CHAPTER III

Free Groups and Free Products of Groups

§1. Introduction

In the preceding chapters we have introduced the fundamental group of a space and actually determined its structure in some of the simplest cases. In more complicated cases we need a larger vocabulary and a greater knowledge of group theory to describe its structure and actually to make use of its properties. The object of this chapter is to supply this need. We first discuss the case of abelian groups because this case is simpler and more closely related to the student's previous experience. Then we discuss the general case of not necessarily abelian groups. Here the results are entirely analogous to the abelian case, but the possibilities are more varied and less intuitive.

The three main group theoretic concepts introduced in this chapter are the following: free group, free product of groups, and presentation of a group by generators and relations. These concepts will be used throughout the next two chapters. The definition of a free group or a free product of groups involves a mathematical concept of wide application, the so-called "universal mapping problem," which is also a basic concept in Chapter IV.

§2. The Weak Product of Abelian Groups

We assume the student is familiar with the concept of the *direct product* of a finite number of groups,

$$G = G_1 \times G_2 \times \cdots \times G_n.$$

The elements of G are ordered n-tuples

$$g = (g_1, g_2, \ldots, g_n),$$

W. S. Massey, *A Basic Course in Algebraic Topology*
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where $g_i \in G_i$ for $i = 1, 2, \ldots, n$, with multiplication defined componentwise:

$$(g_1, g_2, \ldots, g_n)(g'_1, g'_2, \ldots, g'_n) = (g_1g'_1, g_2g'_2, \ldots, g_ng'_n).$$

It is easy to extend this definition to the case of an infinite collection of groups $\{G_i : i \in I\}$. Here I is an index set, which may be countable or uncountable. The *direct product* of such a collection is denoted by

$$\prod_{i \in I} G_i.$$

Its elements are functions g which assign to each index $i \in I$ an element $g_i \in G_i$. These elements are multiplied componentwise: if g and h are elements of the direct product, then

$$(gh)_i = (g_i)(h_i)$$

for any $i \in I$.

Let $\{G_i : i \in I\}$ be any collection of groups, and let

$$G = \prod_{i \in I} G_i$$

be their product.

Definition. The *weak product*[1] of the collection $\{G_i : i \in I\}$ is the subgroup of their product G consisting of all elements $g \in G$ such that g_i is the identity element of G_i for all except a finite number of indices i.

Obviously, if $\{G_i : i \in I\}$ is a finite collection of groups, then the product and weak product are the same.

If G denotes either the product or weak product of the collection $\{G_i : i \in I\}$, then, for each index $i \in I$, there is a *natural monomorphism* $\varphi_i : G_i \to G$ defined by the following rule: For any element $x \in G_i$ and any index $j \in I$,

$$(\varphi_i x)_j = \begin{cases} x & \text{if } j = i \\ 1 & \text{if } j \neq i. \end{cases}$$

In the case where each G_i is an *abelian* group, the following theorem gives an important characterization of their weak product G and the monomorphisms φ_i.

Theorem 2.1. *If $\{G_i : i \in I\}$ is a collection of abelian groups and G is their weak product, then for any abelian group A and any collection of homomorphisms*

$$\psi_i : G_i \to A, \quad i \in I,$$

[1] When each group G_i is abelian and the group operation is addition, it is customary to call the weak product the "direct sum." In this definition, we do not require that any two groups in the collection $\{G_i\}$ be nonisomorphic. In fact, it may even occur that all of the groups of the collection are isomorphic to some given group.

there exists a unique homomorphism $f: G \to A$ such that for any $i \in I$ the following diagram is commutative:

$$\begin{tikzcd} & G \arrow[dd, "f"] \\ G_i \arrow[ur, "\varphi_i"] \arrow[dr, "\psi_i"'] & \\ & A \end{tikzcd}$$

PROOF. Given the ψ_i's, define f by the following rule: For any $x \in G$, $f(x)$ will be the product of the elements $\psi_i(x_i)$ for all $i \in I$. Because $x_i = 1$ for all except a finite number of indices i, this product is really a finite product; and because all the groups involved are abelian, the order of multiplication is immaterial. Thus, $f(x)$ is well defined, and it is readily verified that f is a homomorphism, which renders the given diagram commutative. It is easy to see that f is the unique homomorphism having this property. Q.E.D.

Our next proposition states that this theorem actually characterizes the weak product of abelian groups.

Proposition 2.2. *Let $\{G_i\}$, G, and $\varphi_i: G_i \to G$ be as in Theorem 2.1; let G' be any abelian group and let $\varphi_i': G_i \to G'$ be any collection of homomorphisms such that the conclusion of Theorem 2.1 holds with G' and φ_i' substituted for G and φ_i, respectively. Then, there exists a unique isomorphism $h: G \to G'$ such that the following diagram is commutative for any $i \in I$:*

$$\begin{tikzcd} & G \arrow[dd, "h"] \\ G_i \arrow[ur, "\varphi_i"] \arrow[dr, "\varphi_i'"'] & \\ & G' \end{tikzcd}$$

PROOF. The existence of a *homomorphism* $h: G \to G'$ making the required diagram commutative is assured by Theorem 2.1. Because Theorem 2.1 also applies to G' and the φ_i' (by hypothesis), there exists a unique homomorphism $k: G' \to G$ such that the following diagram is commutative for any index $i \in I$:

$$\begin{tikzcd} & G' \arrow[dd, "k"] \\ G_i \arrow[ur, "\varphi_i'"] \arrow[dr, "\varphi_i"'] & \\ & G \end{tikzcd}$$

From these facts, we readily conclude that the following two diagrams are commutative for any $i \in I$:

$$
\begin{tikzcd}
 & G \arrow[dd, "kh"] \\
G_i \arrow[ur, "\varphi_i"] \arrow[dr, "\varphi_i"'] & \\
 & G
\end{tikzcd}
\qquad
\begin{tikzcd}
 & G' \arrow[dd, "hk"] \\
G_i \arrow[ur, "\varphi_i'"] \arrow[dr, "\varphi_i'"'] & \\
 & G'
\end{tikzcd}
$$

However, these two diagrams would also be commutative if we replaced kh by the identity map $G \to G$ in the first, and hk by the identity map $G' \to G'$ in the second. We now invoke the uniqueness statement in the conclusion of Theorem 2.1 to conclude that kh and hk are both identity maps. Hence, h and k are inverse isomorphisms of each other. Q.E.D.

The student should reflect on the significance of the characterization of the weak product given by Theorem 2.1. We may consider any other abelian group A with definite homomorphisms $\psi_i : G_i \to A$ as a candidate for some kind of a "product" of the abelian groups G_i; then this theorem asserts that the weak product G is the "freest" among all such candidates in the sense that there exists a homomorphism of G into A commuting with φ_i and ψ_i for all i. Here we use the word "freest" in the sense of "fewest possible relations imposed," and the general philosophy is that if certain relations hold for the group G, they also hold for any homomorphic image of G; of course, additional relations may hold for the homomorphic image. This same philosophy also holds for other kinds of algebraic objects, such as rings, etc.

As we shall see, the argument used to prove Proposition 2.2 applies almost verbatim to many other cases.

Since the weak product G of a collection $\{G_i\}$ of abelian groups is completely characterized by the properties of the monomorphisms $\varphi_i : G_i \to G$ stated in Theorem 2.1, we could just as well ignore the fact that G is a subgroup of the product

$$\prod_{i \in I} G_i$$

and focus our attention instead on the group G and the homomorphisms φ_i. Furthermore, because each φ_i is a monomorphism, we can identify G_i with its image in G under φ_i, and consider φ_i as an inclusion map, if this is convenient. In this case, we say that G is the weak product of the subgroups G_i, it being understood that each φ_i is an inclusion map.

§3. Free Abelian Groups

We recall that, if S is a subset of a group G, then S is said to *generate* G in case every element of G can be written as a product of positive and negative powers of elements of S. (An equivalent condition is the following: S is not contained in any proper subgroup of G.) For example, if G is a cyclic group of order n,

$$G = \{x, x^2, x^3, \ldots, x^n = 1\},$$

then the set $S = \{x\}$ generates G.

If the set S generates the group G, certain products of elements of S may be the identity element of G. For example,

(a) If $x \in S$, then $xx^{-1} = 1$.
(b) If G is a cyclic group of order n generated by $\{x\}$, then $x^n = 1$.

Any such product of elements of S that is equal to the identity is often called a *relation* between the elements of the generating set S. Roughly speaking, we may distinguish between two types of relations between generators: *trivial relations*, as in Example (a), which are a direct consequence of the axioms for a group and thus hold no matter what the choice of G and S, and *nontrivial relations*, such as Example (b), which are not a consequence of the axioms for a group, but depend on the particular choice of G and S.

These notions lead naturally to the following definition: Let S be a set of generators for the group G. We say that G is *freely generated* by S or a *free group* on S in case there are no nontrivial relations between the elements of S. For example, if G is an infinite cyclic group consisting of all positive and negative powers of the element x, then G is a free group on the set $S = \{x\}$.

These notions also lead to the idea that we can completely prescribe a group by listing the elements of a generating set S and listing the nontrivial relations between them.

The ideas described in the preceding paragraphs have been current among group theorists for a long time. Unfortunately, when stated as above, these ideas are lacking in mathematical precision. For example, what precisely is a nontrivial relation? It cannot be an element of G, because considered as elements of G, all relations give the identity. Also, under what conditions are two relations to be considered the same? For example, in a cyclic group of order n, are the relations

$$x^n = 1,$$

$$x^{n+1}x^{-1} = 1$$

to be considered the same or different?

We should emphasize that it was not an easy matter for mathematicians to find an entirely satisfactory and precise way of treating these questions. Fortunately, such a treatment has been found in recent years. This treatment has the advantage that it applies not only to groups, but also to other algebraic structures such as rings, and even to many situations in other branches of mathematics. As so often happens in mathematics, the method of definition finally chosen seems rather roundabout and nonobvious.[2] This method of definition depends on the following rather simple observations:

[2] An analogous situation occurs in the problem of precisely defining limits in the calculus. The $\varepsilon - \delta$ technique which is standard today seems rather far removed from our intuitive notion of a variable quantity approaching a limit.

(1) Let S be a set of generators for G, and let $f: G \to G'$ be an epimorphism; i.e., G' is a homomorphic image of G. Then, the set $f(S)$ is a set of generators for G'. Moreover, *any relation which holds between the elements of S also holds between the elements of $f(S)$*. Thus, the group G' satisfies at least as many relations as or more relations than G.

(2) Let S be a set of generators for G, and let $f: G \to G'$ be an arbitrary homomorphism. Then, f is completely determined by its restriction to the set S. However, we do not assert that any map $g: S \to G'$ can be extended to a homomorphism $f: G \to G'$ (the student should give a counterexample). The intuitive reason for this is clear: Given a map $g: S \to G'$ there may be nontrivial relations between the elements of S which do *not* hold between the elements of $g(S)$.

We shall now give a precise definition of a free *abelian* group on a given set S; in §5 we shall discuss the case of general (i.e., not necessarily abelian) groups. The case of abelian groups is discussed first because it is simpler.

Definition. Let S be an arbitrary set. A *free abelian group* on the set S is an abelian group F together with a function $\varphi: S \to F$ such that the following condition holds: For any abelian group A and any function $\psi: S \to A$, there exists a unique homomorphism $f: F \to A$ such that the following diagram is commutative:

$$\begin{tikzcd} & F \arrow[dd, "f"] \\ S \arrow[ur, "\varphi"] \arrow[dr, "\psi"'] & \\ & A \end{tikzcd}$$

First, we show that this definition does indeed characterize free abelian groups on a given set S.

Proposition 3.1. *Let F and F' be free abelian groups on the set S with respect to the functions $\varphi: S \to F$ and $\varphi': S \to F'$, respectively. Then, there exists a unique isomorphism $h: F \to F'$ such that the following diagram is commutative:*

$$\begin{tikzcd} & F \arrow[dd, "h"] \\ S \arrow[ur, "\varphi"] \arrow[dr, "\varphi'"'] & \\ & F' \end{tikzcd}$$

PROOF. The proof is completely analogous to that of Proposition 2.2, and may be left to the reader.

Let us emphasize that all we have done so far is make a definition; given the set S, it is not at all clear that there exists a free abelian group F on the set S. Moreover, even if F exists, it is conceivable that the map φ need not be one-to-one, or that F may not be generated by the subset $\varphi(S)$ in the sense of the definition at the beginning of this section. We shall clarify all these points by actually proving the existence of F and elucidating its structure completely.

Exercises

3.1. Prove directly from the definition that $\varphi(S)$ generates F. [HINT: Assume not; consider the subgroup F' generated by $\varphi(S)$.]

As a first step, we consider the following situation. Assume that $\{S_i : i \in I\}$ is a family of nonempty subsets of S, which are pairwise disjoint and such that

$$S = \bigcup_{i \in I} S_i.$$

For each index $i \in I$, let F_i be a free abelian group on the set S_i with respect to a function $\varphi_i : S_i \to F_i$. Let F denote the weak product of the groups F_i for all $i \in I$, and let $\eta_i : F_i \to F$ denote the natural monomorphism. Since the S_i are pairwise disjoint, we can define a function $\varphi : S \to F$ by the rule

$$\varphi | S_i = \eta_i \varphi_i.$$

Proposition 3.2. *Under the above hypotheses, F is a free abelian group on the set S with respect to the function $\varphi : S \to F$.*

Roughly speaking, this proposition means that the weak product of any collection of free abelian groups is a free abelian group.

Proof. Let A be an abelian group and let $\psi : S \to A$ be a function. We have to prove the existence of a unique homomorphism $f : F \to A$ such that $\psi = f\varphi$. For each index i, let $\psi_i : S_i \to A$ denote the restriction of ψ to the subset S_i. Because F_i is a free abelian group on the set S_i, there exists a unique homomorphism $f_i : F_i \to A$ such that the following diagram is commutative:

$$\begin{tikzcd} & F_i \arrow[d, "f_i"] \\ S_i \arrow[ur, "\psi_i"] \arrow[r, "\psi_i"'] & A \end{tikzcd} \tag{3.3.1}$$

We now invoke the fundamental property of the weak product of groups contained in Theorem 2.1 to conclude that there exists a unique homomorphism $f : F \to A$ such that the following diagram is commutative for any index i:

$$\begin{tikzcd} & F \arrow[d, "f"] \\ F_i \arrow[ur, "\eta_i"] \arrow[r, "f_i"'] & A \end{tikzcd} \tag{3.3.2}$$

We can put these two commutative diagrams together into a single diagram as follows:

$$\begin{tikzcd} S_i \arrow[r, "\varphi_i"] \arrow[dr, "\psi_i"'] & F_i \arrow[r, "\eta_i"] \arrow[d, "f_i"] & F \arrow[dl, "f"] \\ & A & \end{tikzcd} \tag{3.3.3}$$

Because $\varphi | S_i = \eta_i \varphi_i$, we conclude that the following diagram is commutative for each index i.

$$\begin{tikzcd} S_i \arrow[rr, "\varphi_i | S_i"] \arrow[dr, "\psi_i"'] & & F \arrow[dl, "f"] \\ & A & \end{tikzcd} \tag{3.3.4}$$

Finally, because $\psi_i = \psi | S_i$ for each i and $S = \bigcup S_i$, we conclude that $\psi = f\varphi$, as required.

To prove uniqueness, let f be any homomorphism $F \to A$ having the required property. Define $f_i : F_i \to A$ by $f_i = f\eta_i$. With this definition, it follows that diagram (3.3.1) is commutative for each index i; for,

$$\begin{aligned} f_i \varphi_i = f\eta_i \varphi_i = f(\varphi | S_i) &= (\psi | S_i) \\ &= \psi_i. \end{aligned}$$

Because F_i is the free abelian group on S_i (with respect to φ_i), it follows that each f_i is unique. Then because (3.3.2) is commutative for each i, and F is the weak product of the F_i, it follows that f is unique. Q.E.D.

We now apply this theorem as follows: Suppose that

$$S = \{x_i : i \in I\}.$$

For each index i, let S_i denote the subset $\{x_i\}$ having only one element, and let F_i be an infinite cyclic group consisting of all positive and negative powers of the element x_i:

$$F_i = \{x_i^n : n \in \mathbf{Z}\}.$$

Let $\varphi_i : S_i \to F_i$ denote the inclusion map, i.e., $\varphi_i(x_i) = x_i^1$. It is clear that F_i is a free abelian group on the set S_i. Therefore, all the hypotheses of Proposition 3.2 are satisfied. Thus, we conclude that a free abelian group on any set S is a weak product of a collection of infinite cyclic groups, with the cardinal number of the collection equal to that of S.

Because F is the weak product of the F_i, any element $g \in F$ is of the following form: For any index i, the ith component $g_i = x_i^{n_i}$ where each $n_i \in \mathbf{Z}$ and $n_i = 0$ for all but a finite number of indices i. Moreover, the function φ is defined by the following rule: For any index $j \in I$,

$$(\varphi x_i)_j = \begin{cases} x_i^1 & \text{if } i = j \\ x_j^0 & \text{if } i \neq j. \end{cases}$$

From this formula, it is clear that φ is a one-to-one map.

As φ is a one-to-one map, if we wish, we can identify each $x_i \in S$ with its image $\varphi(x_i) \in F$. Then S becomes a subset of F, and it is clear that we can express each element $g \neq 1$ of F uniquely in the following form:

$$g = x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_k}^{n_k}, \tag{3.3.5}$$

where the indices $i_1, i_2, \ldots, i_k$ are all distinct, and $n_1, n_2, \ldots, n_k$ are nonzero integers. This expression for the element g is unique except for the order of the factors. Moreover, each such product of the x_i's represents a unique element $g \neq 1$ of F. From this it is clear that F is generated by the subset $S = \varphi(S)$.

This identification of S and $\varphi(S)$ is quite customary in the discussion of free abelian groups. When this is done, $\varphi : S \to F$ becomes an inclusion map, and often it is not even mentioned in the discussion.

An alternative approach to the topic of free abelian groups would be to *define* an abelian group F to be free on the subset $\{x_i : i \in I\} \subset F$ if every element $g \neq 1$ of F admits an expression of the form (3.3.5), which is unique up to order of the factors. Actually, this procedure would be somewhat quicker and easier than the one we have chosen. However, it would suffer from the disadvantage that it could not be generalized to non-abelian groups and other situations which will actually be our main concern.

One reason for the importance of free abelian groups is the following proposition.

Proposition 3.3. *Any abelian group is the homomorphic image of a free abelian group; i.e., given any abelian group A, there exists a free abelian group F and an epimorphism $f : F \to A$.*

PROOF. The proof is very simple. Let $S \subset A$ be a set of generators for A (e.g., we could take $S = A$), and let F be a free group on the set S with respect to a function $\varphi : S \to F$. Let $\psi : S \to A$ denote the inclusion map. By definition, there exists a homomorphism $f : F \to A$ such that $f\varphi = \psi$. It is clear that f must be an epimorphism, since S was chosen to be a set of generators for A.

Q.E.D.

This proposition enables us to attach a precise meaning to the notion "nontrivial relation between the generators S," mentioned earlier. Let A, S, F, and f have the meaning just described; then we define any element $r \neq 1$ of kernel f to be a nontrivial relation between the set of generators S. If $\{r_i : i \in I\}$ is any collection of such relations, and r is an element of the subgroup of F generated by the r_i's, then the relation r is said to be a *consequence* of the relations r_i. This implies that r can be expressed as a product of the r_i's and their inverses. If the collection $\{r_i : i \in I\}$ generates the kernel of f, then the group A is completely determined up to isomorphism by the set of generators S and the set of relations $\{r_i : i \in I\}$; A is isomorphic to the quotient group of F modulo the subgroup generated by the r_i's.

It is clear that, if S and S' are sets having the same cardinal number, and F and F' are free abelian groups on S and S', respectively, then F and F' are isomorphic. We shall now show that the converse of this statement is true, at least for the case of finite sets. For this purpose, we make the following definition. If G is any group, and n is any positive integer, then G^n denotes the subgroup of G generated by the set

$$\{g^n : g \in G\}.$$

If the group G is abelian, then the set $\{g^n : g \in G\}$ is actually already a subgroup.

Lemma 3.4. *Let F be a free abelian group on a set consisting of k elements. Then, the quotient group F/F^n is a finite group of order n^k.*

Proof. We leave the proof to the reader; it is not difficult if one makes use of the explicit structure of free abelian groups described above.

Corollary 3.5. *Let S and S' be finite sets whose cardinals are not equal, and let F and F' be free abelian groups on S and S', respectively. Then, F and F' are nonisomorphic.*

Proof. The proof is by contradiction. Any isomorphism between F and F' would induce an isomorphism between the quotient groups F/F^n and F'/F'^n, which is impossible by the lemma.

Exercises

3.2. Prove that the statement of this corollary is still true if S is a finite set and S' is an infinite set.

Let F be a free abelian group on a set S. The cardinal number of the set S is called the *rank* of F. We have proved that *two free abelian groups are isomorphic if and only if they have the same rank*, at least in the case where one of them has finite rank.

We shall conclude this section on abelian groups with a brief discussion of the structure of finitely generated abelian groups. Let A be an abelian group; the set of all elements of A which have finite order is readily seen to be a subgroup, called the *torsion subgroup of A*. When the torsion subgroup consists of the element 1 alone, A is called a *torsion-free* abelian group. On the other hand, if every element of A has finite order, then A is called a torsion group. If we denote the torsion subgroup by T, then the quotient group A/T

is obviously torsion free. It is clear that, if A and A' are isomorphic, then so are their torsion subgroups, T and T', and their torsion-free quotient groups, A/T and A'/T'. However, the converse is not true in general; we cannot conclude that A is isomorphic to A' if $T \approx T'$ and $A/T \approx A'/T'$. However, for abelian groups which are generated by a finite subset we have the following theorem which describes their structure completely:

Theorem 3.6. (a) *Let A be a finitely generated abelian group and let T be its torsion subgroup. Then, T and A/T are also finitely generated, and A is isomorphic to the direct product $T \times A/T$. Hence, the structure of A is completely determined by its torsion subgroup T and its torsion-free quotient group A/T.* (b) *Every finitely generated torsion-free abelian group is a free abelian group of finite rank.* (c) *Every finitely generated torsion abelian group T is isomorphic to a product $C_1 \times C_2 \times \cdots \times C_n$, where each C_i is a finite cyclic group of order ε_i such that ε_i is a divisor of ε_{i+1} for $i = 1, 2, \ldots, n-1$. Moreover, the integers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are uniquely determined by the torsion group T and they completely determine its structure.*

The numbers $\varepsilon_1, \ldots, \varepsilon_n$ are called the *torsion coefficients* of T, or more generally, if T is the torsion subgroup of A, they are called the torsion coefficients of A. Similarly, the rank of the free group A/T is called the rank of A. With this terminology, we can summarize Theorem 3.6 by stating that the rank and torsion coefficients are a complete set of invariants of a finitely generated abelian group. Theorem 3.6 asserts that every finitely generated abelian group is a direct product of cyclic groups, but it also asserts much more. Note that a finitely generated torsion group is actually of finite order.

A word of explanation about the various isomorphisms mentioned in Theorem 3.6 seems in order here. These isomorphisms are not *natural*, or uniquely determined in any way. In each case, there are usually many different choices for the isomorphism in question and one choice is as good as another.

Theorem 3.7. *Let F be a free abelian group on a set S, and let F' be a subgroup of F. Then, F' is a free abelian group on a certain set S', and the cardinal of S' is less than or equal to that of S.*

Although the proofs of Theorems 3.6 and 3.7 are not difficult, we shall not give them here, because they properly belong in the study of linear algebra and modules over a principal ideal domain.

Exercises

3.3. Give an example of a torsion-free abelian group which is not free.

3.4. Let A be an abelian group which is a direct product of two cyclic groups of orders 12 and 18, respectively. What are the torsion coefficients of A? (Note that the torsion coefficients are required to satisfy a divisibility condition.)

3.5. Give an example to show that in Theorem 3.7 the subset $S \subset F$ and the subgroup $F' \subset F$ may be disjoint, even in the case where the cardinals of S and S' are equal.

§4. Free Products of Groups

The free product of a collection of groups is the exact analog for arbitrary (i.e., not necessarily abelian) groups of the weak product for abelian groups. (It should be emphasized that any groups considered in this section may be either abelian or non-abelian, unless the contrary is explicitly stated.)

Definition. Let $\{G_i : i \in I\}$ be a collection of groups, and assume there is given for each index i a homomorphism φ_i of G_i into a fixed group G. We say that G is the *free product* or *coproduct* of the groups G_i (with respect to the homomorphisms φ_i) if and only if the following condition holds: For any group H and any homomorphisms

$$\psi_i : G_i \to H, \quad i \in I,$$

there exists a unique homomorphism $f : G \to H$ such that for any $i \in I$, the following diagram is commutative:

$$\begin{tikzcd} & G \arrow[dd, "f"] \\ G_i \arrow[ur, "\varphi_i"] \arrow[dr, "\psi_i"'] & \\ & H \end{tikzcd}$$

First, we have the following uniqueness proposition about free products:

Proposition 4.1. *Assume that G and G' are free products of a collection $\{G_i : i \in I\}$ of groups (with respect to homomorphisms $\varphi_i : G_i \to G$ and $\varphi'_i : G_i \to G'$, respectively). Then, there exists a unique isomorphism $h : G \to G'$ such that the following diagram is commutative for any $i \in I$:*

$$\begin{tikzcd} & G \arrow[dd, "h"] \\ G_i \arrow[ur, "\varphi_i"] \arrow[dr, "\varphi'_i"'] & \\ & G' \end{tikzcd}$$

Proof. The proof is almost word for word that of Proposition 2.2.

Although we have defined free products of groups and proved their uniqueness, it still remains to prove that they always exist. We shall also show that each of the homomorphisms φ_i occurring in the definition is a monomorphism, that the free product is generated by the union of the images $\varphi_i(G_i)$, and get more detailed insight into the algebraic structure of a free product.

Theorem 4.2. *Given any collection $\{G_i : i \in I\}$ of groups, their free product exists.*

Proof. We define a *word* in the G_i's to be a finite sequence $(x_1, x_2, \ldots, x_n)$ where each x_k belongs to one of the groups G_i, *any two successive terms in the sequence belong to different groups, and no term is the identity element of any G_i*. The integer n is the *length* of the word. We also include the empty word, i.e., the unique word of length 0. Let W denote the set of all such words.

For each index i, we now define left operations of the group G_i on the set W (see Appendix B). Let $g \in G_i$ and $(x_1, \ldots, x_n) \in W$; we must define $g(x_1, \ldots, x_n)$.

Case 1: $x_1 \notin G_i$. Then, if $g \neq 1$,

$$g(x_1, \ldots, x_n) = (g, x_1, \ldots, x_n).$$

We shall also define the action of g on the empty word by a similar formula, i.e., $g(\ ) = (g)$. If $g = 1$, then,

$$g(x_1, \ldots, x_n) = (x_1, \ldots, x_n).$$

Case 2: $x_1 \in G_i$. Then,

$$g(x_1, \ldots, x_n) = \begin{cases} (gx_1, x_2, \ldots, x_n) & \text{if } gx_1 \neq 1 \\ (x_2, \ldots, x_n) & \text{if } gx_1 = 1. \end{cases}$$

[Where $gx_1 = 1$ and $n = 1$, it is understood, of course, that $g(x_1)$ is the empty word.]

We must now verify that the requirements for left operations of G_i on W are actually satisfied; i.e., for any word w,

$$1w = w,$$

$$(gg')w = g(g'w).$$

This verification is a trivial checking of various cases.

It is clear that each of the groups G_i acts effectively. Thus, each element g of G_i may be considered as a permutation of the set W, and G_i may be considered as a subgroup of the group of all permutations of W (see Appendix B). Let G denote the subgroup of the group of all permutations of W which is generated by the union of the G_i's. Then, G contains each G_i as a subgroup; we let

$$\varphi_i : G_i \to G$$

denote the inclusion map.

Any element of G may be expressed as a finite product of elements from the various G_i's. If two consecutive factors in this product come from the same G_i, it is clear that they may be replaced by a single factor. Thus, any element

$g \neq 1$ of G may be expressed as a finite product of elements from the G_i's in *reduced form*, i.e., so no two consecutive factors belong to the same group, and so no factor is the identity element. We now assert that *the expression of any element $g \neq 1$ of g in reduced form is unique*: If

$$g = g_1 g_2 \cdots g_m = h_1 h_2 \cdots h_n$$

with both products in reduced form, then $m = n$ and $g_i = h_i$ for $1 \leqq i \leqq m$. To see this, consider the effect of the permutations $g_1 g_2 \cdots g_m$ and $h_1 h_2 \cdots h_n$ on the empty word; the results are the words $(g_1, g_2, \ldots, g_m)$ and $(h_1, h_2, \ldots, h_n)$, respectively. Because these two words must be equal, the conclusion follows.

It is clear how to form the inverse of an element of G written in reduced form, and how to form the product of two such elements.

It is now an easy matter to verify that G is actually the free product of the G_i's with respect to the φ_i's. For, let H be any group and let $\psi_i : G_i \to H$, $i \in I$, be any collection of homomorphisms. Define a function $f : G \to H$ as follows. Express any given $g \neq 1$ in reduced form,

$$g = g_1 g_2 \cdots g_m, \quad g_k \in G_{i_k}, 1 \leqq k \leqq m,$$

and then set

$$f(g) = (\psi_{i_1} g_1)(\psi_{i_2} g_2) \cdots (\psi_{i_m} g_m).$$

We also set $f(1) = 1$, of course. It is clear that f is a homomorphism, and that f makes the required diagrams commutative. It is also clear that f is the only homomorphism that makes these diagrams commutative. Q.E.D.

Because the homomorphisms $\varphi_i : G_i \to G$ are monomorphisms, it is customary to identify each group G_i with its image under φ_i, and to regard it as a subgroup of the free product G. Then, φ_i becomes an inclusion map, and it is not usually necessary to mention it explicitly.

The two most important facts to remember from the proof of Theorem 4.2 are the following:

(a) Any element $g \neq 1$ of the free product can be expressed uniquely as a product in reduced form of elements from the groups G_i.
(b) The rules for multiplying two such products in reduced form (or for forming their inverses) are the obvious and natural ones.

These facts give one great insight into the structure of a free product of groups.

Examples

4.1. Let G_1 and G_2 be cyclic groups of order 2, $G_1 = \{1, x_1\}$ and $G_2 = \{1, x_2\}$. Then, any element $g \neq 1$ of their free product can be written uniquely as a product of x_1 and x_2, with the factors x_1 and x_2 alternating. For example, the following are such elements:

$$x_1, x_1 x_2, x_1 x_2 x_1, x_1 x_2 x_1 x_2, \text{etc.},$$

or

$$x_2, x_2x_1, x_2x_1x_2, x_2x_1x_2x_1, \text{etc.}$$

Note that the elements x_1x_2 and x_2x_1 are both of infinite order, and they are different. Note also the great difference between the direct product or weak product of G_1 and G_2 and their free product in this case. The direct product is an abelian group of order 4, whereas the free product is a non-abelian group with elements of infinite order.

Notation: We denote the free product of groups $G_1, G_2, \ldots, G_n$ by $G_1 * G_2 * \cdots * G_n$ or

$$\prod_{1 \leq i \leq n}^{*} G_i.$$

The free product of the family of groups $\{G_i : i \in I\}$ is denoted by

$$\prod_{i \in I}^{*} G_i.$$

Exercises

4.1. Let $\{G_i : i \in I\}$ be a collection containing more than one group, each of which has more than one element. Prove that their free product is non-abelian, contains elements of infinite order, and that its center consists of the identity element alone.

4.2. For each index i, let G'_i be a subgroup of G_i (proper or improper). Prove that the free product of the collection $\{G'_i : i \in I\}$ may be considered as a subgroup of the free product of the G_i.

4.3. Let $\{G_i : i \in I\}$ and $\{G'_i : i \in I\}$ be two families of groups indexed by the same set I. Assume that for each index $i \in I$ there is given a homomorphism $f_i : G_i \to G'_i$. Prove that there exists a unique homomorphism $f : G \to G'$ of the free product of the first family of groups into the free product of the second family such that the following diagram is commutative for each index i:

$$\begin{tikzcd} G_i \arrow[r, "\varphi_i"] \arrow[d, "f_i"'] & G \arrow[d, "f"] \\ G'_i \arrow[r, "\varphi'_i"'] & G' \end{tikzcd}$$

Show that if each f_i is a monomorphism (respectively, epimorphism), then f is a monomorphism (respectively, epimorphism).

4.4. Prove that if an element x of the free product $G * H$ has finite order, then x is an element of G or H, or is conjugate to an element of G or H. (HINT: Express x as a word in reduced form; then make the proof by induction on the length of the word.) Deduce that if G and H are cyclic groups of orders m and n, respectively, where $m > 1$ and $n > 1$, then the maximum order of any element of $G * H$ of finite order is $\max(m, n)$.

4.5. Let $\{G_i : i \in I\}$ be a collection of abelian groups, and let G be their free product with respect to homomorphisms $\varphi_i : G_i \to G$. Let $G' = G/[G, G]$ be the quotient of G by its commutator[3] subgroup and let $\varphi'_i : G_i \to G'$ be the composition of φ_i with the natural homomorphism $G \to G'$. Prove that G' is a weak product of the groups $\{G_i\}$ with respect to the homomorphisms φ'_i (i.e., the conclusion of Proposition 2.1 holds).

4.6. Let G, H, G', and H' be cyclic groups of orders m, n, m', and n', respectively. If $G * H$ is isomorphic to $G' * H'$, then $m = m'$ and $n = n'$ or else $m = n'$ and $n = m'$. (HINT: Apply Exercise 4.5 to $G * H$ and $G' * H'$; thus we see that, if we "abelianize" $G * H$ and $G' * H'$, we obtain finite abelian groups of orders mn and $m'n'$, respectively. Now apply Exercise 4.4.)

4.7. Let H and H' be conjugate subgroups of G. Prove that if f is any homomorphism of G into some other group such that $f(H) = 1$, then $f(H') = 1$ also.

4.8. Let G be the free product of the family of groups $\{G_i : i \in I\}$, where it is assumed that $G_i \neq \{1\}$ for any index i. Prove that, for any two distinct indices i and $i' \in I$, the subgroups G_i and $G_{i'}$ of G are not conjugate. (HINT: Apply Exercise 4.7. Use Exercise 4.3 to construct a homomorphism f of G into another free product with the required properties.)

4.9. Let $G = G_1 * G_2$, and let N be the least normal subgroup of G which contains G_1. Prove that G/N is isomorphic to G_2. (HINT: Use Exercise 4.3. Let $G'_1 = \{1\}$, $G'_2 = G_2$, $f_1 : G_1 \to G'_1$ be the trivial homomorphism, and let $f_2 : G_2 \to G'_2$ be the identity map. Prove that N is the kernel of the induced homomorphism $f : G \to G'$.)

4.10. Let G admit two different decompositions as a free product:

$$G = G_0 * \left(\prod_{i \in I}^* G_i\right) = H_0 * \left(\prod_{i \in I}^* H_i\right)$$

with the same index set I. Assume that, for each index $i \in I$, G_i and H_i are conjugate subgroups of G. Prove that G_0 and H_0 are isomorphic. (HINT: The method of proof is similar to that of Exercise 4.9.)

§5. Free Groups

As the reader may have guessed, the definition of a free group is entirely analogous to that of a free abelian group.

Definition. Let S be an arbitrary set. A *free group on the set* S (or a *free group generated by* S) is a group F together with a function $\varphi : S \to F$ such that the following condition holds: For any group H and any function $\psi : S \to H$, there exists a unique homomorphism $f : F \to H$ such that the following diagram is

[3] This terminology and notation is explained in the following section just before the statement of Proposition 5.3.

commutative:

$$\begin{tikzcd} & F \arrow[d, "f"] \\ S \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & H \end{tikzcd}$$

Exactly as in the previous cases we have encountered, this definition completely characterizes a free group. To be precise:

Proposition 5.1. *Let F and F' be free groups on the set S with respect to functions $\varphi : S \to F$ and $\varphi' : S \to F'$, respectively. Then, there exists a unique isomorphism $h : F \to F'$ such that the following diagram is commutative:*

$$\begin{tikzcd} & F \arrow[d, "h"] \\ S \arrow[ur, "\varphi"] \arrow[r, "\varphi'"'] & F' \end{tikzcd}$$

It still remains to prove that, given any set S, there exists a free group on the set S, and to establish its principal properties. We shall do this by exactly the same method as that used for the case of free abelian groups.

Assume, then, that

$$S = \bigcup_{i \in I} S_i,$$

where the subsets S_i are disjoint and nonempty. For each index i, let F_i be a free group on the set S_i with respect to a function $\varphi_i : S_i \to F_i$. Let F denote the free product of the groups F_i with respect to homomorphisms $\eta_i : F_i \to F$ (recall that we have proved that each η_i is actually a monomorphism!). Because the subsets S_i are pairwise disjoint, we can define a function $\varphi : S \to F$ by the rule

$$\varphi | S_i = \eta_i \varphi_i.$$

Proposition 5.2. *Under the above hypotheses, F is the free group on the set S with respect to the function $\varphi : S \to F$.*

The proof of this proposition is the same as that of Proposition 3.2 except for obvious modifications. Hence, it is not necessary to go through these details again. This proposition may be restated as follows: The free product of any collection of free groups is a free group.

We shall now apply this proposition to prove the existence of free groups exactly as we applied Proposition 3.2 to prove the existence of free abelian groups. The details are as follows: Let $S = \{x_i : i \in I\}$ be an arbitrary nonempty set, and, for each index i, let $S_i = \{x_i\}$. Let F_i denote an infinite cyclic group generated by x_i,

$$F_i = \{x_i^n : n \in \mathbf{Z}\},$$

and let $\varphi : S_i \to F_i$ denote the inclusion map. Then, F_i is readily seen to be a free group on the set S_i with respect to the map φ_i (as we shall see later, this case, where S has only one element, is the only one where the free group on a set S and the free abelian group on S are the same). The hypotheses of Proposition 5.2 are all satisfied; we conclude that F is a free group on the set S with respect to the functon $\varphi : S \to F$. Note that F is a free product of infinite cyclic groups. From what we have learned about free products, we see that *every element $g \neq 1$ of the free group F can be expressed uniquely in the form*

$$g = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

where $x_1, x_2, \ldots, x_k$ are elements of S such that any two successive elements are different, and $n_1, n_2, \ldots, n_k$ are nonzero integers, positive or negative. Such an expression for g is called a *reduced word* in the elements of S. To avoid exceptions, we say that the identity 1 is represented by the *empty word.* The rules for forming inverses and products of reduced words are the obvious ones.

From these facts, it is clear that the function $\varphi : S \to F$ is one-to-one, and that F is actually generated by the subset $\varphi(S)$ in the sense defined earlier.

In many cases it is convenient to take S to be a subset of F and φ to be the inclusion map. If this is the case, we may as well omit any mention of φ.

EXERCISES

5.1. Prove that a free group on a nonempty set S is abelian if and only if S has exactly one element.

5.2. Prove that the center of a free group on a set having more than one element consists of the identity element alone.

5.3. Let g and h be two elements of a free group on a set S having more than one element. Give a necessary and sufficient condition for g and h to be conjugate in terms of their expressions as reduced words. (HINT: Consider cyclic permutations of the factors of a reduced word.)

We shall conclude this section by considering the relation between free groups and free abelian groups. Recall that, if x and y are any two elements of a group G, the notation $[x, y]$ denotes the element $xyx^{-1}y^{-1} \in G$, and it is called the *commutator* of x and y (in the given order). The notation $[G, G]$ denotes the subgroup of G generated by all commutators; it is called the *commutator* subgroup and is readily verified to be a normal subgroup. The quotient group $G/[G, G]$ is abelian. Conversely, if N is any normal subgroup of G such that G/N is abelian, then $N \supset [G, G]$.

Proposition 5.3. *Let F be a free group on the set S with respect to a function $\varphi : S \to F$, and let $\pi : F \to F/[F, F]$ denote the natural projection of F onto the quotient group. Then, $F/[F, F]$ is a free abelian group on S with respect to the function $\pi\varphi : S \to F/[F, F]$.*

The proof is a nice exercise in the use of the definitions and the facts stated in the preceding paragraph.

Corollary 5.4. *If F and F' are free groups on finite sets S and S', then F and F' are isomorphic if and only if S and S' have the same cardinal number.*

PROOF. Any isomorphism of F onto F' would induce an isomorphism of the quotient groups, $F/[F, F]$ and $F'/[F', F']$. We now reach a contradiction by using the preceding proposition and Corollary 3.5. This proves the "only if" part of the corollary. The proof of the "if" part is trivial.

EXERCISES

5.4. Prove that this corollary is still true if S is a finite set and S is an arbitrary set.

If F is a free group on a set S, the cardinal number of S is called the *rank* of F. Corollary 5.4 shows that the rank is an invariant of the group at least in the case of free groups of finite rank. It can also be proved that the rank of a free group is an invariant even in the case where it is an infinite cardinal. The proof is more of an exercise in the arithmetic of cardinal numbers than in group theory, and we shall not give it here.

If F is a free group on the set S with respect to the function $\varphi : S \to F$, because φ is one-to-one it is usually convenient to consider S as a subset of F and φ as an inclusion map, as we mentioned above. With this convention, S is called a *basis* for F. In other words, a basis for F is any subset S of F such that F is a free group on S with respect to the inclusion map $S \to F$. A free group has many different bases.

§6. The Presentation of Groups by Generators and Relations

We begin with a result that is the analog for arbitrary groups of Proposition 3.3.

Proposition 6.1. *Any group is the homomorphic image of a free group. To be precise, if S is any set of generators for the group G, and F is a free group on S, then the inclusion map $S \to G$ determines a unique epimorphism of F onto G.*

The proof is the same as that of Proposition 3.3. This proposition enables us to give a mathematically precise meaning to the term "nontrivial relation between generators" by a method analogous to that used in the case of abelian groups. There is one slight difference between the abelian case and the present case because, in the case of abelian groups, any subgroup can be the kernel of a homomorphism, whereas in the case of non-abelian groups, only a *normal* subgroup can be a kernel. For this reason we shall give a complete discussion of this case.

Let S be a set of generators for the group G, let F be a free group on the set S with respect to a map $\varphi : S \to F$, let $\psi : S \to G$ be the inclusion map, and let $f : F \to G$ be the unique homomorphism such that $f\varphi = \psi$. Any element $r \neq 1$ of the kernel of f is (by definition) a *relation* between the generators of S for the group G. In view of what we have proved, r can be expressed uniquely as a reduced word in the elements of S. Because every element of S is also an element of G, this reduced word can also be considered as a product in G; however, in G, this product reduces to the identity element. Thus, by this device of introducing the free group F on the set S, we have given the relation r a "place to live," to use a figure of speech. If $\{r_j\}$ is any collection of relations, then any other relation r is said to be a *consequence* of the relations r_j if and only if r is contained in the least *normal* subgroup of F which contains the relation r_j. In the case where every relation is a consequence of the set of relations $\{r_j\}$, the kernel of f is completely determined by the set $\{r_j\}$; it is the intersection of all *normal* subgroups of F which contain the set $\{r_j\}$. In this case, the group G is completely determined up to isomorphism by the set of generators S and the set of relations $\{r_j\}$, because it is isomorphic to the quotient of F modulo the least normal subgroup containing the set $\{r_j\}$. Such a set of relations is called a *complete* set of relations.

Definition. A *presentation* of a group G is a pair $(S, \{r_j\})$ consisting of a set of generators for G and a complete set of relations between these generators. The presentation is said to be *finite* in case both S and $\{r_j\}$ are finite sets, and the group G is said to be *finitely presented* in case it has at least one finite presentation.

Let us emphasize that any group admits many different presentations, which may look quite different. Conversely, given two presentations $(S, \{r_j\})$ and $(S', \{r'_k\})$, it is often nearly impossible to determine whether or not the two groups thus defined are isomorphic.

Examples

6.1. A cyclic group of order n admits a presentation with one generator x and one relation x^n.

6.2. We shall prove later that the fundamental group of the Klein bottle admits the following two different presentations (among others):

(a) Two generators a and b and one relation $baba^{-1}$.
(b) Two generators a and c and one relation a^2c^2.

The relationship between the two presentations in this case is fairly simple: $c = ba^{-1}$ or $b = ca$. To be precise, let $F(a, b)$ and $F(a, c)$ denote free groups on the sets $\{a, b\}$ and $\{a, c\}$, respectively. Define homomorphisms $f: F(a, b) \to F(a, c)$ and $g: F(a, c) \to F(a, b)$ by the following conditions:

$$f(a) = a, \qquad f(b) = ca,$$
$$g(a) = a, \qquad g(c) = ba^{-1}.$$

It follows directly from the definition of a free group that these equations define unique homomorphisms. We compute that

$$g[f(a)] = a, \qquad g[f(b)] = b,$$
$$f[g(a)] = a, \qquad f[g(c)] = c.$$

Therefore, gf is the identity map of $F(a, b)$, and fg is the identity map of $F(a, c)$. Hence, f and g are isomorphisms which are the inverse of eac other. Next, we check that

$$a^2c^2 = c^{-1}[f(baba^{-1})]c,$$
$$baba^{-1} = (ba^{-1})[g(a^2c^2)](ba^{-1})^{-1}.$$

Therefore, the normal subgroup of $F(a, b)$, generated by $baba^{-1}$, and the normal subgroup of $F(a, c)$, generated by a^2c^2, correspond under the isomorphisms f and g. Hence, f and g induce isomorphisms of the corresponding quotient groups.

Note that the essence of the above argument is contained in the following two simple calculations:

(a) If $b = ca$, then $baba^{-1} = ca^2c$ and $a^2c^2 = c^{-1}[baba^{-1}]c$.
(b) If $c = ba^{-1}$, then $a^2c^2 = a^2ba^{-1}ba^{-1}$ and $baba^{-1} = (ba^{-1})(a^2c^2)(ba^{-1})^{-1}$.

6.3. Consider the following two group presentations:

(a) Two generators a and b and one relation a^3b^{-2}.
(b) Two generators x and y and one relation $xyxy^{-1}x^{-1}y^{-1}$.

We assert that these are presentations of isomorphic groups. The relationship between the two different pairs of generators is given by the following system of equations:

$$a = xy, \qquad b = xyx,$$
$$x = a^{-1}b, \qquad y = b^{-1}a^2.$$

We leave it to the reader to work out the details. We shall see in Section IV.6 that this is a presentation of the fundamental group of the complement of a certain knotted circle in Euclidean 3-space.

In dealing with groups presented by means of generators and relations, it is often convenient to take a more informal approach. To illustrate what we mean, consider the first presentation in Example 6.3. The group G under consideration is the quotient of a free group F on two generators a and b by the least normal subgroup containing the element a^3b^{-2}. Let us denote the image of the generators a and b in the group G by the same symbols. Then, $a^3b^{-2} = 1$ in G, or $a^3 = b^2$. When computing with elements of G (which are products of powers of a and b) we can use the equation $a^3 = b^2$ in whatever way is convenient.

EXERCISES

6.1. Suppose we are given presentations of two groups G_1 and G_2 by means of generators and relations. Show how to obtain from this a presentation of the direct product $G_1 \times G_2$, the free product $G_1 * G_2$, and the commutator quotient group $G_1/[G_1, G_1]$.

§7. Universal Mapping Problems

In the preceding sections of this chapter we have defined and studied the following types of algebraic objects: weak products of abelian groups, free abelian groups, free products of groups, and free groups. In each of these cases, the algebraic object in question was actually a system consisting of two things with a mapping between them, e.g., $\varphi : S \to G$. This system consisting of two things and a mapping between them was characterized by a certain triangular diagram, e.g.,

$$\begin{tikzcd} & G \arrow[d, "f"] \\ S \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & H \end{tikzcd}$$

As the reader will recall, the object H and the map ψ in this diagram could be chosen in a fairly arbitrary manner, subject only to minor restrictions. It was then required that there exist a unique map f making the diagram commutative.

This method of characterizing the system $\varphi : S \to G$ is usually referred to by the statement that $\varphi : S \to G$ (or for brevity, G) is the solution of a "universal mapping problem." We shall see another important example of such a universal mapping problem in the next chapter. Defining or characterizing mathematical objects as the solution to a universal mapping problem has become very common in recent years. For example, one of the most prominent contemporary algebraists (C. Chevalley) has written a textbook on algebra [6] that has universal mapping problems as one of its main themes.

If a mathematical object is defined or characterized as being the solution to a universal mapping problem, it follows easily (by the method used to prove Proposition 2.2) that this object is unique up to an isomorphism. In fact, the isomorphism is even uniquely determined! However, the *existence* of an object satisfying a given universal mapping problem is another question. The reader will note that in the four cases discussed in this chapter, at least three different constructions were given to prove the existence of a solution. However, in each case, the existence proof carried with it a bonus, in that it gave great insight into the actual structure of the desired mathematical object.

There exists a rather general method for proving the existence of solutions of universal mapping problems (see [5], [7]). However, this general method gives absolutely no insight into the mathematical structure of the solution. It is a pure existence proof.

We now give two more examples of the characterization of mathematical objects as solutions of universal mapping problems. The examples are given for illustrative purposes only and will not be used in any of the succeeding chapters.

Examples

7.1. *Free commutative ring with a unit.* Let $\mathbf{Z}[x_1, x_2, \ldots, x_n]$ denote, as usual, the ring of all polynomials with integral coefficients in the "variables" or "indeterminates" $x_1, x_2, \ldots, x_n$. Each nonzero element of this ring can be expressed uniquely as a finite linear combination with integral coefficients of the monomials $x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$, where $k_1, k_2, \ldots, k_n$ are non-negative integrs. This ring may be considered to be the free commutative ring with unit generated by the set $S = \{x_1, \ldots, x_n\}$. We make this assertion precise, as follows: Let $\varphi : S \to \mathbf{Z}[x_1, \ldots, x_n]$ denote the inclusion map. Then, for any commutative ring R (with unit) and any function $\psi : S \to R$, there exists a unique ring homomorphism $f : \mathbf{Z}[x_1, \ldots, x_n] \to R$ [with $f(1) = 1$] such that the following diagram is commutative:

$$\begin{tikzcd} & \mathbf{Z}[x_1, \ldots, x_n] \arrow[d, "f"] \\ S \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & R \end{tikzcd}$$

7.2. *The Stone–Čech Compactification.* For any Tychonoff space X, there is defined a certain compact Hausdorff space $\beta(X)$ which contains X as an everywhere dense subset; it is called the Stone–Čech Compactification of X. Let $\varphi : X \to \beta(X)$ denote the inclusion map. Then, we have the following characterization: For any compact Hausdorff space Y and any continuous map $\psi : X \to Y$, there exists a unique continuous map $f : \beta(X) \to Y$ such that the following diagram is commutative:

$$\begin{tikzcd} & \beta(X) \arrow[d, "f"] \\ X \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & Y \end{tikzcd}$$

For a more complete discussion see J. L. Kelley, *General Topology*. Princeton, N.J.: Van Nostrand, 1955. pp. 152–153. (GTM 27, Springer)

For a precise, axiomatic treatment of universal mapping problems and further examples, see references [5, 7].

NOTES

Definition of free groups, free products, etc.

The concepts of free abelian group, free group, free product of groups, etc., are rather old. The main difference between a modern treatment of the subject and one of the older treatments is the method of defining these algebraic objects. Formerly, they were defined in terms of what are now considered some of their characteristic properties. For example, a free group on set S was defined to be the collection of all equivalence classes of "words" formed from the elements of S. From a strictly logical point of view, there can be no objection to this procedure. However, from a conceptual point of view, it has the disadvantage that the definition of each type of free object requires new insight and ingenuity, and may be a difficult problem. The idea of defining free objects as solutions to universal mapping problems, which gradually evolved during the time of World War II and immediately thereafter, seems to be one of the important unifying ideas in modern mathematics.

The elegant proof given in the text for the existence of free products of groups (Theorem 4.2), which is simpler than the older proofs, is due to B.L. Van der Waerden (*Am. J. Math.* **70** (1948), 527–528). In a more recent paper (*Proc. Kon. Ned. Akad. Weten.* (series A) **69** (1966), 78–83), Van der Waerden has pointed out how the basic idea of the procedure used for the proof of Theorem 4.2 is applicable to prove the existence of solutions to universal mapping problems in many other algebraic situations.

Different levels of abstraction in mathematics

The first time the student encounters the material in this chapter, it may seem rather foreign to him. The probable reason is that it is on a higher level of abstraction than any of his previous studies in mathematics. To make this point clearer, we shall try to describe briefly the different levels of abstraction that seem to occur naturally in mathematics.

The lowest level of abstraction is the level of most high school and beginning undergraduate mathematics courses. This level is characterized by a concern with a few very explicit mathematical objects, e.g., the integers, rational numbers, real numbers, the complex numbers, the Euclidean plane, etc. The next level of abstraction occurs when certain properties common to several different concrete mathematical objects are isolated and studied for their own sake. This leads to the study of such abstract and general mathematical systems as groups, rings, fields, vector spaces, topological spaces, etc. Ordinarily the mathematics student makes the transition to this level of abstraction some time in this undergraduate career.

The material of this chapter provides an introduction to the next higher level of abstraction. As was pointed out in Example 4.1, the weak direct product of two abelian groups, G_1 and G_2, and their free product $G_1 * G_2$, are quite different types of groups. Yet there is a strong analogy between the weak direct product of abelian groups and the free product of arbitrary groups. To perceive this analogy, it is necessary to consider the category of all abelian groups and the category of all (i.e., not necessarily abelian) groups, respectively. This is characteristic of this next level of abstraction: the simultaneous consideration of all mathematical systems (e.g., groups, rings, or topological spaces) of a certain kind, and the study of the properties of such a collection of mathematical systems.

The history of mathematics in the last two hundred years or so has been characterized by the considerations of mathematical systems on ever higher levels of abstraction. Presumably this trend will continue in the future. It should be emphasized strongly, however, that this movement is not a case of abstraction for the sake of abstraction itself. Rather, it has been forced on mathematicians for various reasons, such as bringing out the analogies between seemingly quite different phenomena.

Presentations of groups by generators and relations

Let us emphasize that the specification of a group by means of generators and relations is very unsatisfactory in many respects, because some of the most natural problems that arise in connection with group presentations are very difficult or impossible. For a further discussion of this point, see the texts by Kurosh [1, Chap. X] or Rotman [4, Chap. 12].

That part of group theory which is concerned with groups presented by generators and relations is called "Combinatorial Group Theory." The standard introductory text on this subject is Magnus, Karrass, and Solitar [3]. A more advanced treatise is Lyndon and Schupp [2].

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