6
Group algebras

The group algebra of a finite group G is a vector space of dimension $|G|$ which also carries extra structure involving the product operation on G. In a sense, group algebras are the source of all you need to know about representation theory. In particular, the ultimate goal of representation theory – that of understanding all the representations of finite groups – would be achieved if group algebras could be fully analysed. Group algebras are therefore of great interest.

After defining the group algebra of G, we shall use it to construct an important faithful representation, known as the regular representation of G, which will be explored in greater detail later on.

The group algebra of *G*

Let G be a finite group whose elements are $g_1, \ldots, g_n$, and let F be $\mathbb{R}$ or $\mathbb{C}$.

We define a vector space over F with $g_1, \ldots, g_n$ as a basis, and we call this vector space FG. Take as the elements of FG all expressions of the form

$$\lambda_1 g_1 + \ldots + \lambda_n g_n \quad (\text{all } \lambda_i \in F).$$

The rules for addition and scalar multiplication in FG are the natural ones: namely, if

$$u = \sum_{i=1}^{n} \lambda_i g_i \text{ and } v = \sum_{i=1}^{n} \mu_i g_i$$

are elements of FG, and $\lambda \in F$, then

$$u + v = \sum_{i=1}^{n} (\lambda_i + \mu_i) g_i \text{ and } \lambda u = \sum_{i=1}^{n} (\lambda \lambda_i) g_i.$$

With these rules, FG is a vector space over F of dimension n, with basis $g_1, \ldots, g_n$. The basis $g_1, \ldots, g_n$ is called the *natural* basis of FG.

6.1 Example

Let $G = C_3 = \langle a\colon a^3 = e \rangle$. (To avoid confusion with the element 1 of F, we write e for the identity element of G, in this example.) The vector space $\mathbb{C}G$ contains

$$u = e - a + 2a^2 \text{ and } v = \tfrac{1}{2}e + 5a.$$

We have

$$u + v = \tfrac{3}{2}e + 4a + 2a^2,\ \tfrac{1}{3}u = \tfrac{1}{3}e - \tfrac{1}{3}a + \tfrac{2}{3}a^2.$$

Sometimes we write elements of FG in the form

$$\sum_{g \in G} \lambda_g g \quad (\lambda_g \in F).$$

Now, FG carries more structure than that of a vector space – we can use the product operation on G to define multiplication in FG as follows:

$$\left(\sum_{g \in G} \lambda_g g \right) \left(\sum_{h \in G} \mu_h h \right) = \sum_{g, h \in G} \lambda_g \mu_h (gh) = \sum_{g \in G} \sum_{h \in G} (\lambda_h \mu_{h^{-1}g}) g$$

where all $\lambda_g, \mu_h \in F$.

6.2 Example

If $G = C_3$ and u, v are the elements of $\mathbb{C}G$ which appear in Example 6.1, then

$$\begin{aligned} uv &= (e - a + 2a^2)(\tfrac{1}{2}e + 5a) \\ &= \tfrac{1}{2}e + 5a - \tfrac{1}{2}a - 5a^2 + a^2 + 10a^3 \\ &= \tfrac{21}{2}e + \tfrac{9}{2}a - 4a^2. \end{aligned}$$

6.3 Definition
The vector space FG, with multiplication defined by

$$\left(\sum_{g\in G}\lambda_g g\right)\left(\sum_{h\in G}\mu_h h\right)=\sum_{g,h\in G}\lambda_g\mu_h(gh)$$

($\lambda_g, \mu_h \in F$), is called the *group algebra* of G over F.

The group algebra FG contains an identity for multiplication, namely the element $1e$ (where 1 is the identity of F and e is the identity of G). We write this element simply as 1.

6.4 Proposition
Multiplication in FG satisfies the following properties, for all $r, s, t \in FG$ *and* $\lambda \in F$:

(1) $rs \in FG$;
(2) $r(st) = (rs)t$;
(3) $r1 = 1r = r$;
(4) $(\lambda r)s = \lambda(rs) = r(\lambda s)$;
(5) $(r + s)t = rt + st$;
(6) $r(s + t) = rs + rt$;
(7) $r0 = 0r = 0$.

Proof (1) It follows immediately from the definition of rs that $rs \in FG$.
(2) Let

$$r=\sum_{g\in G}\lambda_g g,\ s=\sum_{g\in G}\mu_g g,\ t=\sum_{g\in G}\nu_g g,$$

($\lambda_g, \mu_g, \nu_g \in F$). Then

$$\begin{aligned}(rs)t &= \sum_{g,h,k\in G}\lambda_g\mu_h\nu_k(gh)k\\ &= \sum_{g,h,k\in G}\lambda_g\mu_h\nu_k g(hk)\\ &= r(st).\end{aligned}$$

We leave the proofs of the other equations as easy exercises. ■

In fact, any vector space equipped with a multiplication satisfying properties (1)–(7) of Proposition 6.4 is called an *algebra*. We shall be concerned only with group algebras, but it is worth pointing out that the axioms for an algebra mean that it is both a vector space and a ring.

The regular *FG*-module

We now use the group algebra to define an important FG-module.

Let $V = FG$, so that V is a vector space of dimension n over F, where $n = |G|$. For all $u, v \in V$, $\lambda \in F$ and $g, h \in G$, we have

$$vg \in V,$$

$$v(gh) = (vg)h,$$

$$v1 = v,$$

$$(\lambda v)g = \lambda(vg),$$

$$(u + v)g = ug + vg,$$

by parts (1), (2), (3), (4) and (5) of Proposition 6.4, respectively. Therefore V is an FG-module.

6.5 Definition

Let G be a finite group and F be $\mathbb{R}$ or $\mathbb{C}$. The vector space FG, with the natural multiplication vg $(v \in FG, g \in G)$, is called the *regular FG-module*.

The representation $g \to [g]_{\mathscr{B}}$ obtained by taking $\mathscr{B}$ to be the natural basis of FG is called the *regular representation* of G over F.

Note that the regular FG-module has dimension equal to $|G|$.

6.6 Proposition

The regular FG-module is faithful.

Proof Suppose that $g \in G$ and $vg = v$ for all $v \in FG$. Then $1g = 1$, so $g = 1$, and the result follows. ■

6.7 Example

Let $G = C_3 = \langle a\colon a^3 = e\rangle$. The elements of FG have the form

$$\lambda_1 e + \lambda_2 a + \lambda_3 a^2 \quad (\lambda_i \in F).$$

We have

$$(\lambda_1 e + \lambda_2 a + \lambda_3 a^2)e = \lambda_1 e + \lambda_2 a + \lambda_3 a^2,$$
$$(\lambda_1 e + \lambda_2 a + \lambda_3 a^2)a = \lambda_3 e + \lambda_1 a + \lambda_2 a^2,$$
$$(\lambda_1 e + \lambda_2 a + \lambda_3 a^2)a^2 = \lambda_2 e + \lambda_3 a + \lambda_1 a^2.$$

By taking matrices relative to the basis e, a, a^2 of FG, we obtain the regular representation of G:

$$e \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ a \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\ a^2 \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

FG acts on an *FG*-module

You will remember that an FG-module is a vector space over F, together with a multiplication vg for $v \in V$ and $g \in G$ (and the multiplication satisfies various axioms). Now, it is sometimes helpful to extend the definition of the multiplication so that we have an element vr of V for all elements r in the group algebra FG. This is done in the following natural way.

6.8 Definition

Suppose that V is an FG-module, and that $v \in V$ and $r \in FG$; say $r = \sum_{g \in G} \mu_g g$ $(\mu_g \in F)$. Define vr by

$$vr = \sum_{g \in G} \mu_g (vg).$$

6.9 Examples

(1) Let V be the permutation module for S_4, as described in Example 4.9. If

$$r = \lambda(1\ 2) + \mu(1\ 3\ 4) \quad (\lambda, \mu \in F)$$

then

$$v_1 r = \lambda v_1(1\ 2) + \mu v_1(1\ 3\ 4) = \lambda v_2 + \mu v_3,$$

$$v_2 r = \lambda v_1 + \mu v_2,$$

$$(2v_1 + v_2)r = \lambda v_1 + (2\lambda + \mu)v_2 + 2\mu v_3.$$

(2) If V is the regular FG-module, then for all $v \in V$ and $r \in FG$, the element vr is simply the product of v and r as elements of the group algebra, given by Definition 6.3.

Compare the next result with Proposition 6.4.

6.10 Proposition
Suppose that V is an FG-module. Then the following properties hold for all $u, v \in V$, all $\lambda \in F$ and all $r, s \in FG$:

(1) $vr \in V$;
(2) $v(rs) = (vr)s$;
(3) $v1 = v$;
(4) $(\lambda v)r = \lambda(vr) = v(\lambda r)$;
(5) $(u + v)r = ur + vr$;
(6) $v(r + s) = vr + vs$;
(7) $v0 = 0r = 0$.

Proof All parts except (2) are straightforward, and we leave them to you. We shall give a proof of part (2), assuming the other parts.

Let $v \in V$, and let $r, s \in FG$ with

$$r = \sum_{g \in G} \lambda_g g, \; s = \sum_{h \in G} \mu_h h.$$

Then

$$\begin{aligned} v(rs) &= v\left(\sum_{g,h} \lambda_g \mu_h (gh)\right) \\ &= \sum_{g,h} \lambda_g \mu_h (v(gh)) && \text{by (4) and (6)} \\ &= \sum_{g,h} \lambda_g \mu_h ((vg)h) \\ &= \left(\sum_g \lambda_g (vg)\right)\left(\sum_h \mu_h h\right) && \text{by (4), (5), (6)} \\ &= (vr)s. \end{aligned}$$

■

Summary of Chapter 6

1. The group algebra FG of G over F consists of all linear combinations of elements of G, and has a natural multiplication defined on it.
2. The vector space FG, with the natural multiplication vg ($v \in FG$, $g \in G$) is the regular FG-module.
3. The regular FG-module is faithful.

Exercises for Chapter 6

1. Suppose that $G = D_8 = \langle a, b\colon a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$.
   (a) Let x and y be the following elements of $\mathbb{C}G$:
   $$x = a + 2a^2,\ y = b + ab - a^2.$$
   Calculate xy, yx and x^2.
   (b) Let $z = b + a^2b$. Show that $zg = gz$ for all g in G. Deduce that $zr = rz$ for all r in $\mathbb{C}G$.
2. Work out matrices for the regular representation of $C_2 \times C_2$ over F.
3. Let $G = C_2$. For r and s in $\mathbb{C}G$, does $rs = 0$ imply that $r = 0$ or $s = 0$?
4. Assume that G is a finite group, say $G = \{g_1, \ldots, g_n\}$, and write c for the element $\sum_{i=1}^{n} g_i$ of $\mathbb{C}G$.
   (a) Prove that $ch = hc = c$ for all h in G.
   (b) Deduce that $c^2 = |G|c$.
   (c) Let $\vartheta\colon \mathbb{C}G \to \mathbb{C}G$ be the linear transformation sending v to vc for all v in $\mathbb{C}G$. What is the matrix $[\vartheta]_{\mathscr{B}}$, where $\mathscr{B}$ is the basis $g_1, \ldots, g_n$ *of* $\mathbb{C}G$?
5. If V is an FG-module, prove from the definition that
   $$0r = 0 \text{ for all } r \in FG, \text{ and}$$
   $$v0 = 0 \text{ for all } v \in V,$$
   where the symbol 0 is used for the zero elements of V and FG.

   Show that for every finite group G, with $|G| > 1$, there exists an FG-module V and elements $v \in V$, $r \in FG$ such that $vr = 0$, but neither v nor r is 0.

6. Suppose that $G = D_6 = \langle a, b\colon a^3 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$, and let $\omega = e^{2\pi i/3}$. Prove that the 2-dimensional subspace W of $\mathbb{C}G$, defined by

$$W = \mathrm{sp}\,(1 + \omega^2 a + \omega a^2,\ b + \omega^2 ab + \omega a^2 b),$$

is an irreducible $\mathbb{C}G$-submodule of the regular $\mathbb{C}G$-module.

7
FG-homomorphisms

For groups and vector spaces, the ‘structure-preserving’ functions are, respectively, group homomorphisms and linear transformations. The analogous functions for FG-modules are called FG-homomorphisms, and we introduce these in this chapter.

FG-homomorphisms

7.1 Definition
Let V and W be FG-modules. A function $\vartheta: V \to W$ is said to be an *FG-homomorphism* if ϑ is a linear transformation and

$$(vg)\vartheta = (v\vartheta)g \quad \text{for all } v \in V,\ g \in G.$$

In other words, if ϑ sends v to w then it sends vg to wg.

Note that if G is a finite group and $\vartheta: V \to W$ is an FG-homomorphism, then for all $v \in V$ and $r = \sum_{g\in G} \lambda_g g \in FG$, we have

$$(vr)\vartheta = (v\vartheta)r$$

since

$$(vr)\vartheta = \sum_{g\in G} \lambda_g (vg)\vartheta = \sum_{g\in G} \lambda_g (v\vartheta)g = (v\vartheta)r.$$

The next result shows that FG-homomorphisms give rise to FG-submodules in a natural way.

7.2 Proposition
Let V and W be FG-modules and let $\vartheta: V \to W$ be an FG-homomorphism. Then $\operatorname{Ker}\vartheta$ *is an FG-submodule of V, and* $\operatorname{Im}\vartheta$ *is an FG-submodule of W.*