4
FG-modules

We now introduce the concept of an FG-module, and show that there is a close connection between FG-modules and representations of G over F. Much of the material in the remainder of the book will be presented in terms of FG-modules, as there are several advantages to this approach to representation theory.

FG-modules

Let G be a group and let F be $\mathbb{R}$ or $\mathbb{C}$.

Suppose that $\rho: G \to \mathrm{GL}(n, F)$ is a representation of G. Write $V = F^n$, the vector space of all row vectors $(\lambda_1, \ldots, \lambda_n)$ with $\lambda_i \in F$. For all $v \in V$ and $g \in G$, the matrix product

$$v(g\rho),$$

of the row vector v with the $n \times n$ matrix $g\rho$, is a row vector in V (since the product of a $1 \times n$ matrix with an $n \times n$ matrix is again a $1 \times n$ matrix).

We now list some basic properties of the multiplication $v(g\rho)$. First, the fact that ρ is a homomorphism shows that

$$v((gh)\rho) = v(g\rho)(h\rho)$$

for all $v \in V$ and all $g, h \in G$. Next, since 1ρ is the identity matrix, we have

$$v(1\rho) = v$$

for all $v \in V$. Finally, the properties of matrix multiplication give

$$(\lambda v)(g\rho) = \lambda(v(g\rho)),$$

$$(u+v)(g\rho) = u(g\rho) + v(g\rho)$$

for all $u,\ v \in V,\ \lambda \in F$ and $g \in G$.

4.1 Example

Let $G = D_8 = \langle a,\ b\colon a^4 = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle$, and let $\rho\colon G \to \mathrm{GL}(2, F)$ be the representation of G over F given in Example 3.2(1). Thus

$$a\rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ b\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $v = (\lambda_1, \lambda_2) \in F^2$ then, for example,

$$v(a\rho) = (-\lambda_2,\ \lambda_1),$$

$$v(b\rho) = (\lambda_1,\ -\lambda_2),$$

$$v(a^3\rho) = (\lambda_2,\ -\lambda_1).$$

Motivated by the above observations on the product $v(g\rho)$, we now define an FG-module.

4.2 Definition

Let V be a vector space over F and let G be a group. Then V is an *FG-module* if a multiplication vg $(v \in V, g \in G)$ is defined, satisfying the following conditions for all $u,\ v \in V,\ \lambda \in F$ and $g,\ h \in G$:

(1) $vg \in V$;
(2) $v(gh) = (vg)h$;
(3) $v1 = v$;
(4) $(\lambda v)g = \lambda(vg)$;
(5) $(u+v)g = ug + vg$.

We use the letters F and G in the name 'FG-module' to indicate that V is a vector space over F and that G is the group from which we are taking the elements g to form the products vg $(v \in V)$.

Note that conditions (1), (4) and (5) in the definition ensure that for all $g \in G$, the function

$$v \to vg \quad (v \in V)$$

is an endomorphism of V.

4.3 Definition

Let V be an FG-module, and let $\mathscr{B}$ be a basis of V. For each $g \in G$, let

$$[g]_{\mathscr{B}}$$

denote the matrix of the endomorphism $v \to vg$ of V, relative to the basis $\mathscr{B}$.

The connection between FG-modules and representations of G over F is revealed in the following basic result.

4.4 Theorem

(1) *If* $\rho\colon G \to \mathrm{GL}(n, F)$ *is a representation of* G *over* F, *and* $V = F^n$, *then* V *becomes an* FG-*module if we define the multiplication* vg *by*

$$vg = v(g\rho) \quad (v \in V, g \in G).$$

Moreover, there is a basis $\mathscr{B}$ *of* V *such that*

$$g\rho = [g]_{\mathscr{B}} \quad \textit{for all } g \in G.$$

(2) *Assume that* V *is an* FG-*module and let* $\mathscr{B}$ *be a basis of* V. *Then the function*

$$g \to [g]_{\mathscr{B}} \quad (g \in G)$$

is a representation of G *over* F.

Proof (1) We have already observed that for all u, $v \in F^n$, $\lambda \in F$ and g, $h \in G$, we have

$$v(g\rho) \in F^n,$$
$$v((gh)\rho) = (v(g\rho))(h\rho),$$
$$v(1\rho) = v,$$
$$(\lambda v)(g\rho) = \lambda(v(g\rho)),$$
$$(u + v)(g\rho) = u(g\rho) + v(g\rho).$$

Therefore, F^n becomes an FG-module if we define

$$vg = v(g\rho) \quad \text{for all } v \in F^n, \mathrm{g} \in G.$$

Moreover, if we let $\mathscr{B}$ be the basis

$$(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1)$$

of F^n, then $g\rho = [g]_{\mathscr{B}}$ for all $g \in G$.

(2) Let V be an FG-module with basis $\mathscr{B}$. Since $v(gh) = (vg)h$ for all $g, h \in G$ and all v in the basis $\mathscr{B}$ of V, it follows that

$$[gh]_{\mathscr{B}} = [g]_{\mathscr{B}}[h]_{\mathscr{B}}.$$

In particular,

$$[1]_{\mathscr{B}} = [g]_{\mathscr{B}}[g^{-1}]_{\mathscr{B}}$$

for all $g \in G$. Now $v1 = v$ for all $v \in V$, so $[1]_{\mathscr{B}}$ is the identity matrix. Therefore each matrix $[g]_{\mathscr{B}}$ is invertible (with inverse $[g^{-1}]_{\mathscr{B}}$).

We have proved that the function $g \to [g]_{\mathscr{B}}$ is a homomorphism from G to $\mathrm{GL}(n, F)$ (where $n = \dim V$), and hence is a representation of G over F. ∎

Our next example illustrates part (1) of Theorem 4.4.

4.5 Examples

(1) Let $G = D_8 = \langle a, b \colon a^4 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle$ and let ρ be the representation of G over F given in Example 3.2(1), so

$$a\rho = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ b\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Write $V = F^2$. By Theorem 4.4(1), V becomes an FG-module if we define

$$vg = v(g\rho) \quad (v \in V, g \in G).$$

For instance,

$$(1, 0)a = (1, 0)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (0, 1).$$

If v_1, v_2 is the basis $(1, 0), (0, 1)$ of V, then we have

$$\begin{aligned} v_1a &= v_2, & v_1b &= v_1, \\ v_2a &= -v_1, & v_2b &= -v_2. \end{aligned}$$

If $\mathscr{B}$ denotes the basis v_1, v_2, then the representation

$$g \to [g]_{\mathscr{B}} \quad (g \in G)$$

is just the representation ρ (see Theorem 4.4(1) again).

(2) Let $G = Q_8 = \langle a, b \colon a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1} \rangle$. In Example

1.2(4) we defined Q_8 to be the subgroup of $\mathrm{GL}(2,\mathbb{C})$ generated by

$$A = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so we already have a representation of G over $\mathbb{C}$. To illustrate Theorem 4.4(1) we must this time take $F = \mathbb{C}$. We then obtain a $\mathbb{C}G$-module with basis v_1, v_2 such that

$$\begin{aligned} v_1 a &= \mathrm{i}v_1, & v_1 b &= v_2, \\ v_2 a &= -\mathrm{i}v_2, & v_2 b &= -v_1. \end{aligned}$$

Notice that in the above examples, the vectors v_1a, v_2a, v_1b and v_2b determine vg for all $v \in V$ and $g \in G$. For instance, in Example 4.5(1),

$$\begin{aligned} (v_1 + 2v_2)ab &= v_1ab + 2v_2ab \\ &= v_2b - 2v_1b \\ &= -v_2 - 2v_1. \end{aligned}$$

A similar remark holds for all FG-modules V: if $v_1, \ldots, v_n$ is a basis of V and $g_1, \ldots, g_r$ generate G, then the vectors v_ig_j $(1 \leqslant i \leqslant n,$ $1 \leqslant j \leqslant r)$ determine vg for all $v \in V$ and $g \in G$.

Shortly, we shall show you various ways of constructing FG-modules directly, without using a representation. To do this, we turn a vector space V over F into an FG-module by specifying the action of group elements on a basis $v_1, \ldots, v_n$ of V and then extending the action to be linear on the whole of V; that is, we first define v_ig for each i and each g in G, and then define

$$(\lambda_1 v_1 + \ldots + \lambda_n v_n)g \quad (\lambda_i \in F)$$

to be

$$\lambda_1(v_1g) + \ldots + \lambda_n(v_ng).$$

As you might expect, there are restrictions on how we may define the vectors v_ig. The next result will often be used to show that our chosen multiplication turns V into an FG-module.

4.6 Proposition

Assume that $v_1, \ldots, v_n$ is a basis of a vector space V over F. Suppose that we have a multiplication vg for all v in V and g in G which *satisfies the following conditions for all i with $1 \leqslant i \leqslant n$, for all $g, h \in G$, and for all $\lambda_1, \ldots, \lambda_n \in F$:*

(1) $v_i g \in V$;
(2) $v_i(gh) = (v_i g)h$;
(3) $v_i 1 = v_i$;
(4) $(\lambda_1 v_1 + \ldots + \lambda_n v_n)g = \lambda_1(v_1 g) + \ldots + \lambda_n(v_n g)$.

Then V is an FG-module.

Proof It is clear from (3) and (4) that $v1 = v$ for all $v \in V$.

Conditions (1) and (4) ensure that for all g in G, the function $v \to vg$ $(v \in V)$ is an endomorphism of V. That is,

$$vg \in V,$$

$$(\lambda v)g = \lambda(vg),$$

$$(u + v)g = ug + vg,$$

for all $u, v \in V$, $\lambda \in F$ and $g \in G$. Hence

$$(4.7) \qquad (\lambda_1 u_1 + \ldots + \lambda_n u_n)h = \lambda_1(u_1 h) + \ldots + \lambda_n(u_n h)$$

for all $\lambda_1, \ldots, \lambda_n \in F$, all $u_1, \ldots, u_n \in V$ and all $h \in G$.

Now let $v \in V$ and $g, h \in G$. Then $v = \lambda_1 v_1 + \ldots + \lambda_n v_n$ for some $\lambda_1, \ldots, \lambda_n \in F$, and

$$\begin{aligned} v(gh) &= \lambda_1(v_1(gh)) + \ldots + \lambda_n(v_n(gh)) \quad \text{by condition (4)} \\ &= \lambda_1((v_1 g)h) + \ldots + \lambda_n((v_n g)h) \quad \text{by condition (2)} \\ &= (\lambda_1(v_1 g) + \ldots + \lambda_n(v_n g))h \quad \text{by (4.7)} \\ &= (vg)h \quad \text{by condition (4).} \end{aligned}$$

We have now checked all the axioms which are required for V to be an FG-module. ∎

Our next definitions translate the concepts of the trivial representation and a faithful representation into module terms.

4.8 Definitions

(1) The *trivial FG*-module is the 1-dimensional vector space V over F with

$$vg = v \quad \text{for all } v \in V, g \in G.$$

(2) An FG-module V is *faithful* if the identity element of G is the only element g for which

$$vg = v \quad \text{for all } v \in V.$$

For instance, the FD_8-module which appears in Example 4.5(1) is faithful.

Our next aim is to use Proposition 4.6 to construct faithful FG-modules for all subgroups of symmetric groups.

Permutation modules

Let G be a subgroup of S_n, so that G is a group of permutations of $\{1, \ldots, n\}$. Let V be an n-dimensional vector space over F, with basis $v_1, \ldots, v_n$. For each i with $1 \leqslant i \leqslant n$ and each permutation g in G, define

$$v_i g = v_{ig}.$$

Then $v_i g \in V$ and $v_i 1 = v_i$. Also, for g, h in G,

$$v_i(gh) = v_{i(gh)} = v_{(ig)h} = (v_i g)h.$$

We now extend the action of each g linearly to the whole of V; that is, for all $\lambda_1, \ldots, \lambda_n$ in F and g in G, we define

$$(\lambda_1 v_1 + \ldots + \lambda_n v_n)g = \lambda_1(v_1 g) + \ldots + \lambda_n(v_n g).$$

Then V is an FG-module, by Proposition 4.6.

4.9 Example

Let $G = S_4$ and let $\mathscr{B}$ denote the basis v_1, v_2, v_3, v_4 of V. If $g = (1\ 2)$, then

$$v_1 g = v_2,\ v_2 g = v_1,\ v_3 g = v_3,\ v_4 g = v_4.$$

And if $h = (1\ 3\ 4)$, then

$$v_1 h = v_3,\ v_2 h = v_2,\ v_3 h = v_4,\ v_4 h = v_1.$$

We have

$$[g]_{\mathscr{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ [h]_{\mathscr{B}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

4.10 Definition
Let G be a subgroup of S_n. The FG-module V with basis $v_1, \ldots, v_n$ such that

$$v_i g = v_{ig} \quad \text{for all } i, \text{ and all } g \in G,$$

is called the *permutation module* for G over F. We call $v_1, \ldots, v_n$ the *natural basis* of V.

Note that if we write $\mathscr{B}$ for the basis $v_1, \ldots, v_n$ of the permutation module, then for all g in G, the matrix $[g]_{\mathscr{B}}$ has precisely one non-zero entry in each row and column, and this entry is 1. Such a matrix is called a *permutation matrix*.

Since the only element of G which fixes every v_i is the identity, we see that the permutation module is a faithful FG-module. If you are aware of the fact that every group G of order n is isomorphic to a subgroup of S_n, then you should be able to see that G has a faithful FG-module of dimension n. We shall go into this in more detail in Chapter 6.

4.11 Example
Let $G = C_3 = \langle a \colon a^3 = 1 \rangle$. Then G is isomorphic to the cyclic subgroup of S_3 which is generated by the permutation (1 2 3). This alerts us to the fact that if V is a 3-dimensional vector space over F, with basis v_1, v_2, v_3, then we may make V into an FG-module in which

$$v_1 1 = v_1,\ v_2 1 = v_2,\ v_3 1 = v_3,$$

$$v_1 a = v_2,\ v_2 a = v_3,\ v_3 a = v_1,$$

$$v_1 a^2 = v_3,\ v_2 a^2 = v_1,\ v_3 a^2 = v_2.$$

Of course, we define vg, for v an arbitrary vector in V and $g = 1$, a or a^2, by

$$(\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3)g = \lambda_1(v_1 g) + \lambda_2(v_2 g) + \lambda_3(v_3 g)$$

for all $\lambda_1, \lambda_2, \lambda_3 \in F$. Proposition 4.6 can be used to verify that V is an FG-module, but we have been motivated by the definition of permutation modules in our construction.

FG-modules and equivalent representations

We conclude the chapter with a discussion of the relationship between FG-modules and equivalent representations of G over F. An FG-

module gives us many representations, all of the form

$$g \to [g]_{\mathscr{B}} \quad (g \in G)$$

for some basis $\mathscr{B}$ of V. The next result shows that all these representations are equivalent to each other (see Definition 3.3); and moreover, any two equivalent representations of G arise from some FG-module in this way.

4.12 Theorem

Suppose that V is an FG-module with basis $\mathscr{B}$, and let ρ be the representation of G over F defined by

$$\rho\colon g \to [g]_{\mathscr{B}} \quad (g \in G).$$

(1) *If $\mathscr{B}'$ is a basis of V, then the representation*

$$\phi\colon g \to [g]_{\mathscr{B}'} \quad (g \in G)$$

of G is equivalent to ρ.

(2) *If σ is a representation of G which is equivalent to ρ, then there is a basis $\mathscr{B}''$ of V such that*

$$\sigma\colon g \to [g]_{\mathscr{B}''} \quad (g \in G).$$

Proof (1) Let T be the change of basis matrix from $\mathscr{B}$ to $\mathscr{B}'$ (see Definition 2.23). Then by (2.24), for all $g \in G$, we have

$$[g]_{\mathscr{B}} = T^{-1}[g]_{\mathscr{B}'}T.$$

Therefore ϕ is equivalent to ρ.

(2) Suppose that ρ and σ are equivalent representations of G. Then for some invertible matrix T, we have

$$g\rho = T^{-1}(g\sigma)T \quad \text{for all } g \in G.$$

Let $\mathscr{B}''$ be the basis of V such that the change of basis matrix from $\mathscr{B}$ to $\mathscr{B}''$ is T. Then for all $g \in G$,

$$[g]_{\mathscr{B}} = T^{-1}[g]_{\mathscr{B}''}T,$$

and so $g\sigma = [g]_{\mathscr{B}''}$. ■

4.13 Example

Again let $G = C_3 = \langle a\colon a^3 = 1\rangle$. There is a representation ρ of G which is given by

$$1\rho = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ a\rho = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},\ a^2\rho = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

(To see this, simply note that $(a\rho)^2 = a^2\rho$ and $(a\rho)^3 = I$; see Exercise 3.2.)

If V is a 2-dimensional vector space over $\mathbb{C}$, with basis v_1, v_2 (which we call $\mathscr{B}$), then we can turn V into a $\mathbb{C}G$-module as in Theorem 4.4(1) by defining

$$\begin{aligned} &v_1 1 = v_1, \quad v_1 a = v_2, \qquad v_1 a^2 = -v_1 - v_2, \\ &v_2 1 = v_2, \quad v_2 a = -v_1 - v_2, \quad v_2 a^2 = v_1. \end{aligned}$$

We then have

$$[1]_{\mathscr{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ [a]_{\mathscr{B}} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},\ [a^2]_{\mathscr{B}} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now let $u_1 = v_1$ and $u_2 = v_1 + v_2$. Then u_1, u_2 is another basis of V, which we call $\mathscr{B}'$. Since

$$\begin{aligned} &u_1 1 = u_1, \quad u_1 a = -u_1 + u_2, \quad u_1 a^2 = -u_2, \\ &u_2 1 = u_2, \quad u_2 a = -u_1, \quad u_2 a^2 = u_1 - u_2, \end{aligned}$$

we obtain the representation ϕ: $g \to [g]_{\mathscr{B}'}$ where

$$[1]_{\mathscr{B}'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ [a]_{\mathscr{B}'} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix},\ [a^2]_{\mathscr{B}'} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Note that if

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

then for all g in G, we have

$$[g]_{\mathscr{B}} = T^{-1}[g]_{\mathscr{B}'}T,$$

and so ρ and ϕ are equivalent, in agreement with Theorem 4.12(1).

Summary of Chapter 4

1. An FG-module is a vector space over F, together with a multiplication by elements of G on the right. The multiplication satisfies properties (1)–(5) of Definition 4.2.

2. There is a correspondence between representations of G over F and FG-modules, as follows.
   (a) Suppose that $\rho\colon G \to \mathrm{GL}(n, F)$ is a representation of G. Then F^n is an FG-module, if we define
   $$vg = v(g\rho) \quad (v \in F^n,\ g \in G).$$
   (b) If V is an FG-module, with basis $\mathscr{B}$, then $\rho\colon g \to [g]_{\mathscr{B}}$ is a representation of G over F.
3. If G is a subgroup of S_n, then the permutation FG-module has basis $v_1, \ldots, v_n$, and $v_i g = v_{ig}$ for all i with $1 \leqslant i \leqslant n$, and all g in G.

Exercises for Chapter 4

1. Suppose that $G = S_3$, and that $V = \mathrm{sp}(v_1, v_2, v_3)$ is the permutation module for G over $\mathbb{C}$, as in Definition 4.10. Let $\mathscr{B}_1$ be the basis v_1, v_2, v_3 of V and let $\mathscr{B}_2$ be the basis $v_1 + v_2 + v_3$, $v_1 - v_2$, $v_1 - v_3$. Calculate the 3×3 matrices $[g]_{\mathscr{B}_1}$ and $[g]_{\mathscr{B}_2}$ for all g in S_3. What do you notice about the matices $[g]_{\mathscr{B}_2}$?
2. Let $G = S_n$ and let V be a vector space over F. Show that V becomes an FG-module if we define, for all v in V,
   $$vg = \begin{cases} v, & \text{if } g \text{ is an even permutation,} \\ -v, & \text{if } g \text{ is an odd permutation.} \end{cases}$$
3. Let $Q_8 = \langle a, b\colon a^4 = 1,\ b^2 = a^2,\ b^{-1}ab = a^{-1}\rangle$, the quaternion group of order 8. Show that there is an $\mathbb{R}Q_8$-module V of dimension 4 with basis v_1, v_2, v_3, v_4 such that
   $$v_1 a = v_2, \quad v_2 a = -v_1, \quad v_3 a = -v_4, \quad v_4 a = v_3, \text{ and}$$
   $$v_1 b = v_3, \quad v_2 b = v_4, \quad v_3 b = -v_1, \quad v_4 b = -v_2.$$
4. Let A be an $n \times n$ matrix and let B be a matrix obtained from A by permuting the rows. Show that there is an $n \times n$ permutation matrix P such that $B = PA$. Find a similar result for a matrix obtained from A by permuting the columns.