$, where $\alpha_k \neq 0_F$.

Then,

$$\alpha_k v_k = -\alpha_1 v_1 - \alpha_2 v_2 \ldots - \alpha_{k-1} v_{k-1}$$

or $$v_k = \frac{-\alpha_1}{\alpha_k} v_1 - \frac{\alpha_2}{\alpha_k} v_2 \ldots - \frac{\alpha_{k-1}}{\alpha_k} v_{k-1}$$

show... as a linear combination of the preceding vectors $v_1, v_2, \ldots, v_{k-1}$ in S.

...e converse, if $v_k = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{k-1} v_{k-1}$ then $v_k = \alpha_1 v_1 + \alpha_2 v_2$ ... $v_{k-1} + 0_F v_{k+1} + \ldots + 0_F v_m$ and then set $\{v_1, v_2, \ldots 1_F v_k, v_{k+1} \ldots v_m\}$ ...cause of $1_F \neq 0_F \in F$.

...OTE If $k = 1$, then we have $\alpha_1 v_1 = 0_V$, $v_1 \neq 0_V \Rightarrow v_1 = 0_V$, which provides contradiction to the supposition of the non-zero vectors.

6.2.8 LINEARLY INDEPENDENT (L.I) SUBSET OF VECTOR SPACE V(F)

We know that a subset S of vector space V(F) is either linearly dependent ...n not. If $S = \{v_1, v_2, \ldots, v_m\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_m$ are scalars of F, such that their ...inear combination is a null-vector,

i.e., $$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0_V \qquad \ldots\ldots (1)$$

S is a L.D subset of V(F), if and only if equation (1) has a non zero ...olution i.e., $(\alpha_1, \alpha_2 \ldots \alpha_m) \neq (0, 0, \ldots, 0)$ exists . If S is not a L.D subset of V(F), ...hen equation (1) has only its zero solution, which is trivially a solution of ...quation (1). In such a case the set S is understood as a linearly independent ...ubset of V(F).

6.2.9 DEFINITION

In a vector space V(F), a finite subset $S = \{v_1, v_2, \ldots, v_m\}$ of V(F) is called **linearly independent** in V(F) if, there exist scalars, $\alpha_1, \alpha_2, \ldots, \alpha_m \in F$ such that the equation $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0_V$ has only its zero solution and no non-zero solution exists at all.

6.2.10 IMMEDIATE OBSERVATIONS FROM THE DEFINITION

In a vector space V(F),

1. If a finite subset S of V(F) is not L.D, then S is L.I.
2. The empty subset $\varnothing$ of V(F) is always taken, by definition, a linearly independent subset.
3. Any non-zero singleton subset of V(F) is L.I.
4. A finite subset $T = \{v_1, v_2, \ldots, v_n\}$ of a vector space V(F) is L.I, if none of the vectors of T is a linear combination of the remaining vectors.
5. Every subset of a linearly independent (L.I) set S of a vector space V(F) is L.I.
6. Every finite subset S of non-zero vectors of vector space V(F), which is L.D contains a subset which is L.I in V(F).
7. Every L.I subset S of any subset T of V(F), remains L.I in V(F).
8. If $S = \{v_1, v_2, \ldots, v_k\}$ is a L.I subset of a vector space V(F), then each vector v of V(F) which is contained in the linear span L(S) of S, is uniquely expressed as a linear combination of the vectors of S.

PROOF

Suppose that $v \in L(S)$ and S is a L.I subset of V(F). Then there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in F$ such that,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k \quad \cdots\cdots (a)$$

If (a) is not unique, then suppose that v is expressed as a linear combination of the vectors of S in another way.

i.e., there exist $\beta_1, \beta_2, \ldots, \beta_k \in F$, not all $\alpha_i = \beta_i$ for each i, such that

$$v = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_k v_k \quad \cdots\cdots (b)$$

Since expressions (a) and (b) are of the same vector, therefore,

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_k v_k$$

$$\Rightarrow \quad (\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \ldots + (\alpha_k - \beta_k)v_k = 0_V \quad \ldots\ldots (c)$$

As the vectors $v_1, v_2, \ldots, v_k$ are L.I the equation (c) has only the zero solution in scalars.

i.e. $(\alpha_1 - \beta_1) = 0 = (\alpha_2 - \beta_2) = \ldots = (\alpha_k - \beta_k)$, which gives $\alpha_i = \beta_i$, for all $i = 1, 2, \ldots, k$,

which contradicts to the fact that $\alpha_i \neq \beta_i$, for all i. Hence, one concludes that v has a unique expression as a linear combination of L.I vectors $v_1, v_2, \ldots, v_k$ of V(F).

6.2.11 THEOREM

Let V(F) be a vector space over the field F. Let $S = \{v_1, v_2, v_3, \ldots, v_k\}$ be a subset of V(F) which contains $v_1, v_2, \ldots, v_r$ as L.I vectors for maximum $r < k$. Then,

$$L\{v_1, v_2, \ldots, v_r\} = L\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_k\}.$$

PROOF

Let $u \in L\{v_1, v_2, \ldots, v_r\}$. Then, there exist $\alpha_1, \alpha_2, \ldots, \alpha_r \in F$, such that,

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r$$

Also $u = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r + 0v_{r+1} + \ldots + 0v_k$,

which implies that $u \in L\{v_1, v_2, \ldots, v_r, \ldots, v_k\}$

and hence $L\{v_1, v_2, \ldots, v_r\} \subseteq L\{v_1, v_2, \ldots v_r, \ldots, v_k\}$. .... (a)

Suppose that $u_1 \in L\{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_k\}$. Then, there exist $\alpha_1, \alpha_2, \ldots, \alpha_r, \ldots, \alpha_k \in F$, such that

$$u_1 = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r + \ldots + \alpha_k v_k$$

Since the set $\{v_1, v_2, \ldots, v_r, v_{r+1}\}$ is L.D., therefore v_{r+1} is a linear combination of the L.I vectors $\{v_1, v_2, \ldots, v_r\}$ and hence $v_{r+1} \in L\{v_1, v_2, \ldots, v_r\}$, and similarly, $v_{r+2}, \ldots, v_k \in L\{v_1, v_2, v_3, \ldots, v_r\}$.

Hence $u_1 \in L\{v_1, v_2, \ldots, v_r\}$ and consequently

$$L(S) \subseteq L\{v_1, v_2, \ldots, v_r\} \quad \ldots\ldots (b)$$

The equations (a) and (b) imply the equality that

$$L\{v_1, v_2, \ldots, v_r\} = L\{v_1, v_2, \ldots, v_r, \ldots, v_k\}$$

6.2.12 BASIS OF A VECTOR SPACE

Let V(F) be a vector space over the field F. We know that L(V) = V a the whole space forms a L.D subset of itself. If we delete one by one the vect of V which are linearly dependent on the remaining vectors and continue th process of deletion till we find a finite number of elements $\{v_1, v_2, \ldots, v_n\}$ of such that $L\{v_1, v_2, \ldots, v_n\} = V$ and the vectors $v_1, v_2, \ldots, v_n$ are L.I in V ov F. Such a finite subset exists in V(F), which is labelled to be called a basis of t vector space V(F).

6.2.13 DEFINITION

Let V(F) be a vector space over the field F. A subset B of V(F) is said form a **basis** of the vector space V(F), if

(i) L(B) = V(F)

i.e., linear span of B is the whole of the space V(F) and

(ii) the set B is a linearly independent subset of V(F).

6.2.14 DEFINITION

A vector space V(F) with basis B ($\subseteq$V(F)) is called m-dimensional (or dimension m) vector space if number of vectors in the basis B are finite number equal to m. Otherwise if there exists no such finite m, V(F) understood infinite dimensional.

6.2.15 OBSERVATIONS FROM THE DEFINITIONS

1. If a finite number of vectors span vector space V(F) then vector sp V(F) is said to be called a finite dimensional vector space.
2. Each subset S of a vector space V(F) which spans V(F) cannot be a b of V(F) unless S is a linearly independent subset of V(F).
3. Each L.I subset S of V(F) does not form a basis of vector space V unless S spans V(F).
4. There can be more than one basis of a vector space V(F).

5. If V(F) is a finite dimensional vector space then each basis has the same finite number of vectors of V(F).

6. Every subset of a basis of a vector space V(F) is L.I subset of V(F) which spans a vector space within V(F).

7. If V(F) is an n-dimensional vector space, then
   (i) Any subset of (n+1) vectors of V(F) is L.D.
   (ii) No set of (n−1) vectors can span the whole space V(F) but does span a vector space within V(F).
   (iii) Any subset of a set of n L.I vectors of V(F) forms a basis of a vector space properly contained within V(F).

ortant Note:

(i) The null subspace $\{0_v\}$ has zero dimension.

(ii) The empty subset Φ of V(F) forms a basis of the null subspace.

(iii) The space R^n is finite dimensional of dimension n.

(iv) The vector space $M_{m\times n}$ (F) of all m × n matrices is of dimension m.n and the vector space $M_{n\times n}$ (F) is of dimension n^2 (the number of entries in the matrix).

(v) Every field F over itself is a vector space of dimension one.

Some fundamental results are proved in the following theorems, which help understanding the basis of a vector space and its dimension.

6.2.16 THEOREM (BASIS THEOREM)

Any finite dimensional vector space V(F) contains a finite basis.

PROOF

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a subset of V(F) containing n vectors such that L(S) = V(F).

If $S = \{v_1, v_2, \ldots, v_n\}$ is a L.I subset of V(F), then S forms a basis of V(F) and hence V(F) contains a finite basis.

If $S = \{v_1, v_2, \ldots, v_i, \ldots, v_n\}$ is L.D, then there exists at least one vector, say v_i, of S, which is expressed as a linear combination of its preceding vectors. Eliminating such a vector v_i from S and relabelling the remaining vectors we obtain the set $S_1 = \{v_1, v_2, \ldots, v_{n-1}\}$ of V(F) containing (n−1) vectors. Also $L(S_1)$ = V(F). If S_1 is L.I, then S forms a basis of V(F). If S_1 is again L.D, then there exists a vector v_j say, in S_1, which is a linear combination of its preceding vectors. Eliminating v_j again from S_1 and relabelling the remaining n −2 vectors we form a set S_2 of (n−2) vectors which spans V(F). If S_2 is L.I then S_2 forms a basis of V(F). If S_2 is again L.D, we continue the process of elimination of dependent vectors till we arrive at a subset $S_r = \{v_1, v_2, \ldots, v_r\}$, $1 \leq r \leq n$, which is L.I and spans V(F) which is a finite basis of V(F).

COR

The dimension of a finite dimensional vector space cannot exceed the number of elements in a spanning set of that space.

6.2.17 THEOREM

The dimension of a finite dimensional vector space is unique.

PROOF

Let us suppose that $B_1 = \{v_1, v_2, \ldots, v_m\}$ and $B_2 = \{u_1, u_2, \ldots, u_n\}$ be two basis of a vector space V(F), where $n \neq m$.

Since B_1 spans V(F) and B_2 is a basis of V(F), therefore $n \leq m$, by the basis theorem 5.2.16. ...... (a)

Similarly if, B_2 spans V(F) and B_1 forms a basis of V(F) then $m \leq n$ .... (b)

(a) and (b) imply that $m = n$

Hence all the basis of a vector space have the same number of vectors of V(F) and uniqueness of the dimension of a vector space follows.

COR.

Let U be a proper subset of a finite dimensional vector space V(F)which spans a vector space within V(F). Then dim (U) ≤ dim V. i.e., the dimension of the linear span of U as a vector space can not exceed the dimension of the vector space.

6.2.18 THEOREM (EXTENSION OF BASIS)

Every L.I subset of a finite dimensional vector space V(F) can be extended to be a basis of V(F).

OR

Let V(F) be a vector space of dimension $n > 1$. If $S = \{v_1, v_2, \ldots, v_r\}$, $1 \leq r < n$ is a subset of V(F), which is L.I, then there exist vectors $\{v_{r+1}, v_{r+2}, \ldots, v_n\}$ of

V(F) such that the extended set $S = \{v_1, v_2, v_3, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ *forms a basis of V(F).*

PROOF

Let V(F) be a n-dimensional space. Since $r < n$, therefore the subset $S = \{v_1, v_2, \ldots, v_r\}$ of V(F) does not form a basis of V(F) and hence cannot generate the whole of the vector space V(F). Then there exists a vector $v_{r+1} \in V(F)$ such that $V_{r+1} \notin L(S)$. Including v_{r+1} in S, we get a subset $S_1 = \{v_1, v_2, \ldots, v_r, v_{r+1}\}$ which is L.I in V(F).

If $r + 1 < n$, we repeat this argument and continue combining one vector *in S, in each repetition,* untill we obtain the set $\bar{S} = \{v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ *of V(F) which is* L.I and spans V(F). Hence S forms a basis of V(F).

COR

Every L.I. subset of V(F) is contained in a basis of V(F). Hence there are as many bases of V(F), as many choices of basis vectors are there.

6.3 SOME USEFUL EXAMPLES

EXAMPLE 1 EVERY FIELD IS A VECTOR SPACE OVER ITSELF

The sets $(\Re, +, \cdot)$ $(\mathbb{C}, + \cdot)$ and $(Q, +, \cdot)$ together with their usual respective binary operations of addition (+) and scalar multiplication (.) form vector spaces over the same respective fields. They are respectively named by real, complex and rational vector spaces.

EXAMPLE 2 THE VECTOR SPACES OF $\Re^2, \Re^3, \ldots, \Re^n$

We have studied vectors as ordered pairs, ordered triples, ... etc., *the* ordered n-tuples together with operations and their structural interactions *as in chapter 2.* Such models are realised as vector spaces over the real field $\Re$. *The* action *of the* field elements on the elements of $\Re^2, \Re^3, \ldots \Re^n$ has already *been termed* as a scalar multiplication in chapter 2. Such models are closely *related to the physical* environments and remain connected to the geometrical *structures of vector spaces.*

Since, all the vector spaces of $\Re^2, \Re^3, \ldots, \Re^n$ have similar structures, *therefore the space* $\Re^2$ *is thought sufficient* here to be revealed for discussion.

Consider the set $\Re^2 = \{(x,y) : x,y \in \Re\}$. By the usual addition of ordered pairs,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in \Re^2, \; x_1, x_2, y_1, y_2 \in \Re,$$

$(\Re^2, +)$ forms an abelian group, which we recall from chapter 2.

The scalar multiplication is a function from the set $\Re \times \Re^2$ into the abelian group $\Re^2$, under the rule defined by,

$$\alpha (x, y) = (\alpha x, \alpha y) \in \Re^2,$$

which observes the vector space axioms, for all $\alpha, \beta \in \Re$.

(1) $\alpha [(x_1, y_1) + (x_2, y_2)] = \alpha [(x_1 + x_2, y_1 + y_2)]$

$= [\alpha(x_1+x_2), \alpha(y_1 + y_2)]$

$= [\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2]$

$= (\alpha x_1, \alpha y_1) + (\alpha x_2, + \alpha y_2)$

$= \alpha (x_1, y_1) + \alpha (x_2, y_2)$

(2) $(\alpha + \beta) (x, y) = ((\alpha + \beta)x, (\alpha+\beta)y) = (\alpha x + \beta x, \alpha y + \beta y)$

$= (\alpha x, \alpha y) + (\beta x, \beta y)$

$= \alpha (x, y) + \beta (x, y).$

(3) $(\alpha\beta) (x, y) = ((\alpha\beta)x, (\alpha\beta)y) = (\alpha(\beta x), \alpha(\beta y)) = \alpha (\beta x, \beta y) = \alpha (\beta (x,y))$.

By the associative and commutative property of multiplication of real numbers, it gives that

$$(\alpha\beta) (x,y) = (\beta\alpha) (x,y) = \beta (\alpha (x,y))$$

(4) $1.(x,y) = (1.x, 1.y) = (x,y)$, where 1 is the real number unity.

Thus $\Re^2$ is a vector space over $\Re$.

Similarly $\Re^3, \Re^4, \ldots, \Re^n$ are Vector spaces over $\Re$.

NOTE

(i) $\mathbb{C}^2, \mathbb{C}^3 \ldots \mathbb{C}^n$ are vector spaces over the complex field $\mathbb{C}$.

(ii) $Q^2, Q^3 \ldots Q^n$ are vector spaces over the rational field Q.

(iii) $F, F^2, \ldots, F^n$ are vector spaces over any field F (finite or infinite).

EXAMPLE 3 VECTOR SPACE OF MATRICES

The set $M_{m \times n}(\Re) = \{A : A = [a_{ij}]$ is a $m \times n$ matrix over the real field $\Re\}$ as a vector space over $\Re$.

We recall from chapter three that $M_{m \times n}(\Re)$ is an abelian group under the usual addition of matrices. If $A, B, C \in M_{m \times n}(\Re)$, the action of scalar multiplication $(\cdot)$ by the elements of the real field on the abelian group $M_{m \times n}(\Re)$ over $\Re$ is defined by, $r \times [a_{ij}] = [ra_{ij}], \forall r \in \Re, [a_{ij}] \in M_{m \times n}(\Re)$, which observes the following properties:

1. $r([a_{ij}] + [b_{ij}]) = r[a_{ij} + b_{ij}] = [r(a_{ij} + b_{ij})]$

$= [ra_{ij} + rb_{ij}] = [ra_{ij}] + [rb_{ij}]$, by the addition of matrices

$= r[a_{ij}] + r[b_{ij}], \forall r \in R.$

2. $r_2)\ [a_{ij}] = [(r_1 + r_2)a_{ij}] = [r_1 a_{ij} + r_2 a_{ij}]$

$= [r_1 a_{ij}] + [r_2 a_{ij}]$

$= r_1[a_{ij}] + r_2[a_{ij}], \forall r_1, r_2 \in R.$

$(r_1 r_2)[a_{ij}] = [r_1 r_2 a_{ij}] = [r_1(r_2 a_{ij})] = r_1(r_2[a_{ij}])$

$= r_1[r_2 a_{ij}] = [r_2 r_1 a_{ij}] = [r_2(r_1 a_{ij})] = r_2(r_1[a_{ij}])$

$1[a_{ij}] = [1a_{ij}] = [a_{ij}],$

$\forall [a_{ij}], [b_{ij}] \in M_{m \times n},$

where 1 is the multiplicative identity of the real field $\Re$. Thus, $M_{m \times n}(\Re)$ is a vector space for any m, n as positive integers either $m = n$ or $m \neq n$.

NOTE Similarly the vector spaces $M_{m \times n}(Q)$, $M_{m \times n}(\mathbb{C})$ and $M_{m \times n}(F)$ over any field F are constructed.

EXAMPLE 4 VECTOR SPACE OF POLYNOMIALS

Let the set $P_n(x)$ be of all polynomials of degree $\leq n$ in variable x, over the real field R.

Let $p_n(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \ (a_n \neq 0)$

and $g_n(x) = \sum_{i=0}^{n} b_i x^i$ ($b_n \neq 0$) be two members of $P_n(x)$.

$P_n(x)$, under the usual addition of polynomials form an abelian group (ref. chapter 4)

By the scalar multiplication of the real field elements to the elements of $P_n(x)$, for $r \neq 0$ in $\Re$, is defined by

$$r.\, f_n(x) = r \sum_{i=0}^{n} a_i x^i \; (a_n \neq 0)$$

$$= \sum_{i=0}^{n} (ra_i) x^i \in P_n(x),\ \forall\, r \in \Re,$$

and observes the following properties:

- $r\,[p_n(x) + g_n(x)] = r\,[\sum_{i=0}^{n} a_i x^i + \sum_{i=0}^{n} b_i x^i] \qquad a_n \neq 0,\ b_n \neq 0$

$$= r \sum_{i=0}^{n} (a_i + b_i)\, x^i$$

$$= \sum_{i=0}^{n} r(a_i + b_i)\, x^i$$

$$= \sum_{i=0}^{n} [ra_i\, x^i + rb_i\, x^i]$$

$$= \sum_{i=0}^{n} (ra_i)\, x^i + \sum_{i=0}^{n} (rb_i) x^i$$

$$= r \sum_{i=0}^{n} a_i x^i + r \sum_{i=0}^{n} b_i\, x^i = rp_n(x) + rg_n(x)$$

- If $r_1, r_2 \in \Re - \{0\}$, then

$$(r_1 + r_2)\, p_n(x) = (r_1 + r_2) \sum_{i=0}^{n} a_i x^i$$

$$= \sum_{i=0}^{n} (r_1 + r_2)\, a_i\, x^i$$

$$= \sum_{i=0}^{n} (r_1 a_i + r_2 a_i)\, x^i$$

$$= \sum_{i=0}^{n} [r_1 a_i x_i + r_2 a_i x^i]$$

$$= \sum_{i=0}^{n} (r_1 a_i)x^i + \sum_{i=0}^{n} (r_2 a_i)x^i$$

$$= r_1 \sum_{i=0}^{n} a_i x^i + r_2 \sum_{i=0}^{n} a_i x^i = r_1 p_n(x) + r_2 g_n(x)$$

$$(r_1 r_2)\, p_n(x) = (r_1 r_2) \sum_{i=0}^{n} a_i x^i \ (a_n \neq 0)\ \forall r_1, r_2 \in \Re - \{0\}$$

$$= \sum_{i=0}^{n} (r_1 r_2) a_i x^i = \sum_{i=0}^{n} r_1 (r_2 a_i) x^i = \sum_{i=0}^{n} r_2 (r_1 a_i) x^i$$

$$= r_1 \sum_{i=0}^{n} r_2 a_i x^i = r_2 \sum_{i=0}^{n} (r_1 a_i) x^i$$

$$= r_1 (r_2 p_n(x)) = r_2 (r_1 p_n(x))$$

$$1.\ p_n(x) = 1. \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} (1.\ a_i) x^i = \sum_{i=0}^{n} a_i x^i = p_n(x),$$

for $p_n(x) \in P_n(x)$ and the multiplicative identity $1 \in \Re$.

It, thus proves that $P_n(x)$ forms a vector space over the real field $\Re$, $\mathbb{C}$ and Q for n = 1, 2, 3, . . . , n, for any finite positive integer n.

Similarly $P_n(x)$ is a vector space over a finite field F of order p^n, p, a prime number.

EXAMPLE 5 VECTOR SPACE OF FUNCTIONS

Let F = {f : f is a real valued continuous function on [0, 1]}.

Then, $\forall m \in [0, 1]$ and f, g $\in$ F are defined from [0, 1] into R i.e., $f : [0, 1] \longrightarrow \Re$ by $f(m) \in \Re$, $\forall m \in [0, 1]$

The functions are added by the rule, if f, g, h $\in$ F

$$[f + g]\ (m) = f(m) + g(m)$$

$$[(f + g) + h]\ (m) = [f + g]\ (m) + h(m)$$

$$= (f(m) + g(m)) + h(m) = f(m) + [g(m) + h(m)]$$

$$= [f + (g + h)]\ (m)$$

$\Rightarrow (f + g) + h = f + (g + h)$ (associative property)

There exists $0_F \in F$, which is continuous on [0,1] with real values, and is defined by,

$$0_F(m) = 0, \forall m \in [0, 1]$$

0_F observes the property of identity function (additively), where

$$\begin{aligned}[f + 0_F](m) &= f(m) + 0_F(m) \\ &= f(m) + 0 = 0 + f(m) = 0_F(m) + f(m) \\ &= f(m)\ [0_F + f]\ (m) \quad \text{(identity property)} \\ \Rightarrow \quad f + 0_F &= f = 0_F + f\end{aligned}$$

$\forall f \in F$, there exists $(-f) \in F$ such that $(-f)(m) = -f(m)$, $m \in [0, 1]$. $(-f)$ is the inverse (additive) of f in F, since,

$$\begin{aligned}[f + (-f)]\ (m) &= f(m) + (-f)\ (m) = f(m) - f(m) = 0 \\ &= 0_F(m), \forall m \in [0, 1] \\ &= -f(m) + f(m) = [(-f) + f]\ (m) \\ \Rightarrow \quad f + (-f) = 0_F &= (-f) + f, \forall f \in F \quad \text{(inverse property)}\end{aligned}$$

Also $\qquad f_1 + f_2 = f_2 + f_1$ (commutative property)

as induced by the commutative property of addition of real numbers.

Consequently (F, +) forms an abelian group. Scalar multiplication of the real field elements to the elements of F is a functional activity from, $\Re \times F \longrightarrow F$, defined by, where $(r.f)\ (m) = (rf)(m) = r(f(m)) \in \Re$:

- $$\begin{aligned}[r\ (f_1 + f_2)]\ (m) &= [r\ (f_1 + f_2)]\ (m) = r\ [(f_1 + f_2)\ (m)] \\ &= r\ [f_1(m) + f_2(m)] = rf_1(m) + rf_2(m) \\ &= [rf_1 + rf_2]\ (m), \forall m \in [0, 1] \\ \Rightarrow r\ [f_1 + f_2] &= rf_1 + rf_2\end{aligned}$$
- $$\begin{aligned}[(r_1 + r_2)\ f]\ (m) &= (r_1 + r_2)\ f(m) \\ &= [r_1f + r_2f]\ (m) \\ &= [r_1\ f + r_2\ f]\ (m), \forall m \in [0, 1] \\ \Rightarrow (r_1 + r_2)\ f &= r_1\ f + r_2\ f\end{aligned}$$

- $(r_1 r_2) f = r_1 (r_2 f) = r_1 (r_2 f)$
  $= (r_2 r_1) f = r_2 (r_1 f) = r_2 (r_1 . f)$
- $[1_F . f] (m) = (1f) (m) = f(m), \forall m \in [0, 1]$

$\Rightarrow$ $1_F . f = f, \forall f \in F.$

Thus, it proves that $F(\Re)$ forms a vector space

EXAMPLE 6 VECTOR SPACE OF SOLUTIONS OF A DIFFERENTIAL EQUATION

Consider a differential equation,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(a)}$$

which is homogeneous of order two and degree one. We solve the equation and find solutions $y = e^{2x}$ and e^x, of the given equation (a).

...en their sum $y = e^{2x} + e^x$ is also a solution. $y = 0$ is a solution as well ...ntity (additive) of the set of all solutions of equation (a). If e^{2x} is a ...hen $-e^{2x}$ is also a solution, which is the inverse (additive) of the ... e^{2x} of (a)

Sum is clearly commutative.

Thus, the set S of all solutions of (a) forms the structure of an abelian ...oup (additive).

For the scalar multiplication of the complex field to the group S of solutions of the equation (a) if, $\alpha, \beta \in \Re$ or $\mathbb{C}$ then,

- $\alpha (e^{x_1} + e^{x_2}) = \alpha e^{x_1} + \alpha e^{x_2}$
- $(\alpha + \beta)e^x = \alpha e^x + \beta e^x$
- $(\alpha\beta) e^x = \alpha (\beta . e^x) = \beta . (\alpha e^x) = (\beta\alpha)e^x$
- $1 . e^x = e^x$

Thus the set S of all solutions of the equation (a) forms a vector space over the complex field or real field according as α, β are elements of $\mathbb{C}$ or $\Re$ respectively.

A PIECE OF ADVICE

The students are advised to show that,

* The set $V = \{a + b\sqrt{3} : a, b \in \Re\}$ forms a vector space over the real field $\Re$.
* The set = $\{x + iy : x, y \in \Re, i = \sqrt{-1}\}$ forms a vector space over the field of real numbers.
* The set $M_2(\Re)$ of all 2×2 matrices with real entries forms a vector space over the real field.
* The set $P_2[x]$ of all polynomids of degree ≤ 2, over the field $F = \{0, 1\}$ of two elements, forms a vector space of order $2^3 = 8$

EXERCISES 6A

Q.1 Let V be a vector space over the field F. Prove that

(i) $-(-v) = v, v \in V$.

(ii) $-(v_1 + v_2) = -v_1 - v_2, v_1, v_2 \in V$

(iii) $-\alpha(-v) = \alpha v, \alpha \in F, v \in V$

(iv) $(\alpha - \beta)v = \alpha v - \beta v, \alpha, \beta \in F, v \in V$

Q.2 Let V(F) be a vector space and S be a subset of V, then, prove that,

(i) If S is singleton then, S is L.D if and only if S is a null set.

(ii) If S contains two non-zero vectors, then S is L.D if, and only if, one of them is scalar multiple of other.

(iii) If S contains more than two non-zero vectors then S is L.D if and only if at least one of the vectors of S is a linear combination of some of the vectors of S.

(iv) If S is not L.D, then S is L.I.

(v) If S is not L.I then S is L.D.

(vi) $\mathbb{C}$ is a vector space over the real field R.

(vii) $\Re$ is not a vector space over the complex field.

(viii) Q is not a vector space over the real or complex field.

Q.3 Let (x_1, x_2) and (y_1, y_2) be two non-zero vectors in $\Re^2$. Prove that they are L.I if, and only if $x_1y_2 - x_2y_1 \neq 0$ $\left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \neq 0\right)$.

Q.4 For what value of $\lambda \in \Re$, the vectors (1, 1, 2), (2,3,5) and $(1,3,\lambda)$ of vector space $\Re^3$ are L.D?

Q.5 Show that the vectors (1,0,0), $(y_1, 1, 0)$ and $(z_1, z_2, 1)$ of $\Re^3$ are L.I for scalars $y_1, z_1, z_2 \in \Re$.

Q.6 Show that the vectors $(3 + \sqrt{2}, 1 + \sqrt{2})$, $(7, 1 + 2\sqrt{2})$ in $\Re^2$ are L.D over $\Re$ but L.I over Q.

Q.7 Show that the vectors (1, – i, I) and (2, – 1 + I) are L.D over $\Re$ or L.I over $\Re$.

Q.8 The subset S = {(1, 0, 0), (0, 1, 0), (0, 0, 1)} of vector space $\Re^3$ is linearly independent (L.I) over $\Re$, prove that.

Q.9 The subset S = {1, 1, 0}, (2,3,0), (1,0,1)} of vector space $\Re^3$ is L.I over $\Re$, prove that.

Q.10 Determine $k \in \Re$ so that the set {(1, – 1, k – 1), (2, k, – 8)}), (0, 2 –k, –8)} of R^3 is L.D over $\Re$.

6.4 SUBSPACE OF A VECTOR SPACE

The structure of a vector space V(F) is discussed and examplified in a variety of aspects of mathematical modeling. The local search continues within V(F) as a super structure of vector space. Any vector space lying within V(F) is called subspace of V(F), under the same operations and over the same field F as that of V(F). It is meant to study the **local behaviour** of V(F) over its subsets. It is easy to understand that a subset U of V(F), which observes all the properties of a vector space, is said to form a subspace of V(F). We recall that V(F) is a vector space, if (i) (V, +) is an abelian group and (ii) (V, ·) is closed under the scalar multiplication (·) of field elements to those of V by, $\alpha.v \in V$, $\forall \alpha \in F$, $\forall v \in V$, with additional reference axioms (1 – 4) in 6.1.1. It is important to note that some of the axioms which carry inheritance to the subsets of V(F) are needless to mention in the definition of the concept of a subspace of a vector space. *It is important to note that every subset of a vector space V(F) is not a subspace of V(F).*

6.4.1 DEFINITION

Let V(F) be a vector space over a Field F. A subset U of V(F) is called a **subspace** of V(F), if,

(i) U is a subgroup of V(F).

and (ii) U is closed under the field multiplication, i.e., $\alpha . u \in U, \forall \alpha \in F, \forall u \in U$.

Noted that the other axioms (1 – 4) of scalar multiplication in 6.1.1 are hereditary from V(F) to U.

6.4.2 DEFINITION

A subset U of a vector space V(F) over the field F, is said to form a subspace of V(F), if,

(i) $u_1 - u_2 \in U, \forall u_1, u_2 \in U$ (subgroup criterion)

and (ii) $\alpha u \in U, \forall \alpha \in F, \forall u \in U$. (closure of scalar multiplication)

The definition of a subspace of a vector space is refined further by the following, which is termed to be the **criterion of subspace**.

6.4.3 DEFINITION (SUBSPACE CRITERION)

A subset U of a vector space V(F) is called a subspace of V(F) if, $\forall \alpha, \beta \in F$ and $\forall u, v \in U, \alpha u + \beta v \in U$.

It is very easy to show that all the three definitions 6.4.1, 6.4.2 and 6.4.3 of a subspace of a vector space are equivalent to each other, by making variations in the choices of α and β from F.

6.4.4 IMMEDIATE DEDUCTIONS FROM THE DEFINITION

D_1 If V(F) is a vector space, then each subgroup U of abelian group (V, +) is a subspace of V(F), which is invariant under the scalar multiplication by the elements of F.

D_2 Subspace must be a vector space over the same field as that of its super space V.

D_3 The vector space V(F) itself is a subspace of its own vector space.

D_4 The zero-vector of V(F) is contained in each subspace of V(F).

D_5 The subset $\{0_v\}$ of V(F) forms a subspace of V(F). It is the only subspace containing one vector of V(F). It is called the zero-space or **null space**.

D_6 The subspaces $\{0_V\}$ and the space V(F) are called **trivial** subspaces of V(F) and other subspaces, if they exist, are called **proper** subspaces.

D_7 If $S = \{u\}$ is a singleton subset of a non-zero vector u of V(F), then $L(u) = \{ \alpha u : \alpha \in F \}$ is a subspace of V(F), generated (or spanned) by u and denoted by $< u >$.

i.e., $< u > = L(u) = \{\alpha u : \alpha \in F\}$.

D_8 If $u\ (\neq 0) \in V(F)$ then $L(u) \cong F$, the field over which L(u) is a vector space.

D_9 $L(1) \cong R$, over the field R and $L(1) \cong \mathbb{C}$, over the field $\mathbb{C}$.

LEMMA 6.4.5

D_{10} If U_1 and U_2 are two subspaces of a vector space V(F), then $U_1 \cap U_2$ is also a subspace of V(F).

PROOF

Take $u, v \in U_1 \cap U_2$. Then $u, v \in U_1$ and $u, v \in U_2$.

If $u, v \in U_1$ and U_1 is a subspace of V(F), then for all $\alpha, \beta \in F$, $\alpha u + \beta v \in U_1$ .... (a)

If $u, v \in U_2$ and U_2 is a subspace of V(F), then for $\alpha, \beta \in F$, $\alpha u + \beta v \in U_2$ ...(b)

From (a) and (b), it implies that $\alpha u + \beta v \in U_1 \cap U_2$, $\forall \alpha, \beta \in F$ and $\forall u, v \in U_1 \cap U_2$.

Hence $U_1 \cap U_2$ forms a subspace of V(F)

This way $((U_1 \cap U_2) \cap U_3) \cap \ldots \cap U_n$, etc forms a subspace of V(F), if $U_1, U_2, U_3, \ldots, U_n$ are subspaces.

D_{11} If U_1 and U_2 are two subspaces of a vector space V(F), then U_1 and U_2 have to be abelian subgroups of V(F) but $U_1 \cup U_2$ is not always a subgroup of V(F), therefore $U_1 \cup U_2$ is not a subspace of V(F), in general.

6.4.6 THEOREM

Let U_1 and U_2 be two subspaces of a vector space V(F). Then the subset,

$U_1 + U_2 = \{u_1 + u_2 : u_1 \in u_1, u_2 \in U_2\}$ forms a subspace of V(F).

PROOF

Let $x = u_1 + u_2$ and $y = \bar{u}_1 + \bar{u}_2$ be two elements of $U_1 + U_2$, *where* $u_1, \bar{u}_1 \in U_1$ and $u_2, \bar{u}_2 \in U_2$.

For arbitrary $\alpha, \beta \in F$,

$$\alpha x + \beta y = \alpha (u_1 + u_2) + \beta(\bar{u}_1 + \bar{u}_2)$$

$$= \alpha u_1 + \alpha u_2 + \beta\bar{u}_1 + \beta\bar{u}_2$$

$$= (\alpha u_1 + \beta\bar{u}_1) + (\alpha u_2 + \beta\bar{u}_2),$$ by the commutative property of addition.

Since U_1 is subspace, therefore $\alpha u_1 + \beta\bar{u}_1 \in U_1$. Similarly $\alpha u_2 + \beta\bar{u}_2 \in U_2$ Hence $\alpha x + \beta y \in U_1 + U_2$, for all $x, y \in U_1 + U_2$ and thus $U_1 + U_2$ forms a subspace of V(F), which is understood as **sum of the subspaces** of U_1 and U_2 of V(F).

6.4.7 REMARKS

1. The subspace $U_1 + U_2$ contains the subsets U_1 and U_2 both and consequently $U_1 \cup U_2$
2. $U_1 + U_2$ is the smallest subspace which contains $U_1 \cup U_2$.
3. Infact $L(U_1 \cup U_2) = U_1 + U_2$
4. For a finite family $W_1, W_2, \ldots, W_n$ of subspaces of a vector space V(F), $W_1 + W_2 + \ldots + W_n = \sum_{i=1}^{n} W_i$ is a subspace of V(F), which is sum of the subspaces W_i $(i = 1,2, \ldots, n)$ of V(F).
5. $U + U = U$, for each subspace U of vector space V(F) i.e., for every $u \in U$, and $\alpha \in F$,

$$\alpha u + (1 - \alpha) u = u = \beta u + (1 - \beta)u \quad \text{and} \quad \alpha u + \beta u = (\alpha + \beta)u$$

6.4.8 LEMMA

Let S be a non-empty subset of a vector space V(F). Then the linear span L(S) in V(F) is a subspace of V(F).

PROOF

If $S = \{s_1, s_2, \ldots, s_n\}$ then $L(S) = \{\alpha_1 s_1 + \alpha_2 s_2 + \ldots + \alpha_n s_n : \alpha_i \in F\}$

$$= \left\{\sum_{i=1}^{n} \alpha_i s_i : \alpha_i \in F\right\}_n$$

For arbitrary $u = \sum_{i=1}^{n} \alpha_i s_i$ and $v = \sum_{i=1}^{n} \beta_i s_i$, $\gamma, \delta \in F$,

$$\gamma u + \delta v = \gamma \sum_{i=1}^{n} \alpha_i s_i + \delta \sum_{i=1}^{n} \beta_i s_i = \sum_{i=1}^{n} (\gamma\alpha_i) s_i + \sum_{i=1}^{n} (\delta\beta_i) s_i$$

$= \sum_{i=1}^{n} (\gamma\alpha_i + \delta\beta_i) s_i \in L(S)$, which implies $L(S)$ is a subspace of $V(F)$, for each $S \subseteq V(F)$.

COR. 1

$L(s_1) = L(s_2) = \ldots = L(s_n) \cong F$, for each non-zero singletion $\{s_i\} \in V(F)$.

COR. 2

$L(s_i)$ is a subspace of $V(F)$ of dimension 1 for each i, with basis $\{s_i\}$.

COR 3

$L(S) = L(s_1) \oplus L(s_2) \oplus \ldots \oplus L(s_n)$, if $S = \{s_1, s_2, \ldots, s_n\}$ is L.I.

COR. 4

$L(s_i)$ is a line through the origin for each i.

6.4.9 DEFINITION

Let $W = S + T$, where S and T are subspaces of the vector space $W(F)$. If every element $w \in W$ is expressed uniquely as a sum of the elements of S and T (or $S \cap T = \{0\}$) then, the space W is called the **internal direct sum** of the subspaces S and T of $W(F)$.

It shall be denoted by $W = S \oplus T$.

If $W = W_1 \oplus W_2 \oplus \ldots \oplus W_n$, then W is called the **Internal direct sum** of a family of subspaces $W_1, W_2, \ldots, W_n$ of vector space $V(F)$.

Note that, $\text{Dim}(W) = \dim(W_1) + \dim(W_2) + \ldots + \dim(W_n)$

6.4.10 THEOREM

Let V(F) be a vector space and U(F) be a subspace of V(F). A relation $\underset{\sim}{\mathbb{R}}$ is defined on the elements of V(F) by, "v_1 in V(F), is in relation $\mathbb{R}$ to an element v_2 in V(F), if $v_1 - v_2 \in U(F)$". Then $\mathbb{R}$ is an equivalence relation on V(F).

PROOF

We know that $0_v \in U(F)$ and $v_1 - v_1 = 0_v \in U(F)$. Then v_1 is related to itself by the relation $\mathbb{R}$ i.e., $v_1 \underset{\sim}{\mathbb{R}} v_1, \forall v_1 \in V(F)$. Thus, the relation $\mathbb{R}$ is reflexive on V(F).

In addition, if $v_1 \underset{\sim}{\mathbb{R}} v_2$ then $v_1 - v_2 \in U(F)$, and $-(-(v_2 - v_1)) = v_2 - v_1 \in$ U(F), (by definition) which implies that $v_2 \underset{\sim}{\mathbb{R}} v_1$. It shows that the relation $\mathbb{R}$ is symmetric on V(F).

If $v_1 \underset{\sim}{\mathbb{R}} v_2$ and $v_2 \underset{\sim}{\mathbb{R}} v_3$, for $v_1, v_2, v_3 \in U(F)$, then $v_1 - v_2 \in U(F)$ and $v_2 - v_3 \in$ U(F) and consequently their addition, $v_1 - v_2 + v_2 - v_3 = v_1 - v_3 \in U(F)$. It shows that the relation $\mathbb{R}$ is transitive on V(F) i.e., $v_1 \underset{\sim}{\mathbb{R}} v_3$. By the reflexive, symmetric and transitive properties of $\mathbb{R}$ on V(F), $\mathbb{R}$ is an equivalence relation on V(F).

COR. 1

The relation $\mathbb{R}$ on V(F) subdivides the vector space V(F) into subsets which are mutually disjoint and their union produces the whole of the vector space V(F). The subdivided subsets of V(F) formed by an equivalence relation $\mathbb{R}$ on V(F), are called the equivalence classes by $\mathbb{R}$ on V(F).

COR 2

The relation $\mathbb{R}$ on V(F) defined by ($v_1 \underset{\sim}{\mathbb{R}} v_2$ if $v_1 - v_2 \in U$) is understood by

$v_1 \equiv v_2$ (modulo U), the congruence relation $\equiv$ under which, the subsets

$$v_1 + U = v_2 + U \Leftrightarrow v_1 \mathbb{R} v_2 \text{ in } V(F) \text{ or } v_1 \in v_2 + U \text{ and } v_2 \in v_1 + U$$

COR 3

The equivalence classes formed by the relation $\mathbb{R}$ are subsets $v_i + U = \{v_i + u : u \in U\}$, which contain respective v_i, for each i and are mutually disjoint. The classes are called cosets of V(F) by U(F) (ref. Chapter 4).

COR. 4

The coset $v_i + U = U$, if $v_i \in U$. Thus U itself is one of the cosets of V(F) by U(F) if $v_i \in U$.

COR. 5

The set of all cosets of V(F) by a subspace U(F) is called the **quotient set** of V(F) by U(F) and is denoted by the quotient set,

$$\frac{V}{U} = \{v_1 + U, v_2 + U, \ldots, v_n + U, \ldots\}$$ and denoted by, in short as,

$$= \{[v_1], [v_2], \ldots, [v_n] \ldots\},$$

which is finite if V is finite and or its index $[V : U] = \left|\frac{V}{U}\right| = \frac{|V|}{|U|}$ of cosets of V(F)/U(F) is finite.

COR. 6

If any element of V(F) is common in any two of the cosets of V by U then the cosets coincide.

PROOF

Let $v_1 + U, v_2 + U \in V/U$ and $x \in (v_1 + U) \cap (v_2 + U)$. If $x \in v_1 + U$, then $x = v_1 + u_1$ and if $x \in v_2 + U$, then $x = v_2 + u_2$, where $u_1, u_2 \in U$. Thus $v_1 + u_1 = v_2 + u_2 = x$ which gives that $v_1 + U \subseteq v_2 + U$. Similarly $v_2 + U \subseteq v_1 + U$ and consequently $v_1 + U = v_2 + U_2$.

REMARKS

1. In case of vector space $\Re^2$, each line passing through the origin is a subspace of $\Re^2$ but $\Re^2$ is internal direct sum of the subspaces of the lines of the coordinate axes of the plane $\Re^2$.
2. In case of vector space $\Re^3$, each line through the origin is subspace of $\Re^3$ and each plane which passes through the origin forms a subspace of $\Re^3$ but $\Re^3$ is internal direct sum of three subspaces of the lines of three coordinate axes of the space $\Re^3$.

6.4.11 THEOREM

Let V(F) be a vector space and W, W_1, W_2 be subspaces of V(F). Then,

(a) $v \in u + W$, if and only if $v + W = u + W$

(b) $v \in W$, if and only if $v + W = W$

(c) If $u + W_1 \subseteq u + W_2$, then $W_1 \subseteq W_2$

(d) If $u + W_1 \subset u + W_2$, then $W_1 \subset W_2$

(e) If $u + W_1 = u + W_2$, then $W_1 = W_2$

PROOF

By the use of the concept of *cosets of V(F) by its subspaces, reader is* advised to prove as an exercise.

5.4.12 THEOREM

Let W be a subspace of a vector space V(F). Then the quotient set V/W = $\{[v_i] : v_i + W : v_i \in V\}$ of all cosets of V(F) by W, forms a vector space over the same field F as that of V(F).

PROOF

The addition (+) of the cosets $[v_i]$ of V(F) by W is defined by $[v_1] + [v_2] = [v_1 + v_2] \in V/W$ and the scalar multiplication of coset $[v_i] = v_i + W$, by scalar $\alpha \in F$ is defined by $\alpha\,[v_i] = [\alpha v_i] \in V/W$.

The addition of cosets is associative, since,

(i) $$([v_1] + [v_2]) + [v_3] = [v_1 + v_2] + [v_3] = [(v_1 + v_2) + v_3]$$
$$= [v_1 + (v_2 + v_3)],\ v_1, v_2, v_3 \in V$$
$$= [v_1] + [v_2 + v_3]$$
$$= [v_1] + ([v_2] + [v_3]),$$

(ii) Cosets addition is commutative, since,

$$[v_1] + [v_2] = [v_1 + v_2] = [v_2 + v_1],$$
$$= [v_2] + [v_1],\ v_1, v_2 \in V$$

(iii) $0_v + W = [0_v] = W$ is the additive identity since,

$$[0_v] + [v_i] = [0_v + v_i] = [v_i] = [v_i] + [0_v],\ \forall v_i \in V$$

(iv) $[-v_i] = -[v_i]$, the inverse of $[v_i]$, because,

$$-[v_i] + [v_i] = [-v_i] + [v_i]$$
$$= [v_i - v_i] = [v_i] + (-[v_i]) = [0_v]$$

Thus the quotient set (V/W, +)is an abelian group.

The scalar multiplication to the cosets by the field elements is defined by, $\alpha[v] = [\alpha v] \in V/W$, which observes the following properties;

1. $$\alpha([v_1] + [v_2]) = \alpha[v_1 + v_2] = [\alpha(v_1 + v_2)] = [\alpha v_1 + \alpha v_2]$$
$$= [\alpha v_1] + [\alpha v_2]$$
$$= \alpha[v_1] + \alpha[v_2], \forall [v_1], [v_2] \in V/W$$

2. $$(\alpha + \beta)[v_1] = [(\alpha + \beta)v_1] = [\alpha v_1 + \beta v_1] = [\alpha v_1] + [\beta v_1]$$
$$= \alpha[v_1] + \beta[v_1], \forall [v_1], [v_2] \in V/W$$

For $\alpha, \beta \in F$,

3. $$(\alpha\beta)[v_1] = [\alpha\beta v_1] = [\alpha(\beta v_1)] = \alpha[\beta v_1]$$
$$= (\beta\alpha)[v_1] = \beta(\alpha[v_1]), \qquad \forall \alpha, \beta \in F, \forall [v_1] \in V/W$$

4. $$1_F[v_1] = [1_F v_1] = [v_1] \qquad \forall [v_1] \in V/W$$

Thus the quotient set V/W forms a vector space over the same field F, over which V is a vector space.

The quotient set forms the quotient space V/W of V by W.

REMARKS

1. For each subspace W of V(F), there is a quotient space V/W over the same field F.
2. $V/V = 0_v + V = [0_v]$, the zero quotient space.
3. $V/[0_v] = V$, the quotient space of V by its zero subspace.

6.4.13 LEMMA

4. $\dfrac{V}{W_1 \cap W_2} = V/W_1 \cap V/W_2$, if W_1 and W_2 are two subspaces of V over the field F.

PROOF

Let $[\bar{v}] \in \frac{V}{W_1 \cap W_2}$. Then $\bar{v} \in v_i + (W_1 \cap W_2)$ which implies

$\bar{v} - v_i \in W_1 \cap W_2 \Rightarrow \bar{v} - v_i \in W_1$ and $\bar{v} - v_i \in W_2$

If $\bar{v} - v_i \in W_1$, then $[\bar{v}] \in V/W_1$

If $\bar{v} - v_i \in W_2$, then $[\bar{v}] \in V/W_2$

Hence $[\bar{v}] \in V/W_1 \cap V/W_2$, which implies that

$$\frac{V}{W_1 \cap W_2} \subseteq V/W_1 \cap V/W_2 \qquad \dots\dots\dots$$

Conversely, if $[\bar{v}] \in V/W_1 \cap V/W_2$, then $[\bar{v}] \in V/W_1$ and $[\bar{v}] \in V/W_2$

$\Rightarrow \quad \bar{v} - v_i \in W_1$ and $\bar{v} - v_i \in W_2$, for $v_i \in V$

$\Rightarrow \quad \bar{v} - v_i \in W_1 \cap W_2$

$\Rightarrow \quad [\bar{v}] \in V/W_1 \cap W_2$, giving that $\frac{V}{W_1} \cap \frac{V}{W_2} \subseteq \frac{V}{W_1 \cap W_2} \qquad \dots\dots\dots$

The expressions (a) and (b) imply the required equality.

6.4.14 THEOREM

Let V(F) be a finite dimensional vector space and S be a subspace. Let V/S forms a vector space over the same field F, called the quotient of V(F) by S. Then,

$$\dim (V/S) = \dim V - \dim S.$$

PROOF

Since V(F) is finite dimensional vector space and S is a subspace therefore S is also finite dimensional and dim S < dim V. If B = {v₁, v

The vectors of $V/S = \{v_i + S : v_i \in V\}$ are the r cosets of V(F) by S, whose union $\bigcup_{i=1}^{r} \{v_i + S\} = V$ and $L(\{v_i + S : v_i \in V\}) = V/S$.

Suppose that, for $\alpha_1, \alpha_2, \ldots, \alpha_r \in F$, $\alpha_1 [w_1] + \alpha_2 [w_2] + \ldots + \alpha_r[w_r] = S$, the zero of V/S. Then,

$\{\alpha_1 w_1] + [\alpha_2 w_2] + \ldots + [\alpha_r w_r] = S$

$\Rightarrow \quad [\alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_r w_r] = S.$

$\Rightarrow \quad (\alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_r w_r) + S = S$

$\Rightarrow \quad (\alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_r w_r) \in S$

$\Rightarrow \quad \alpha w_1 + \alpha_2 w_2 + \ldots + \alpha_r w_r = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_m v_m, \ \beta_i \in F$

$\Rightarrow \quad \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_r w_r - \beta_1 r_1 - \beta_2 r_2 \ \ldots - \beta_m v_m = 0_V$

$\Rightarrow \quad \alpha_1 = \alpha_2 = \ldots = \alpha_r = 0_F = \beta_1 = \beta_2 = \ldots = \beta_m$, since $\overline{B} = \{v_1, v_2, \ldots, v_m, w_1, W_2, \ldots, w_r\}$ is a basis of V(F).

Hence $[w_1], [w_2], \ldots, [w_r]$, are L.I vectors of V/S which span V/S.

Hence, $\dim\left(\frac{V}{S}\right) = r = (m + r) - m = \dim V - \dim S$

COR.

The set of all cosets of V(F) by S forms a basis of V/S and dim V/S = [V : S], the index of V by S.

6.4.15 THEOREM

Let S and T be two subspaces of a finite dimensional vector space V(F). Then $\dim(S+T) = \dim(S) + \dim(T) - \dim(S \cap T)$.

PROOF

Since S and T are two subspaces of a finite dimensional vector space V(F) then $S \cap T$ is also a subspace of V(F) and is a finite dimensional subspace. Let $\{u_1, u_2, \ldots, u_r\}$ be a basis of $S \cap T$ which contains r vectors.

Since $S \cap T$ is a subspace of both of its super subspaces S and T, therefore, the basis $\{u_1, u_2, \ldots, u_r\}$ of $S \cap T$ can be extended to form bases of S and T both by

$B_1 = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s\}$ and $B_2 = \{u_1, u_2, \ldots u_r, w_1, w_2, \ldots,$ containing $(r + s)$ and $(r + t)$ vectors respectively. Since S and T both a subspaces of the further super subspace (S+T) of V(F), therefore, the bases I and B_2 of S and T respectively can be extended to a basis of S+T. Let $B_3 = \{u_1, u$ $\ldots, u_r, v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_t\}$ be a basis of a subspace of V(F) containing $(r + s + t)$ vectors which span (S+T). Thus $\dim (S+T) \leq (r + s + t)$.

Suppose that there exist scalars $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_s, c_1, c_2, \ldots, c_t$ of F such that,

$$a_1u_1 + a_2u_2 + \ldots + a_ru_r + b_1v_1 + b_2v_2 + \ldots + b_sv_s + c_1w_1 + c_2w_2 + \ldots + c_tw_t = 0_V$$

Then,

$$a_1u_1 + a_2u_2 + \ldots + a_ru_r + b_1v_1 + b_2v_2 + \ldots + b_sv_s = - c_1w_1 - c_2w_2 + \ldots - c_tw_t$$

If $x = a_1u_1 + a_2u_2 + \ldots + a_ru_r + b_1v_1 + b_2v_2 + \ldots + b_sv_s = -c_1w_1 - c_2w_2 \ldots - c_tw_t$, then $x \in S \cap T$ and hence ....(i)

$x = d_1u_1 + d_2u_2 + \ldots + d_ru_r$, for $d_i \in F$. ....(ii)

Subtracting (ii) from (i), we get that

$0_V = x - x = (a_1 - d_1) u_1 + (a_2 - d_2) u_2 + \ldots + (a_r - d_r) u_r + b_1v_1 + b_2v_2 + \ldots + b_sv_s$.

As B_1 is a basis of S, therefore B_1 is L.I and hence,

$a_1 - d_1 = 0 = a_2 - d_2 = \ldots = (a_r - d_r) = b_1 = b_2 = \ldots = b_s$

$\Rightarrow$ $a_i = d_i$, $i = 1,2, \ldots$ and $b_1 = b_2 = \ldots = b_s = 0$

But $d_i = 0$, or $i = 1,2, \ldots$, from (ii) and hence $a_i = 0$, for $i = 1,2, \ldots, r$

Thus the set $B_3 = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_t\}$ is L.I and hence is a basis of (S+T) and consequently

$$\begin{aligned}\dim (S+T) &= r + s + t \\ &= (r + s) + (t + r) - r \\ &= \dim (S) + \dim(T) - \dim(S \cap T),\end{aligned}$$

which proves the assertion of the theorem.

COR. 1

If S and T are finite dimensional subspaces of a vector space V, then S+T is finite dimensional. By the group theoretic structure of a vector space, $\frac{S+T}{T} \cong \frac{S}{S \cap T}$, which gives that $\dim \frac{S+T}{T} = \dim \left(\frac{S}{S \cap T}\right)$, implying that dim (S+T) = dim S + dim T − dim (S∩T).

COR. 2

Let V(F) be vector space having two subspaces W_1 and W_2, such that

(i) $W_1 \cap W_2 = \{0_V\}$

and (ii) $W_1 + W_2 = V$

Then $\dim V = \dim W_1 + \dim W_2$

COR. 3

A finite vector space V(F) is internal direct sum of its two subspace W_1 and W_2 if and only if $\dim V = \dim W_1 + \dim W_2$

EXERCISES 6B

1. Mark the following statements true or false. Give a brief explanation in support of your answers.
   (i) If a set $\{u_1, u_2, \ldots, u_n\}$ is L.I in a vector space V(F) then the set $\{u_2, u_3, \ldots, u_n\}$ is also L.I.
   (ii) Any L.I set of vectors in space $\Re^m$ is part of a basis of $\Re^m$.
   (iii) Zero vector of vector space is a linear combination of any sub-collection of vectors.
   (iv) It is possible to find five vectors that span the vector space $\Re^6$.
   (v) Any four vectors of vector space $\Re^3$ are L.D.
   (vi) Every set of L.I vectors forms a basis of a subspace of a finite-dimensional vector space V(F).
   (vii) If dim V = n then, any subset S of n L.I vectors of V(F) spans the vector space V(F).

(viii) If U and V are two subspaces of R^m and dim U = dim V, then U = V.

(ix) If U is a subspace of a finite dimensional vector space W(F), then $\dim\left(\frac{W}{U}\right) = \dim(W) - \dim(W \cap U)$

(x) If V(F) is a vector space of dimension n, then each subset of V(F) containing n vectors forms a basis of V(F).

(xi) Vector space R^m cannot contain a L.I subset of m vectors.

2. In a vector space V(F) show that $\alpha(v-w) \equiv \alpha w$, $\alpha \in F$.

3. Let F be the field of all real numbers and $V = \{(a_1, a_2, \ldots, a_n, \ldots) : a_i \in F\}$ be the set of all sequences, where equality, addition and scalar multiplication are defined component wise. Prove that V is a vector space.

4. If S and T are subspaces of a vector space V(F), prove that $S + T = \{v \in V : v = s + t : s \in S, t \in T\}$ is a subspace of V(F).

5. Prove that the intersection of three subspaces of V(F) is a subspace of V(F).

6. Show that if S, T and R are three subspaces of a finite dimensional vector space V(F) the dim (S + T + R) = dim S + dim T + dim R – dim $S \cap T$ – dim $T \cup R$

– dim $(S \cap R)$ + dim $(S \cap T \cap R)$.

7. Prove that the vectors (1,1,0,0), (0,1,–1,0), (0,0,0,3) in R^4 are L.I.

8. If V(F) is a vector space of dimension n, show that any set of n L.I. vectors of V(F) forms a basis of V(F).

9. If V(F) is a finite dimensional vector space and W is a subspace of V(F) such that dim V = dim W, then prove that V = W.

10. Prove that the set F[x] of all polynomials in x is not a finite dimensional vector space over the field F.

11. If W is a subspace of a finite dimensional vector space V(F), prove that there is a subspace W_1 of V(F) such that $V = W \oplus W_1$.

12. Let V(F) be the set of real functions y = F (x) satisfying

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$$

Prove that V is a 3-dimensional space.

Examine whether the following sets of vectors form a basis of $\Re^3$.

(i) $\{(1,2,-1), (0,3,1)\}$ (ii) $\{(2,4,-3), (0,1,1), (0,1,-1)\}$

(iii) $\{(2,0,1), (1,1,-1), (0,0,1)\}$

(i) Find a basis of the subspace of vector space $\Re^4$ generated by

$v_1 = (1, 1, 2, 0)$, $v_2 = (2, 1, 0, 2)$ $v_3 = (1, 2, 3, 4)$, $v_4 = (0, 4, 5, 2)$

(ii) Find the real 'a' such that $a\,(1, 1, 0) \in L(\{v_1, v_2, v_3\})$, if possible.

Find a basis of a subspace of $\Re^3$, which is generated by, $v_1 = (1, 0, -1)$, $v_2 = (1, 2, 1)$ and $v = (0, -3, 2)$.

Extend the set $\{(2, 1, 4, 3), (2, 1, 2, 0)\}$ to be a basis of $\Re^4$.

Examine whether the subset $\{(1, 1), (3, 1)\}$ of $\Re^2$, forms a basis of $\Re^2$.

Find the dimensions of the subspaces of $\Re^4$, spanned by

(i) $\{(1, -2, 3, 1), (1, 1, -2, 3)\}$

(ii) $\{(-1, -1, 5, 0), (0, 0, 0, 1)\}$

(iii) $\{(1, 0, 2, 3), (2, 0, 4, 6)\}$

(iv) $\{(1, 1, 1\ 2), (0, 1, 1, 3), (0, 0, 1, 1), (1, 0, 2, 3)\}$

Let $M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, \in \Re \right\}$ be a vector space of dim 4.

Then,

(i) Find a basis of M_2

(ii) Prove that $U = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \Re \right\}$ is a subspace of M_2.

(iii) Show that $V = \left\{ \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} : a, \in \Re \right\}$ is a subspace of M_2.

(iv) Find the dimensions of U and V.

(v) Show that $M_2 = U \oplus V$, the internal direct sum of U and V.

20. Let $\bar{M}_3 = \left\{ \begin{pmatrix} a & b & e \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} : a, b, c, d, e, f \in \Re \right\}$ be a subset of

$$M_3(R) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} : a, b, c, d, e, f, g, h \in \Re \right\}$$

Prove that

(i) $\bar{M}_3$ is a subspace of $M_3(\Re)$

(ii) The dimension of $\bar{M}_3$ is $\frac{3(3+1)}{2}$

21. Show that the field F is a vector space of dimension 1, over the field F itself. Show that any non-zero element of F spans the vector space F..

22. Let $D_3 = \begin{pmatrix} a_{11} & b_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$ be set of all matrices over the real field $\Re$.

Show that D_3 is a vector space over $\Re$. Show also that D_3 contains a subspace of dimension 4.

23. Show that the set of all polynomials $P_3(x)$ of degree $d \leq 3$ in x over the real field forms a vector space. Find its basis and dimension.

6.5 VECTOR SPACE HOMOMORPHISMS

The concept of a homomorphism is naturally attached to any pair of alike algebraic systems. Here it is a special kind of a mapping from one vectorspace to another vector space which preserves the group operations of both the vector spaces and that of the field operation by the elements of the same field on the group structures. It is important because of another attribute that the set of all vector space – homomorphisms, from one vector space to another or on a vector space, forms the model of a vector space over the same field structure of the vector spaces connected by the homomorphisms. If V(F) and W(F) are two vector spaces of a field F, then the set of all homomorphisms from vector space V(F) into W(F) is denoted by $Hom_F(V,W)$, which forms a vector space over the same field F. We would like to introduce this concept in order to carry it out to form a

6.5.1 DEFINITION

Let U(F) and V(F) be two vector spaces over the same field F. Let $T : U \longrightarrow V$, be a mapping from vector space U(F) into vector space V(F). Then T is called a **homomorphism**, if

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$

and (ii) $T(\alpha u) = \alpha(T(u))$,

for all $u_1, u_2, u \in U$, $\forall \alpha \in F$ and $T(u_1), T(u_2), T(u) \in V$

By (i), T preserves the group structure of both the vector spaces.

and by (ii), T preserves the scalar multiplicative action on the elements of the vector spaces.

6.5.2 FURTHER NOTEABLE OBSERVATIONS

A reader is made familiar to the facts that if, in addition

(a) T is one–to–one and onto both, then T is called an isomorphism between U(F) and V(F). In this case the spaces U(F) and V(F) are called isomorphic to each other (or the structure wise photocopies of each other).

(b) If T is one–to–one and onto, then the mapping T is **invertible** and hence is called invertible by this property i.e., T^{-1} exists as a Homomorphism from V(F) into U(F), i.e., $T^{-1} : V(F) \longrightarrow U(F)$ such that $T^{-1}(v) = u$, $\forall v \in V$.

(c) If V(F) = U(F) then $T \in Hom_F(U,U)$ and is known to be called an **endomorphism** of U(F) and if T is one–to–one and onto then T is called an **automorphism** of U(F).

(d) $T \in Hom_F(U, V)$, is infact a group-homomorphism which preserves the field operation.

(e) Since field F is a vector space of dimension one, and $Hom_F(U, F)$ is the set of all homomorphisms from a vector space U(F) into the field space is a vector space of same dimension as that of U.

(f) If V(F) is a finite dimensional vector space over the field F then $Hom_F(V,F)$ exists as a finite dimensional vector space over F.

(g) The elements of Hom(V, F) are called **linear functionals** of V(F) and the vector space $Hom_F(V, F)$ is called the **dual space** of V(F).

(h) $T \in Hom_F(V, V)$ is called a **linear operator** on V.

6.5.3 PROPOSITION

Let V(F) and W(F) be two vector spaces over the same field F, Hom(V,W), then

(i) $T(0_V) = 0_W$

(ii) $T(-v) = -T(v) = -w$, if $T(v) = w \in W$.

(iii) $T(v_1 + v_2) = T(v_1) + T(v_2)$, $\forall\, v_1, v_2 \in V$.

(iv) $T(\alpha v) = \alpha T(v) = \alpha w$, if $T(v) = w \in W$.

PROOF

The proof follows directly from the definition of group homomor (i.e. ref. Chapter 4).

6.5.4 THEOREM

Let V(F) and W(F) be two vector spaces over the same field F. $V(F) \rightarrow W(F)$ is a homomorphism from V(F) into W(F), then

(i) T(S) is a subspace of W(F), if S is a subspace of V(F).

(ii) If T is one–to–one and onto W(F) and W_1 is a subspace of W then $T^{-1}(W_1)$ is a subspace of V(F).

(iii) If $S = \{v_1, v_2, \ldots, v_k\} \subset V(F)$ then $T(S) = \{T(v_1), T(v_2), \ldots, T\ldots$ a subset of W(F), and

$$T(L(S)) \subseteq L(T(S))$$

PROOF

(i) Given that S is a subspace of V(F). If $s_1, s_2 \in S$ and $\alpha, \beta \in F$, then $\beta s_2 \in S$, by definition, for all $\alpha, \beta \in F$ and $\forall s_1, s_2 \in S$.

Since $T : V(F) \longrightarrow W(F)$, from V(F) into W(F), then $T(s_1), T(s_2) \in T(S) \subseteq W(F)$

For $\alpha, \beta \in F$,

$$\alpha T(s_1) + \beta T(s_2) = T(\alpha s_1) + T(\beta s_2)$$

$$= T(\alpha s_1 + \beta s_2) \in T(S), \text{ since S is a subspace of V}$$

Since it holds for all $\alpha, \beta \in F$, therefore T(S) is a subspace of W(F).

(ii) Given that W_1 is a subspace of W(F). If $\bar{s}_1, \bar{s}_2 \in W_1$ and $\alpha, \beta \in F$, then $\alpha\bar{s}_1 + \beta\bar{s}_1 \in W_1$, by definition.

If T is one-to-one and onto W(F) from V(F) then T^{-1} exists as a homomorphism from W(F) into V(F) and

$$T^{-1}(W_1) = \{v \in V(F) : T(v) \in W_1\} \subseteq V(F)$$

If $v_1, v_2 \in T^{-1}(W_1)$ and $\alpha, \beta \in F$, then

$\alpha v_1 + \beta v_2 = \alpha T^{-1}(\bar{w}) + \beta T^{-1}(\bar{\bar{w}})$, where $v_1 = T^{-1}(\bar{w})$

and $v_2 = T^{-1}(\bar{\bar{w}})$, $\bar{w}, \bar{\bar{w}} \in W_1$,

Hence $\alpha v_1 + \beta v_2 = T^{-1}(\alpha \bar{w}) + T^{-1}(\beta \bar{\bar{w}})$

$$= T^{-1}(\alpha \bar{w} + \beta\bar{\bar{w}}) \in T^{-1}(W_1)$$

Hence $T^{-1}(W_1)$ is a subspace of V(F)

(iii) For $\alpha_1, \alpha_2, \ldots, \alpha_k \in F$, and $S = \{v_1, v_2, \ldots, v_k\} \subseteq V$. Then $T(v_1), T(v_2), \ldots, T(v_k) \in T(S)$.

Let $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k \in L(S)$. Then

$$\begin{aligned} T(v) &= T(\alpha v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k) \\ &= T(\alpha_1 v_1) + T(\alpha_2 v_2) + \ldots + T(\alpha_k v_k) \\ &= \alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots + \alpha_k T(v_k) \in L(T(S)), \end{aligned}$$

by the definition of T, (as a homomorphism) and that of linear span. Hence it holds for all such $v \in L(S)$ that $T(L(S)) \subseteq L(T(S))$.

Of course, equality holds, if T is an automorphism of V(F).

6.5.5 THEOREM

Let V(F) and W(F) be two vector spaces over the field F and $T \in$ Hom (V,W). A subset $K = \{v \in V, T(v) = 0_W\}$ of V(F), each element of which is mapped onto the zero-vector 0_W of W, forms a subspace of V(F).

PROOF

Let $x_1, x_2, \in K \subseteq V(F)$. Then $T(x_1) = 0_W = T(x_2)$. If $\alpha, \beta \in F$, then

$$T(\alpha x_1 + \beta x_2) = T(\alpha x_1) + T(\beta x_2)$$
$$= \alpha T(x_1) + \beta T(x_2)$$
$$= \alpha \cdot 0_W + \beta \cdot 0_W = 0_W$$

Hence $\alpha x_1 + \beta x_2 \in K$, which proves that K is a subspace of V(F).

6.5.6 DEFINITION

Let $T : V(F) \longrightarrow W(F)$ be a homomorphism from a vector space V(F) into a vector space W(F). The maximal subset K of V(F), each element of which is mapped onto the zero vector 0_W of W(F), is called the **kernel** of T and is denoted by ker (T).

6.5.7 OBSERVATIONS FROM THE DEFINITION

(1) For every homomorphism T from V(F) into W(F), there is a kernel of T = ker (T)

(2) Ker (T) is a subspace of V(F)

(3) If T is an isomorphism from V(F) into W(F) then ker (T) = $\{0_V\}$ containing the zero vector alone.

(4) Let $T : V(F) \longrightarrow W(F)$ be homomorphism from V(F) onto W(F). Then T is an isomorphism if and only if ker (T) = $\{0_V\}$.

6.5.8 THEOREM

Let V(F) be a n-dimensional vector space and U be a subspace of V(F). Then there exists a homomorphism $n_\varphi : V(F) \longrightarrow V/U$, from V(F) onto the quotient space V/U.

PROOF

We have proved that, for a subspace U of a vector space V(F), there exists a vector space V/U, the quotient space of V(F) by U.

Let $n_\varphi : V(F) \longrightarrow V/U$, be a mapping from V(F) into V/U, which is defined by, $n_\varphi(v) = v + U$, $\forall v \in V$. If $v_1, v_2 \in V$, then

(i) $n_\varphi(v_1 + v_2) = (v_1 + v_2) + U$

$= (v_1 + U) + (v_2 + U)$, by coset addition

$= n_\varphi(v_1) + n_\varphi(v_2), \forall v_1, v_2 \in V(F)$

i.e., n_φ is a group homomorphism.

(ii) $n_\varphi(\alpha v_1) = \alpha v_1 + U = \alpha(v_1 + U)$

$= \alpha n_\varphi(v_1)$

i.e., n_φ preserves scalar multiplication.

Equation (i) and (ii) show that n_φ is a vector space homomorphism from V(F) into V/U.

n_φ is onto, because $\cup\{v_i + U\} = V$ and for every coset $v_i + U \in V/U$, there exists $v_i \in V$ such that $n_\varphi(v_i) = v_i + U$

OBSERVATIONS

(1) $\ker(n_\varphi) = U$.

(2) For every subspace U of V(F), there exists natural homomorphism $n_\varphi : V(F) \longrightarrow V/U$

(3) Each natural homomorphism is epimorphism.

(4) Kernel of n_φ is subspace of V(F).

6.5.9 THEOREM (FUNDAMENTAL THEOREM)

Let $\psi : V(F) \longrightarrow W(F)$, be a vector space homomorphism from V(F) onto W(F). Let $K = \ker \psi$. Then $V/\ker \psi \cong \psi(V)$.

PROOF

Since $\ker \psi$ is a subspace of V, therefore $V/\text{Ker}\,\psi$ is a vector space over the field F. Of course, $\psi(V)$ is a subspace of W(F), as an image space of V(F) within W(F), under ψ.

A map $\eta : V/\ker \psi \longrightarrow \psi(V)$

from V/K into $\psi(V)$ is defined by, $\eta(v + K) = \psi(v)$.

If $v_1 + K \in V/K$, then $\eta(v_1 + K) = \psi(v_1)$, where

$$\eta[(v_1 + K) + (v_2 + K)] = \eta(v_1 + v_2 + K), \text{ where } v_2 + k \in \frac{V}{K}$$

$$= \psi(v_1 + v_2)$$

$$= \psi(v_1) + \psi(v_2), \because \psi \in \text{Hom}(V, W)$$

$$= \eta(v_1 + K) + \eta(v_2 + K),$$

and $\eta(\alpha(v + K))$ $= \eta(\alpha v + K) = \psi(\alpha v)$

$$= \alpha\,\psi(v)$$

$$= \alpha\,\eta(v + K), \text{ for all } \alpha \in F,$$

giving that $\eta \in \text{Hom}(V/K, W)$.

If $\eta(v + k) = \eta(v_1 + k)$, then $\psi(v) = \psi(v_1)$, which

$\Rightarrow$ $\psi(v) - \psi(v_1) = o_W \Rightarrow \psi(v - v_1) = 0_W$

$\Rightarrow$ $v - v_1 \in K$, the kernal of $\psi \Rightarrow (v - v_1) + K = K$

$\Rightarrow$ $v + K = v_1 + K, \forall v, v_1 \in V(F) \Rightarrow \psi$ is a (1 –1) homomorphism.

Further η is onto $\psi(V) \subseteq W(F)$

Hence η is (1 – 1) and onto homomorphism from V/K into $\psi(V)$ a consequently η is an isomorphism and $V/K \cong \psi(V)$

COR. 1

If ψ is a homomorphism from V(F) onto W(F) then $V/K \cong W$.

COR. 2

Let $T \in \text{Hom}(V,V)$ and $K = \ker T$, a subspace of a finite dimensi vector space V(F). Then

$$V/K \cong T(V)$$

And hence $\dim(V/K) = \dim T(V)$

$\Rightarrow$ $\dim V - \dim K = \dim T(V)$

$\Rightarrow$ $\dim V = \dim K + \dim T(V)$

6.5.10 DEFINITION

If V(F) is a finite dimensional space and $T \in Hom_F(V,V)$, with K = ker (T), then dim K is called the **Nullity** of T and dim T(V) is called the **Rank** of T.

COR. 3

dim V = Rank (T) + Nullity (T), for each $T \in Hom_F(V,V)$

or Rank (T) = dim V – Nullity of T

= dim V – dim (Ker T)

Noted that Rank (T) and Nullity of T vary with $T \in$ Hom (V,V)

COR. 4

If $T \in$ Hom(V,F) then Nullity (T) = dim V and Rank (T) = dim 0_v, for each T.

6.5.11 DEFINITION

Let $T \in$ Hom(V,W) If T is invertible then there exists $T_1 \in$ Hom(W,V) such that

(i) $T_1 T(v) = T_1(T(v)) = v$ i.e. T is a left inverse of each T on v.

(ii) $T T_1(w) = T(T_1(w)) = w$, i.e. T is a right inverse of T on w.

for all $v \in V$ and $w \in W$,

where T_1, if it exists when W = V is unique inverse of T and is denoted by $T^{-1} = T_1$.

6.5.12 THEOREM

Let V(F) and W(F) be two finite dimensional vector spaces over the same field F. Then the set Hom(V,W) of all homomorphisms from V(F) into W(F) forms a vector space over the same field F.

PROOF

Let $T_1, T_2, T_3 \in$ Hom(V,W). Let their addition (+) and scalar multiplication (·) by the elements of F be defined by the following:

For $v \in V(F)$, $(T_1 + T_2)(v) = T_1(v) + T_2(v) \in W(F)$ . ... (i)

and $(\alpha T_1)(v) = \alpha(T_1(v)) \in W(F)$, for every $v \in V$ and $\alpha \in F$. .... (ii)

Then $(T_1 + T_2) \in Hom(V,W)$, since,

$$(T_1 + T_2)(v_1 + v_2)$$

$$= T_1(v_1 + v_2) + T_2(v_1 + v_2), \text{ by (i)}$$

$$= T_1(v_1) + T_1(v_2) + T_2(v_1) + T_2(v_2), \qquad T_1, T_2 \in Hom(V, W)$$

$$= T_1(v_1) + T_2(v_1) + T_1(v_2) + T_2(v_2), \text{ by commutativity of W.}$$

$$= (T_1 + T_2)(v_1) + (T_1 + T_2)(v_2), \qquad \ldots\ldots\ldots \text{(iii)}$$

and

$$(\alpha T)(v_1+v_2) = \alpha(T(v_1 + v_2)) = \alpha[T(v_2)]$$

$$= \alpha(T(v_1)) + \alpha[T(v_1) + T(v_2)), \text{ by(ii)}$$

$$= (\alpha T)(v_1) + (\alpha T)(v_2), \qquad \ldots\ldots\ldots \text{(iv)}$$

which implies that $T_1 + T_2$ and $\alpha T \in Hom(V, W)$, for all $T_1, T_2, T \in Hom(V,W)$.

It is easy to exhibit that

$$[(T_1 + T_2) + T_3](v) = (T_1 + T_2)(v) + T_3(v)$$

$$= T_1(v) + T_2(v) + T_3(v)$$

$$= T_1(v) + (T_2 + T_3)(v)$$

$$= [T_1 + (T_2 + T_3)](v), \forall v \in V(F),$$

which proves the associative property of addition of homomorphisms. i.e., $(T_1+ T_2) + T_3 = T_1 + (T_2 + T_3)$, $\forall T_1, T_2, T_3 \in Hom(V,W)$.

There exists, $T_0(v) = 0_W$, $\forall v \in V(F)$, which is a homomorphism from V(F) into W(F), because,

$$T_0(v_1 + v_2) = 0_W = 0_W + 0_W = T_0(v_1) + T_0(v_2)$$

Thus $T_0 \in Hom(V,W)$, which is unique and observes the identity property, by,

$$(T_1 + T_0)(v) = T_1(v) + T_0(v) = T_1(v) + 0_W = T_1(v)$$

$$= 0_W + T_1(v) = T_0(v) + T_1(v)$$

$$= (T_0 + T_1)(v) = \forall v \in V(F),$$

giving that, $T_1 + T_0 = T_I = T_0 + T_1$, $\forall T_1 \in Hom(V, W)$

Similarly, for each $T \in$ Hom (V,W), there exists $(-T) = (-1_F)T$ such that $[(-1_F)T]\,(v) = (-1_F)\,T(v) = -T(v)$, since,

$$\begin{aligned}(-T)\,(v_1 + v_2) &= -\,(T(v_1 + v_2)) = -\,[(T(v_1) + T(v_2)]\\ &= -\,T(v_1) - T(v_2)\\ &= (-T)\,(v_1) + (-T)\,(v_2),\ \forall v_1, v_2 \in V(F),\end{aligned}$$

such that $[T+(-T)]\,(v) = T(v) + (-T)\,(v)$

$$\begin{aligned}&= T(v) - T(v) = 0_W\\ &= T_0(v),\text{ for all } v \in V(F)\\ &= [-T + T]\,(v),\end{aligned}$$

proving that, $T + (-T) = T_0 = (-T) + T$, $\forall T \in$ Hom (V,W) where $(-T)$ is exhibited additive inverse of T, for each $T \in$ Hom (V,W).

The property of additive abelianess does observe on Hom (V, W). It is easy to show that, in addition, $T_1 + T_2 = T_2 + T_1$, for all $T_1, T_2 \in$ Hom (V, W). Thus,

(Hom (V, W), +) forms the structure of an abelian group.

The scalar product (αT) of $\alpha \in F$ to $T \in$ Hom (V, W) is defined by, $(\alpha T)(v) = \alpha(T(v))$, for all $v \in V(F)$, which observes the properties:

(1) $$\begin{aligned}\alpha\,(T_1 + T_2)\,(v) &= \alpha\,[(T_1 + T_2)\,(v)] = \alpha[T_1(v) + T_2(v)]\\ &= \alpha\,(T_1\,(v)) + \alpha\,(T_2(v))\\ &= (\alpha T_1)\,(v) + (\alpha T_2)(v)\\ &= (\alpha T_1 + \alpha T_2)\,(v),\text{ for all } v \in V(F)\end{aligned}$$

Hence $\alpha[T_1 + T_2] = \alpha\,T_1 + \alpha\,T_2$

(2) $$\begin{aligned}(\alpha+\beta)\,T_1(v) = (\alpha+\beta)\,(T_1(v)) &= \alpha\,T_1(v) + \beta\,T_1(v)\\ &= [(\alpha T_1) + (\beta T_1)]\,(v)\end{aligned}$$

$$\Rightarrow\ (\alpha+\beta)\,T_1 = \alpha T_1 + \beta T_2$$

(3) $$(\alpha.\beta)T_1(v) = \alpha(\beta T_1)(v) = \beta(\alpha T_1)\,(v) = (\beta\alpha)\,T_1(v)$$

$$\Rightarrow\ (\alpha\beta)T_1 = \alpha(\beta T_1) = \beta(\alpha T_1) = (\beta\alpha)T_1$$

(4) $1_F(T_1) = (1_F T_1) = T_1$, by definition.

Consequently, (Hom_F (V, W), +, F) forms a vector space over the same field F.

COR. 1

Set $Hom_F(V,W)$ forms a vector space over the field F of V(F) called space of linear transformations of V if W = V.

COR. 2

$Hom_F(V,F)$ forms a vector space of functionals of V.

REMARK

If T is a linear transformation on V(F), then

(i) T preserves collinearity of vectors

(ii) T preserves the parallelograms and triangles made by any two non-collinear vectors of V(F).

(iii) T stretches a vector or contracts (or reverses) it on the same line.

6.5.13 THEOREM

If dim (V) = m and dim (W) = n. Then Hom (V,W) is a vector space of dimension mn over the same field F as those of V and W.

PROOF

Let $\{v_1, v_2, \ldots, v_m\}$ be a basis of V(F) and $\{w_1, w_2, \ldots, w_n\}$ be a basis of W(F).

Let T_{ij} be mn mappings from V(F) into W(F) and $v \in V(F)$ $1 \le i \le m$ and $1 \le j \le n$. Then,

$v = \alpha v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m$, is uniquely expressed as a linear combination of a basis of V(F) and the action of mn $T_{ij} \in$ Hom (V,W), on V(F) is determined uniquely via actions on the basis v_i, i = 1, 2, . . . , m of V(F), which is defined by,

$$T_{ij}(v_k) = \begin{cases} w_j, & \text{if } k = j \\ 0_W, & \text{if } k \neq j \end{cases} \qquad \ldots\ldots\ldots\ldots (a)$$

Let an arbitrary $S \in$ Hom (V,W) and $v_1 \in V(F)$. Then $S(v_1) \in W(F)$,

where, $S(v_1) = \alpha_{11}w_1 + \alpha_{12}w_2 + \ldots + \alpha_{1n}w_n$, $\qquad \alpha_{1j} \in F$

and, $S(v_i) = \alpha_{i1}w_1 + \alpha_{i2}w_2 + \ldots + \alpha_{in}w_n, \quad \alpha_{ij} \in F$ . ...(b)

for i = 1,2, . . . , m. Then

$[\alpha_{11}T_{11} + \alpha_{12}T_{12} + \ldots + \alpha_{1n}T_{1n} + \alpha_{21}T_{21} + \alpha_{22}T_2 + \ldots + \alpha_{2n}T_{2n} + \ldots + \alpha_{i1}T_{i1} + \alpha_{i2}T_{i2} + \ldots + \alpha_{in}T_{in} + \ldots + \alpha_{m1}T_{m1} + \alpha_{m2}T_{m2} + \ldots + \alpha_{mn}T_{mn}] (v_k)$

$= \alpha_{11}T_{11}(v_k) + \alpha_{12}T_{12}(v_k) + \ldots + \alpha_{m1}T_{m1}(v_k) + \ldots \alpha_{mn}T_{mn}(v_k)$

$= \alpha_{k1}w_1 + \alpha_{k2}w_2 + \ldots + \alpha_{kn}w_n$, by using (a)

$= S(v_k)$, by (b).

Thus $\{T_{ij}, 1 \le i \le m, 1 \le j \le n\}$ span Hom(V, W), where arbitrary $S \in < T_{ij} >$ and hence $L(\{T_{ij}\}) = \text{Hom}(V, W)$.

r to show that the set $\{T_{11}, T_{12}, \ldots T_{in} \ldots, T_{m1}, T_{m2}, \ldots T_{mn}\}$ is ndent over F,

$T_{11} + \beta_{12}T_{12} + \ldots + \beta_{in}T_{in} + \ldots + \beta_{m1}T_{m1} + \ldots + \beta_{mn}T_{mn} = T_0, \beta_{ij} \in F$.

n

$[\beta_{11}T_{11} + \beta_{12}T_{12} + \ldots + \beta_{in}T_{in} + \ldots + \beta_{m1}T_{m1} + \ldots + \beta_{mn}T_{mn}] (v_k) = T_0(v_k)$

$\Rightarrow \quad \beta_{k1}w_1 + \beta_{k2}w_2 + \ldots + \beta_{kn}w_n = 0_W$

Since w_i (i = 1,2, . . . , n) are linearly independent therefore

$\beta_{k1} = 0 = \beta_{k2} = \ldots = \beta_{kn}$

$\Rightarrow \quad \beta_{ij} = 0$ for all i,j , which shows that T_{ij} for $1 \le i \le m$, $1 \le j \le n$ are linearly independent over F

Hence $\{T_{ij}\}$ is a basis of Hom (V,W) and dim (V,W) = mn

COR 1

The dimension of vector space $\text{Hom}_F(V,V)$ of linear transformations of V is
$m^2 = (\dim V)^2$

COR 2

The dimension of dual space Hom(V,F) of functionals of V(F) is equal to the dimension of V = m

The dual space Hom (V, F) of V(F) is generally denoted by $\hat{V}$, each element of which is $f \in$ Hom(V,F).

REMARK

We have exhibited that for finite dimensional vector space V(F), a vector space $\hat{V}$ (F) is attached to V(F) under the same field having the same dimension i.e., dim V(F) = dim $\hat{V}$ (F).

QUESTION?

"What is the dual space $\hat{\hat{V}}$ (F) of $\hat{V}$ (F), (the dual space of V(F)), if V(F) is finite dimensional vector space over F?". As an answer of the above question, we prove the following theorem:

6.5.14 THEOREM

Let V be a vector space with dual space $\hat{V} = \{f : f \in \text{Hom}(V,F)\}$.

If $\hat{\hat{V}} = \{T : T \in \text{Hom}(\hat{V}, F)\}$ then $\hat{\hat{V}} \cong V$.

PROOF

Let $v_0 \in$ V(F) be a fixed vector. If $f \in \hat{V}$ (F) then $f(v_0)$ defines a functional f from $\hat{V}$ into F. Let us denote it a function T_{v_0} defined by,

$T_{v_0}(f) = f(v_0)$, for every $f \in \hat{V}$, $T_{v_0} \in \text{Hom}(\hat{V}, F)$, since,

$$T_{v_0}(f + g) = (f + g)(v_0) = f(v_0) + g(v_0)$$

$$= T_{v_0}(f) + T_{v_0}(g), \forall f,g \in \hat{V}$$

Furthermore, $T_{v_0}(\alpha f) = (\alpha f)(v_0) = \alpha(f(v_0)) = (\alpha T_{v_0})(f)$

Thus $T_{v_0} \in \text{Hom}(\hat{V}, F)$, the dual space $\hat{\hat{V}}$ of $\hat{V}$

In general, given a vector $v \in V$, a vector T_v of $\hat{\hat{V}}$ is associated with v.

This association generates a homomorphism ψ from V(F) into $\hat{\hat{V}}$, which is defined by,

$$\psi(v) = T_v \in \hat{\hat{V}} \text{ for every } v \in V. \text{ Then,}$$

$$\psi(v+w) = T_{v+w}, \text{ where, } T_v, T_w \in \hat{\hat{V}}$$

For $f \in \hat{V}$,

$$\begin{aligned} T_{v+w}(f) &= f(v + w) \\ &= f(v) + f(w) \\ &= T_v(f) + T_w(f) \\ &= (T_v + T_w)(f), \ \forall f \in \hat{V} \end{aligned}$$

$$\Rightarrow \quad T_{v+w} = T_v + T_w,$$

and hence $\psi(v + w) = \psi(v) + \psi(w) = T_v + T_w$, of V into $\hat{\hat{V}}$ and $\psi(\lambda v) = \lambda\psi(v)$, $\lambda \in F$.

which proves that ψ is a homomorphism. Infact, ψ is an isomorphism, because, $v \in \ker \psi$ iff $\psi(v) = T_0$, the zero of Hom(V, F). It implies that $T_v = T_0$.

If $T_v = T_0$, then $T_v(f) = f(v) = 0$, for all $f \in \hat{V}$, then $v = 0v$ and $\ker(\psi) = \{T_0\}$

It proves then that ψ is an isomorphism. Hence $\hat{\hat{V}} \equiv V$

Note that,

(i) If V is finite dimensional then ψ is onto $\hat{\hat{V}}$, because $\dim V = \dim \hat{V} = \dim \hat{\hat{V}}$

(ii) If V is infinite dimensional, then V is not onto

(iii) dimensions of V, $\hat{V}$ and $\hat{\hat{V}}$ are equal if V is finite dimensional.

(iv) If $0v \neq v \in V(F)$ then there exists $f \in \hat{V}$ such that $f(v) \neq 0v$

6.6 ANNIHILATOR OF A SUBSPACE OF V(F)

We know that any f in Hom(V,W) acts on V and transforms onto vectors of W. The vectors of V(F) which are transformed onto the zero vector of W(F) constitute the kernel of f. It provides information that how many vectors of V(F) are transformed onto a non-zero vector of W(F). They are equal in number to the vector of the quotient space V/ker f. In other words every homomorphism annihilates its kernel which is subspace of the domain space of the homomorphism f.

A fixed subspace W of a vector space V(F) contained in the kernel of a linear functional $f \in Hom(V,F)$ is said to be an **annihilator** of W.

6.6.1 DEFINITION

If W is a subspace of a vectorspace V(F). Then, the set $A(W) = \{f \in \hat{V} : f(w) = 0w$, for all $w \in W\}$ is said to form the set of annihilators of W.

6.6.2 LEMMA: Let A(W) be the set of all annihilators of W, with W as a subspace of V(F). Then A(W) is a subspace of $\hat{V}$.

PROOF

Let $f, g \in A(W) \subseteq \hat{V}$ and $\alpha, \beta \in F$. Then $f(w) = 0w = g(w)$, for all $w \in W$

Then $(\alpha f + \beta g)(w) = (\alpha f)(w) + (\beta g)(w)$

$$= \alpha(f(w)) + \beta(g(w)) = \alpha 0w + \beta 0w = 0w,$$

which implies that $\alpha f + \beta g \in A(W)$ and hence A (W) forms a subspace of $\hat{V}$

6.6.3 LEMMA

If $U \subset W \subseteq V(F)$, then $A(U) \supset A(W)$.

PROOF

Let $f \in A(W)$. Then $f(w) = 0w$, $\forall w \in W$

Since $U \subset W$ and $u \in U \Rightarrow u \in W$, therefore $f(u) = 0w$, for all $u \in U$. It implies that $f \in A(U)$ which holds for all $f \in A(W)$. Hence $A(W) \subset A(U)$ or $A(U) \supset A(W)$.

6.6.4 THEOREM

Let W be a subspace of a finite dimensional vector space V(F).

Then $\hat{W} \cong \hat{V}/A(W)$ and dimension of $A(W) = \dim(V) - \dim(W) = \dim\left(\frac{V}{W}\right)$

PROOF

Since V(F) is finite dimensional and $W \subseteq V(F)$ therefore W is finite dimensional.

Le* ...d $\bar{f}$ be the restriction of f to W defined by $f(w) = \bar{f}(w)$, for every ...e $f \in \hat{V}$, therefore $\bar{f} \in \hat{W}$. If $T : \hat{V} \longrightarrow \hat{W}$, then T is a ...sm defined by,

$$T(f) = \bar{f}.$$

...$\in \hat{V}$, then $T(f + g) = T(f) + T(g)$ and $T(\lambda f) = \lambda T(f)$,

which shows that $T \in \text{Hom}(\hat{V}, \hat{W})$ such that $\ker T = A(W)$

We intend to show that T is onto $\hat{W}$. For this purpose, let $h \in \hat{W}$. Then h is the restriction of some $f \in \hat{V}$ i.e., $T(h) = \bar{f}$

Let $\{w_1, w_2, \ldots, w_m\}$ be a basis of W, which can be extended to be a basis $\{w_1, w_2, \ldots, w_m, v_1, \ldots, v_r\}$ of V such that $dm(V) = m + r = n$

Let W_1 be a subspace of V spanned by $\{v_1, v_2, \ldots, v_r\}$ giving $V = W \oplus W_1$. If $h \in \hat{W}$ and $v \in V$ such that $v = w + w_1$, $w \in W$ and $w_1 \in W_1$, then $f(v) = h(w)$, $f \in \hat{V}$ such that $\bar{f} = h = T(f)$. Hence T maps $\hat{V}$ onto $\hat{W}$.

Since $\ker T = A(W)$, therefore, by the fundamental theorem of homomorphsims (Chapter 4),

$$\hat{V}/A(W) \cong \hat{W}$$

...e isomorphic spaces have same dimension. Thus,

$$\dim \hat{V} - \dim A(W) = \dim(\hat{W})$$

$\Rightarrow$ $\dim V - \dim A(W) = \dim W$, where $\dim \hat{V} = \dim V$ and $\dim \hat{W} = \dim W$.

$\Rightarrow$ $n - m = \dim A(W) \Rightarrow \dim V - \dim W = \dim(V/W)$,

giving that $\dim A(W) = \dim(V/W)$.

COR

$A(A(W)) = W$, for each subspace W of $V(F)$.

PROOF

Since $W \subset V$ and $A(A(W)) \subset \hat{\hat{V}} \cong V$, therefore $W \subset A(A(W))$, for if $w \in W$ then T_w acts on V by $T_w(f) = f(w) = 0_W$, $\forall f \in A(W)$

However, $\dim A(A(W)) = \dim \hat{V} - \dim A(W)$

or $\dim A(A(W)) = \dim V - (\dim V - \dim W)$

$= \dim W$

Since $W \subset A(A(W))$, and $\dim W = \dim (A(A(W)))$, therefore $W = A(A(W))$.

6.7 MATRIX REPRESENTATIONS OF LINEAR TRANSFORMATIONS

We have shown in this chapter that the set $Hom_F(V, W)$ of all vector space homomorphisms (vector spaces being over the same field) from a vector space $V(F)$ into a vector $W(F)$. The elements of $Hom_F(V, W)$ act as linear transforms from $V(F)$ into $W(F)$ respecting their binary operations of addition and scalar multiplication. i.e.

If $T \in Hom_F(V, W)$ and $v_1, v_2 \in V$, then

(a) $T(v_1 + v_2) = T(v_1) + T(v_2)$

and (b) $T(\alpha v_1) = \alpha T(v_1)$, for all $\alpha \in F$.

(a) and (b) imply that T preserves the geometrical figures of parallelogram of $V(F)$ into $W(F)$, if v_1 and v_2 are not collinear vectors of $V(F)$.
If the vectors v_1 and v_2 are collinear then T preserves the collinearity. T may stretch or contract or reverse a vector along a line.

At this point one desires to develop a connection between the space $Hom_F(V, W)$ of linear transforms and the space $M_{m\times n}$ of m×n matrices over the same field as of V and W of finite dimensions n and m respectively. We are aware of the fact the space $M_{m\times n}(F)$ is of same dimension mn as that of space $Hom_F(V, W)$. For the sake of establishing such a relationship we shall assume that

(1) V and W are finite dimensional vector spaces over the same field F($\Re$ or $\mathbb{C}$)
(2) $\dim V = n$, $\dim W = m$
(3) $V = \Re^n$ and $W = \Re^m$
(4) The elements of $\Re^n$ and $\Re^m$ are Coordinated Column Vectors having n and m components respectively.