Chapter 2
Introduction to Nilpotent Groups

The aim of this chapter is to introduce the reader to the study of nilpotent groups. In Section 2.1, we define a nilpotent group, as well as the lower and upper central series of a group. Section 2.2 contains some classical examples of nilpotent groups. In particular, we prove that every finite p-group is nilpotent for a prime p. In Section 2.3, numerous properties of nilpotent groups are derived. For example, we prove that every subgroup of a nilpotent group is subnormal, and thus satisfies the so-called normalizer condition. Section 2.4 is devoted to the characterization of finite nilpotent groups. In Section 2.5, we use tensor products to show that certain properties of a nilpotent group are inherited from its abelianization. We focus on torsion nilpotent groups in Section 2.6. We prove that every finitely generated torsion nilpotent group must be finite, and that the set of torsion elements of a nilpotent group form a subgroup. Section 2.7 deals with the upper central series and its factors. Among other things, we illustrate how the center of a group influences the structure of the group.

2.1 The Lower and Upper Central Series

In this section, we define a nilpotent group and discuss the lower and upper central series of a group. First, we provide some standard terminology.

2.1.1 Series of Subgroups

Definition 2.1 Let G be a group. A *series* for G is a finite chain of subgroups

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G.$$

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If the subgroups $G_0, \ldots, G_n$ are distinct, then n is called the *length* of the series. The series is called *normal* if $G_i \trianglelefteq G$ for $0 \le i \le n$, and *subnormal* if $G_i \trianglelefteq G_{i+1}$ for $0 \le i \le n-1$. The *factors* of a subnormal series are the quotients G_{i+1}/G_i for $0 \le i \le n-1$.

Clearly, every normal series of a group is subnormal. On the other hand, not every subnormal series is normal. Consider, for example, the symmetric group S_4. Let $G_1 = \{e, (1\ 2)(3\ 4)\}$ and $G_2 = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. It can be shown that the series

$$\{e\} \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq A_4 \trianglelefteq S_4$$

is subnormal. It is not normal, however, because G_1 is not a normal subgroup of S_4. Notice, for instance, that

$$(1\ 2\ 3\ 4)^{-1}(1\ 2)(3\ 4)(1\ 2\ 3\ 4) \notin G_1.$$

Definition 2.2 Let $G_1, G_2, G_3, \ldots$ be a sequence of subgroups of a group G.

(i) If $G_i \le G_j$ for $1 \le i \le j$, then

$$G_1 \le G_2 \le G_3 \le \cdots \tag{2.1}$$

is an *ascending series* (or an *ascending chain of subgroups*).

(ii) If $G_i \ge G_j$ for $1 \le i \le j$, then

$$G_1 \ge G_2 \ge G_3 \ge \cdots \tag{2.2}$$

is a *descending series* (or a *descending chain of subgroups*).

An ascending series may not reach G. If it does, then we say that the series *terminates in* G. Similarly, a descending series which reaches the identity is said to *terminate in the identity*. If there exists an integer $m > 1$ such that $G_{m-1} \ne G_m$ and $G_m = G_{m+1} = G_{m+2} = \cdots$ in either (2.1) or (2.2), then the series is said to *stabilize* in G_m.

2.1.2 Definition of a Nilpotent Group

Definition 2.3 A group G is called *nilpotent* if it has a normal series

$$1 = G_0 \le G_1 \le \cdots \le G_n = G \tag{2.3}$$

such that

$$G_{i+1}/G_i \le Z(G/G_i)$$

for $i = 0, 1, \ldots, n-1$. Such a series (2.3) is called a *central series* for G. The shortest length of all central series for G is called the *nilpotency class*, or simply the *class*, of G.

An equivalent definition of a central series which involves commutators is given in the next lemma.

Lemma 2.1 *Let G be a group with a series*

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G. \tag{2.4}$$

The series (2.4) is central if and only if $[G_{i+1}, G] \leq G_i$ for $0 \leq i \leq n-1$.

Proof If the series (2.4) is central, then setting $H = G_{i+1}$ and $N = G_i$ in Lemma 1.9 yields the desired result.

Conversely, suppose that $[G_{i+1}, G] \leq G_i$ for $1 \leq i \leq n-1$. We claim that (2.4) is a normal series. Let $g \in G$ and $g_i \in G_i$ for some $i = 1, 2, \ldots, n$. By Lemma 1.4 (ii), we have

$$g_i^g = g_i[g_i, g] \in G_i G_{i-1} = G_i.$$

Thus $G_i \trianglelefteq G$. The rest follows from Lemma 1.9. □

The trivial group is regarded as a nilpotent group of class 0, and nontrivial abelian groups are nilpotent of class 1. To see why this is the case, suppose that G is a nontrivial abelian group. Since $Z(G) = G$, the series $1 < G$ is a central series for G of shortest length (simply take $G_1 = G$ in Definition 2.3). More examples of nilpotent groups are given in the next section.

The following lemma shows that the only nilpotent group with trivial center is the trivial group.

Lemma 2.2 *If G is a nontrivial nilpotent group, then $Z(G) \neq 1$.*

Proof Suppose that $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ is a central series for G. There exists an integer $i \geq 0$ such that $G_i = 1$ and $G_{i+1} \neq 1$. Thus, $G_{i+1}/G_i \leq Z(G/G_i)$ becomes $G_{i+1} \leq Z(G)$. And so, $Z(G) \neq 1$. □

Remark 2.1 An important collection of groups which arises in many areas of research (Galois theory, for example) are solvable groups. A group G is *solvable* if it has a subnormal series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

such that G_{i+1}/G_i is abelian for $0 \leq i \leq n-1$. Every nilpotent group is solvable since the series (2.3) is subnormal and each factor is abelian. On the other hand, not every solvable group is nilpotent. For example, S_3 is a solvable group because it has a series $1 \lhd A_3 \lhd S_3$ which is subnormal and has abelian factors. However, Lemma 2.2 shows that S_3 is not nilpotent because it has trivial center (refer to Example 1.2).

2.1.3 The Lower Central Series

One series which is fundamental in the study of nilpotent groups is the lower central series.

Definition 2.4 Let G be a group. The descending series

$$G = \gamma_1 G \geq \gamma_2 G \geq \cdots \tag{2.5}$$

recursively defined by $\gamma_{i+1}G = [\gamma_i G, G]$ for $i \in \mathbb{N}$ is called the *lower central series* of G. Its terms are called the *lower central subgroups* of G.

In particular, $\gamma_2 G = [G, G] = G'$ is the commutator (or derived) subgroup of G. By definition,

$$\gamma_i G = [\underbrace{G, \cdots, G}_{i}]$$

for $i \geq 2$. Thus, $\gamma_i G \trianglelefteq G$ for $i \geq 1$ by Corollary 1.3.

Remark 2.2 Let G be any group.

(i) If $\gamma_i G = 1$ for some $i \geq 1$, then $\gamma_{i+1}G = [\gamma_i G, G] = [1, G] = 1$. It follows by induction on j that $\gamma_j G = 1$ for all $j = i, i+1, \ldots$. In this case, the lower central series of G is a central series in the sense of Definition 2.3 (see Lemma 2.1).
(ii) If $\gamma_2 G = G$, then $\gamma_3 G = [\gamma_2 G, G] = [G, G] = \gamma_2 G = G$. Continuing this argument shows that $\gamma_j G = G$ for all $j \geq 1$.

In the next examples, we give the lower central subgroups of some groups.

Example 2.1 If G is an abelian group, then $[G, G] = 1$ (see Example 1.6). Thus, $\gamma_i G = 1$ for all $i \geq 2$ by Remark 2.2 (i).

Example 2.2 We find the lower central subgroups of A_n, the alternating group on $S = \{1, 2, \ldots, n\}$. By Example 2.1, $\gamma_i A_n = \{e\}$ for $n = 1, 2, 3$ and $i \geq 2$ since A_1, A_2, and A_3 are abelian.

It was shown in Example 1.7 that $[A_4, A_4] = K$. We claim that $\gamma_i A_4 = K$ for $i \geq 3$. It suffices to consider the case $i = 3$. We begin by noting that any nonidentity element of K can be written as

$$(a \;\; d)(b \;\; c) = [(a \;\; c \;\; b), (a \;\; b)(c \;\; d)],$$

where a, b, c, d are distinct elements of $S = \{1, 2, 3, 4\}$. Since A_4 is generated by 3-cycles, $K \leq [K, A_4]$. Consequently, $K = [K, A_4] = \gamma_3 A_4$ as claimed. We conclude that $\gamma_i A_4 = K$ for $i \geq 2$.

If $n \geq 5$, then each lower central subgroup of A_n equals A_n. This is a consequence of Example 1.7 and Remark 2.2 (ii).

This example illustrates that the lower central series (2.5) may not descend to the identity, and consequently, may not be a central series in the sense of Definition 2.3.

Example 2.3 As before, let S_n be the symmetric group on $S = \{1, 2, \ldots, n\}$. Clearly, $\gamma_i S_1 = \gamma_i S_2 = \{e\}$ for $i \geq 2$. We claim that $\gamma_i S_n = A_n$ for $i \geq 2$ and $n \geq 3$. Consider the case $n = 3$. By Example 1.8, $[S_3, S_3] = A_3$. It is easy to verify that

$$[(a \quad c \quad b), (a \quad b)] = (a \quad c \quad b) \text{ for distinct } a, b, c \in S.$$

It follows that $\gamma_3 S_3 = [\gamma_2 S_3, S_3] = [A_3, S_3] = A_3$. And so, $\gamma_i S_3 = A_3$ for $i \geq 2$ as claimed.

Next, consider the case $n = 4$. We found that $[S_4, S_4] = A_4$ in Example 1.8. The computation given in the same example also shows that $A_4 \leq [S_4, A_4]$. Thus,

$$[S_4, A_4] \leq [S_4, S_4] = A_4 \leq [S_4, A_4].$$

We conclude that $A_4 = [S_4, A_4]$. Hence, $\gamma_3 S_4 = A_4$, and thus $\gamma_i S_4 = A_4$ for $i \geq 2$.

Finally, suppose that $n \geq 5$. We know that $[S_n, S_n] = A_n$ whenever $n \geq 5$ from Example 1.8. Furthermore, $A_n = [A_n, A_n]$ whenever $n \geq 5$ from Example 2.2. Thus,

$$A_n = [A_n, A_n] \leq [S_n, A_n] \leq [S_n, S_n] = A_n.$$

This implies that $\gamma_3 S_n = [S_n, A_n] = A_n$, and in general, $\gamma_i S_n = A_n$ for $i \geq 2$.

Example 2.4 We give the lower central subgroups of the dihedral group D_n. See Examples 1.4 and 1.9 for notations and [4] for details.

- If $n \geq 3$ is odd, then $\gamma_2 D_n = gp\left(x^2\right) = gp(x)$ and

$$\gamma_3 D_n = [\gamma_2 D_n, D_n] = [gp(x), D_n] = gp(x).$$

  Thus, $\gamma_i D_n = gp(x)$ for $i \geq 2$.
- If $n = 2^k m$, where $m \geq 3$ is odd and $k \geq 1$, then

$$\gamma_2 D_{2^k m} = gp\left(x^2\right), \ \gamma_3 D_{2^k m} = gp\left(x^4\right), \ \ldots, \ \gamma_i D_{2^k m} = gp\left(x^{2^{i-1}}\right)$$

  for $2 \leq i \leq k+1$. Since x^{2^k} has odd order m, $\gamma_i D_{2^k m} = gp\left(x^{2^k}\right)$ when $i \geq k+1$.
- If $n = 2^k$ for some $k > 1$, then

$$\gamma_2 D_{2^k} = gp\left(x^2\right), \ \gamma_3 D_{2^k} = gp\left(x^4\right), \ \ldots, \ \gamma_i D_{2^k} = gp\left(x^{2^{i-1}}\right)$$

  for $2 \leq i \leq k+1$. In particular, $\gamma_{k+1} D_{2^k} = gp\left(x^{2^k}\right) = 1$, and thus $\gamma_i D_{2^k} = 1$ for $i \geq k+1$ by Remark 2.2. This shows that the lower central series of D_{2^k} is central in the sense of Definition 2.3. Therefore, D_{2^k} is nilpotent.

Example 2.5 Consider the Heisenberg group. It was shown in Example 1.10 that $[\mathscr{H}, \mathscr{H}] = Z(\mathscr{H})$ or, equivalently, $\gamma_2\mathscr{H} = Z(\mathscr{H})$. Clearly,

$$\gamma_3\mathscr{H} = [Z(\mathscr{H}), \mathscr{H}] = I_3.$$

By Remark 2.2 (i), $\gamma_i\mathscr{H} = I_3$ for $i \geq 3$. Hence, the lower central series of $\mathscr{H}$ is central in the sense of Definition 2.3. Therefore, $\mathscr{H}$ is nilpotent.

We give some useful properties enjoyed by the lower central subgroups.

Lemma 2.3 *The lower central subgroups of a group are fully invariant (hence, characteristic).*

Proof Apply Proposition 1.2 (i) repeatedly. □

Lemma 2.4 *If G is any group and $H \leq G$, then $\gamma_i H \leq \gamma_i G$ for each $i \in \mathbb{N}$.*

Proof The proof is done by induction on i. If $i = 1$, then the result is obvious. Assume that $\gamma_i H \leq \gamma_i G$ holds for $i > 1$. Then $\gamma_{i+1}H = [\gamma_i H, H] \leq [\gamma_i G, G] = \gamma_{i+1}G$. □

Lemma 2.5 *Let G and K be groups. If $\varphi : G \to K$ is a homomorphism, then $\varphi(\gamma_i G) = \gamma_i(\varphi(G))$ for each $i \in \mathbb{N}$. Thus, $\varphi(\gamma_i G) \leq \gamma_i K$ with equality when φ is surjective.*

Proof The proof is done by induction on i. If $i = 1$, then

$$\varphi(\gamma_1 G) = \varphi(G) = \gamma_1(\varphi(G)).$$

Assume that $\varphi(\gamma_i G) = \gamma_i(\varphi(G))$ holds for $i > 1$. By Proposition 1.2 (i), we obtain

$$\begin{aligned}\varphi(\gamma_{i+1}G) &= \varphi\left([\gamma_i G, G]\right) = [\varphi(\gamma_i G), \varphi(G)] \\ &= [\gamma_i(\varphi(G)), \varphi(G)] = \gamma_{i+1}(\varphi(G)).\end{aligned}$$

This completes the proof. □

Corollary 2.1 *If G is a group and $N \trianglelefteq G$, then $\gamma_i(G/N) = (\gamma_i G)N/N$ for each $i \in \mathbb{N}$.*

Proof If $\pi : G \to G/N$ is the natural homomorphism, then

$$(\gamma_i G)\,N/N = \pi(\gamma_i G) = \gamma_i(\pi(G)) = \gamma_i(G/N)$$

by Lemma 2.5. □

The lower central subgroups of a group can always be generated by a certain collection of simple commutators.

Lemma 2.6 *Let G be any group. For any $n \in \mathbb{N}$, we have*

$$\gamma_n G = gp\left([g_1, \ldots, g_n] \mid g_i \in G\right). \tag{2.6}$$

Furthermore, if X is a generating set of G, then $\gamma_n G$ is generated by all simple commutators of weight n or more in the elements of X and their inverses.

Proof Corollary 1.6 immediately gives (2.6). Suppose that $G = gp(X)$. Each element of G can be written as a product of the elements of X and their inverses. In particular, we may replace each g_i in a simple commutator $[g_1, \ldots, g_n] \in G$ of weight n by such a product. Since $\gamma_n G$ is generated by such simple commutators, the result follows from repeatedly applying Lemma 1.4. □

Example 2.6 Let G be a group generated by x, y, and z, and consider the simple commutator $\left[x^{-1}y^2,\ z\right]$ of weight 2. By Lemma 2.6, this commutator can be expressed as a product of simple commutators of weight 2 or more in the elements of the set $\left\{x,\ x^{-1},\ y,\ y^{-1},\ z,\ z^{-1}\right\}$. To see how this is done, we use Lemma 1.4 (v) to (vi) and get

$$\begin{aligned}\left[x^{-1}y^2,\ z\right] &= \left[x^{-1},\ z\right]\left[x^{-1},\ z,\ y^2\right]\left[y^2,\ z\right]\\ &= \left[x^{-1},\ z\right]\left[x^{-1},\ z,\ y\right]\left[x^{-1},\ z,\ y\right]\left[x^{-1},\ z,\ y,\ y\right]\left[y,\ z\right]\left[y,\ z,\ y\right]\left[y,\ z\right].\end{aligned}$$

2.1.4 The Upper Central Series

The upper central series plays a key role in the study of nilpotent groups. This series is constructed as follows:

Let G be any group. Set $\zeta_1 G = Z(G)$, and let $\pi_1 : G \to G/\zeta_1 G$ be the natural homomorphism of G onto $G/\zeta_1 G$. Define

$$\zeta_2 G = \pi_1^{-1}(Z(G/\zeta_1 G)),$$

so that $\zeta_2 G/\zeta_1 G = Z(G/\zeta_1 G)$. Observe that $\zeta_2 G \trianglelefteq G$ by the Correspondence Theorem.

Next, take $\pi_2 : G \to G/\zeta_2 G$ to be the natural homomorphism of G onto $G/\zeta_2 G$, and define

$$\zeta_3 G = \pi_2^{-1}(Z(G/\zeta_2 G)).$$

Thus, $\zeta_3 G/\zeta_2 G = Z(G/\zeta_2 G)$. As before, $\zeta_3 G \trianglelefteq G$. Continuing in this way, we obtain the subgroups of the upper central series of G.

Definition 2.5 Let G be any group. The ascending series

$$1 = \zeta_0 G \leq \zeta_1 G \leq \cdots \tag{2.7}$$

recursively defined by $\zeta_{i+1}G/\zeta_i G = Z(G/\zeta_i G)$ for $i \geq 0$ is called the *upper central series* of G, and its terms are called the *upper central subgroups* of G.

If $\pi_i : G \to G/\zeta_i G$ is the natural homomorphism of G onto $G/\zeta_i G$, then

$$\begin{aligned}\zeta_{i+1}G &= \pi_i^{-1}(Z(G/\zeta_i G))\\ &= \{g \in G \mid g\zeta_i G \text{ is central in } G/\zeta_i G\}\\ &= \{g \in G \mid (g\zeta_i G)(h\zeta_i G) = (h\zeta_i G)(g\zeta_i G) \text{ for all } h \in G\}\\ &= \{g \in G \mid [g,\ h] \in \zeta_i G \text{ for all } h \in G\}.\end{aligned}$$

In particular, $\zeta_1 G$ is the center of G. By taking $N = \zeta_i G$ and $H = \zeta_{i+1}G$ in Lemma 1.9, we find that $[\zeta_{i+1}G,\ G] \le \zeta_i G$.

Remark 2.3 Let G be any group.

(i) If $\zeta_i G = G$ for some $i \ge 0$, then

$$\begin{aligned}\zeta_{i+1}G &= \{g \in G \mid [g,\ h] \in \zeta_i G \text{ for all } h \in G\}\\ &= \{g \in G \mid [g,\ h] \in G \text{ for all } h \in G\}\\ &= G.\end{aligned}$$

It follows by induction on j that $\zeta_j G = G$ for $j \ge i$. In this situation, the upper central series of G is a central series in the sense of Definition 2.3.

(ii) If $Z(G) = 1$, then

$$\begin{aligned}\zeta_2 G &= \{g \in G \mid [g,\ h] \in Z(G) \text{ for all } h \in G\}\\ &= \{g \in G \mid [g,\ h] = 1 \text{ for all } h \in G\}\\ &= Z(G).\end{aligned}$$

Thus, $\zeta_2 G = 1$. Continuing in this way, we find that $\zeta_j G = 1$ for $j \ge 0$.

We provide the upper central subgroups of some groups.

Example 2.7 If G is an abelian group, then $\zeta_1 G = G$. Thus, $\zeta_i G = G$ for all $i \ge 1$ by Remark 2.3 (i).

Example 2.8 If $n \ge 3$, then the upper central subgroups of S_n are trivial from Example 1.2 and Remark 2.3 (ii). The same is true for the upper central subgroups of A_n when $n > 3$ (see Example 1.3). This illustrates that the upper central series of a group does not necessarily ascend to the group.

Example 2.9 We find the upper central subgroups of D_n. The last two cases rely on the fact that $D_{2n}/Z(D_{2n})$ is isomorphic to D_n. See [4] for details.

- For all $i \ge 1$, $\zeta_i D_1 = D_1$ and $\zeta_i D_2 = D_2$ since D_1 and D_2 are abelian. This follows from Remark 2.3 (i).
- If $n \ge 3$ is odd, then D_n has trivial center (see Example 1.4). By Remark 2.3 (ii), $\zeta_i D_n = 1$ for all $i \ge 0$.

- If $n = 2^k m$, where $m \geq 3$ is odd and $k \geq 1$, then $\zeta_i D_{2^k m} = gp\left(x^{n/2^i}\right)$ for $1 \leq i \leq k$. In particular,

$$\zeta_k D_{2^k m} = gp\left(x^{n/2^k}\right) = gp\left(x^m\right).$$

It follows that $\zeta_i D_{2^k m} = gp\left(x^m\right)$ for $i \geq k$.

- If $n = 2^k$ for some $k > 1$, then $\zeta_i D_{2^k} = gp\left(x^{n/2^i}\right)$ for $1 \leq i \leq k-1$. In particular,

$$\zeta_{k-1} D_{2^k} = gp\left(x^{n/2^{k-1}}\right) = gp\left(x^2\right).$$

For $i = k$, we get $\zeta_k D_{2^k} = D_{2^k}$, and consequently, $\zeta_i D_{2^k} = D_{2^k}$ for $i \geq k$. Thus, the upper central series of D_{2^k} is central in the sense of Definition 2.3.

Example 2.10 We find the upper central subgroups of the Heisenberg group. By Example 1.10, we know that $Z(\mathscr{H}) = [\mathscr{H}, \mathscr{H}]$. Consequently,

$$\zeta_2 \mathscr{H} = \{g \in \mathscr{H} \mid [g, h] \in Z(\mathscr{H}) \text{ for all } h \in \mathscr{H}\} = \mathscr{H}.$$

By Remark 2.3, $\zeta_i \mathscr{H} = \mathscr{H}$ for $i \geq 2$. We conclude that $\gamma_1 \mathscr{H} = \mathscr{H} = \zeta_2 \mathscr{H}$, $\gamma_2 \mathscr{H} = \zeta_1 \mathscr{H}$, and $\gamma_3 \mathscr{H} = I_3 = \zeta_0 \mathscr{H}$. Thus, the upper and lower central series of $\mathscr{H}$ coincide.

The next lemma deals with epimorphic images of the upper central subgroups of a group.

Lemma 2.7 *If G and H are any groups and $\varphi : G \to H$ is an epimorphism, then $\varphi(\zeta_i G) \leq \zeta_i H$ for $i \geq 0$.*

Proof The proof is done by induction on i. The result is obviously true when $i = 0$. Suppose that $\varphi(\zeta_{i-1} G) \leq \zeta_{i-1} H$ for $i > 0$, and let $g \in \varphi(\zeta_i G)$. We claim that $g \in \zeta_i H$. Since $g \in \varphi(\zeta_i G)$, there exists $x \in \zeta_i G$ such that $g = \varphi(x)$. Suppose that h is any element of H. Since φ is an epimorphism, there exists $y \in G$ such that $h = \varphi(y)$. Now,

$$[g, h] = [\varphi(x), \varphi(y)] = \varphi([x, y])$$

and

$$[x, y] \in [\zeta_i G, G] \leq \zeta_{i-1} G.$$

By induction,

$$\varphi([x, y]) \in \varphi(\zeta_{i-1} G) \leq \zeta_{i-1} H.$$

Thus, $[g, h] \in \zeta_{i-1} H$ and $g \in \zeta_i H$. □

If we put $G = H$ in Lemma 2.7, then we obtain:

Corollary 2.2 *The upper central subgroups of a group are characteristic.*

Remark 2.4 In contrast to the lower central subgroups, the upper central subgroups of a group are not necessarily fully invariant. For instance, let G be a nontrivial abelian group, and let H be a nontrivial group with trivial center. Suppose, in addition, that H contains a subgroup K which is isomorphic to G. We claim that the center of $G \times H$ is not fully invariant. Let $\pi : G \times H \to G$ be the standard projection map, and suppose that α is an isomorphism from G to K. By Lemma 1.2, $Z(G \times H) = G$. However, the endomorphism $\alpha \circ \pi$ of $G \times H$ clearly does not map G to itself. As a particular example, take $G = \mathbb{Z}_2,\ H = S_3$, and $K = \{e,\ (1\ \ 2)\} \cong G$.

2.1.5 Comparing Central Series

The upper central series of a nilpotent group ascends to the group faster than any other central series, whereas its lower central series descends to the identity faster than any other central series. This is highlighted in the next theorem.

Theorem 2.1 *If G is a nilpotent group with a (descending) central series*

$$G = G_1 \geq G_2 \geq \cdots \geq G_n \geq G_{n+1} = 1,$$

then $\gamma_i G \leq G_i$ and $G_{n-j+1} \leq \zeta_j G$ for $1 \leq i \leq n+1$ and $0 \leq j \leq n$.

Proof First, we prove that $\gamma_i G \leq G_i$. If $i = 1$, then $\gamma_1 G = G = G_1$. Let $i > 1$ and assume that $\gamma_{i-1} G \leq G_{i-1}$. By Proposition 1.1 (iii) and Lemma 2.1,

$$\gamma_i G = [\gamma_{i-1} G,\ G] \leq [G_{i-1},\ G] \leq G_i.$$

Next, we show that $G_{n-j+1} \leq \zeta_j G$. If $j = 0$, then $G_{n+1} = 1 = \zeta_0 G$. Let $j > 0$ and assume the result holds for $j - 1$. Lemma 2.1 now gives

$$\left[G_{n-j+1},\ G\right] \leq G_{n-j+2} \leq \zeta_{j-1} G.$$

By setting $H = G_{n-j+1}$ and $N = \zeta_{j-1} G$ in Lemma 1.9, we obtain

$$\left(G_{n-j+1} \zeta_{j-1} G\right) / \zeta_{j-1} G \leq Z\left(G / \zeta_{j-1} G\right) = \zeta_j G / \zeta_{j-1} G.$$

Thus, $G_{n-j+1} \leq \zeta_j G$. □

Remark 2.5 By Theorem 2.1, we have

$$\gamma_{i+1}G \le \zeta_{n-i}G \text{ for } 0 \le i \le n. \tag{2.8}$$

If G has nilpotency class c and we set $i = c-1$ and $n = c$ in (2.8), then $\gamma_c G \le Z(G)$.

Remark 2.6 The proof of Theorem 2.1 shows that if

$$G = G_1 \ge G_2 \ge \cdots \ge G_n \ge \cdots$$

is any descending series such that $[G_i,\ G] \le G_{i+1}$ for $i = 1,\ 2,\ \ldots$, then $\gamma_i G \le G_i$.

Corollary 2.3 *Let G be a group. The following are equivalent:*

(i) *G is nilpotent of class at most c;*
(ii) $\gamma_{c+1}G = 1$;
(iii) $\zeta_c G = G$;
(iv) $[g_1,\ \ldots,\ g_{c+1}] = 1$ *for all* $g_i \in G$.

Proof The result follows from Theorem 2.1 and Lemma 2.6. □

The next theorem is another consequence of Theorem 2.1. It shows that the lengths of the upper and lower central series (when finite) coincide with the nilpotency class of the group, and no other central series has smaller length.

Theorem 2.2 *Let G be a group. The following are equivalent:*

(i) *G is nilpotent of class $c \ge 1$;*
(ii) $\gamma_{c+1}G = 1$ *and* $\gamma_c G \ne 1$;
(iii) $\zeta_c G = G$ *and* $\zeta_{c-1}G \ne G$.

Example 2.11 The dihedral group D_{2^k} $(k > 1)$ has nilpotency class k (see Examples 2.4 and 2.9).

Example 2.12 The Heisenberg group has nilpotency class 2 (see Example 2.10).

We have seen that some groups coincide with their derived subgroup (refer to Example 1.7). This never happens for nontrivial nilpotent groups.

Corollary 2.4 *If G is a nontrivial nilpotent group, then $\gamma_2 G$ is a proper subgroup of G.*

Proof The proof is done by contradiction. If $\gamma_2 G = G$, then $\gamma_i G = G$ for all $i \ge 2$ by Remark 2.2 (ii). However, Theorem 2.2 implies that $\gamma_i G = 1$ for some i since G is nilpotent. Consequently, G must be trivial. □

Remark 2.7 In fact, if G is a nontrivial nilpotent group and $N \ne 1$ is a normal subgroup of G, then $[N,\ G]$ is a proper subgroup of G.

2.2 Examples of Nilpotent Groups

In this section, we give more examples of nilpotent groups.

2.2.1 *Finite p-Groups*

A classical result in finite group theory is that finite p-groups are nilpotent.

Theorem 2.3 *Every finite p-group is nilpotent, where p is any prime.*

Proof We use the fact that the center of a finite p-group is itself a finite p-group. Let G be a finite p-group of order p^n for some $n \in \mathbb{N}$. By Theorem 1.2, $Z(G)$ is nontrivial, and thus $G/Z(G)$ is a finite p-group of order p^r for some $r \in \mathbb{N}$ with $r < n$. Invoking Theorem 1.2 again, we have that $G/Z(G)$ has nontrivial center. Hence, $Z(G/Z(G)) = \zeta_2 G/Z(G)$ is a p-group of order p^s for some $s \in \mathbb{N}$ with $s < r$. This means that $|Z(G)| < |\zeta_2 G|$, so $Z(G) < \zeta_2 G$. By iterating this procedure, we see that $|\zeta_i G| < |\zeta_{i+1} G|$ for $i \geq 0$, and thus $\zeta_i G$ is a proper subgroup of $\zeta_{i+1} G$ for $i \geq 0$. And so, the upper central series for G is strictly increasing. Since G is finite, the series must terminate at $\zeta_k G = G$ for some $k \in \mathbb{N}$. Therefore, G is nilpotent. □

Example 2.13 The dihedral groups D_{2^n} for $n \geq 1$ are finite 2-groups, and thus nilpotent.

Example 2.14 The *quaternion group* Q is the group with presentation

$$Q = \left\langle x, y \;\middle|\; x^4 = 1,\ x^2 = y^2,\ y^{-1}xy = x^{-1} \right\rangle.$$

The elements of Q are $1, x, x^2, x^3, y, xy, x^2y$, and x^3y. Since Q has order $8 = 2^3$, it is nilpotent.

Example 2.15 If G and H are finite groups of orders m and n respectively, then both the direct product $G \times H$ and semi-direct product $G \rtimes_\varphi H$ by φ have order mn. In particular, the direct and semi-direct product of any two finite p-groups is itself a finite p-group.

An important construction of groups is the wreath product. Let A and T be any two groups. For each $s \in T$, let A_s be an isomorphic copy of A, and let a_s denote the isomorphic image of $a \in A$ in A_s. Consider the direct product $B = \prod_{s \in T} A_s$, and define the *standard (or restricted) wreath product* of A by T as

$$W = A \wr T = B \rtimes_\varphi T,$$

where $\varphi : T \to Aut(B)$ is the homomorphism that maps each $t \in T$ to $\varphi(t)$, where $\varphi(t)$ is the automorphism of B induced by the mapping

$$a_s \mapsto a_{st} \text{ for all } a \in A \text{ and } s, t \in T.$$

Thus, T acts on B by permuting its factors. This action can be realized as conjugation, so that $t^{-1}a_s t = a_{st}$ in W for all $a \in A$ and $s,\ t \in T$. A presentation for W is

$$W = A \wr T = \left\langle B,\ T \,\middle|\, t^{-1}a_s t = a_{st} \quad (a \in A,\ s,\ t \in T) \right\rangle.$$

We call W the *unrestricted wreath product* of A by T in case B is an unrestricted direct product of the A_s. In both situations, B is called the *base group*, A is the *bottom group*, and T is the *top group*.

Example 2.16 Suppose that A and T are finite groups of orders m and n, respectively. Using the notation above, we have that $B = \prod_{s \in T} A_s$ is a finite group of order m^n, and thus $A \wr T$ has order nm^n. In particular, if $|A| = p^m$ and $|T| = p^n$ for some prime p, then

$$|A \wr T| = p^n (p^m)^{p^n} = p^{n+mp^n}.$$

Thus, the wreath product of any two finite p-groups is a finite p-group.

Remark 2.8 In contrast to Theorem 2.3, an infinite p-group does not have to be nilpotent. In [2], G. Baumslag showed how to construct infinite p-groups which are not nilpotent using wreath products. Take a nontrivial p-group A and an infinite p-group B, and form the wreath product $W = A \wr B$. Clearly, W is an infinite group. By Corollary 3.2 of [2], W must have trivial center, and thus fails to be nilpotent by Lemma 2.2. Furthermore, W is a p-group since it is the wreath product of two p-groups (see [11]). Thus, W is an infinite p-group that is not nilpotent.

Two groups which are infinite p-groups that are not nilpotent are the wreath products $\mathbb{Z}_p \wr \mathbb{Z}_{p^\infty}$ and $\mathbb{Z}_{p^\infty} \wr \mathbb{Z}_{p^\infty}$, where $\mathbb{Z}_p$ is the cyclic group of order p and

$$\mathbb{Z}_{p^\infty} = \langle x_1,\ x_2,\ \ldots \mid px_1 = 0,\ px_{n+1} = x_n \text{ for } n = 1,\ 2,\ \ldots \rangle \tag{2.9}$$

is an additively written presentation for the *Prüfer p-group* (or *p-quasicyclic group*).

2.2.2 An Example Involving Rings

Nilpotency in ring theory relates to nilpotency in group theory in a natural way.

Definition 2.6 Let R be a ring with unity 1, and let T be a subring of R. For any $k \in \mathbb{N}$, let T^k be the subring of T consisting of all finite sums of the form

$$\sum a_{p_1 \cdots p_k} x_{p_1} \cdots x_{p_k} \quad \left(a_{p_1 \cdots p_k} \in \mathbb{Z},\ \ x_{p_1},\ \ldots,\ x_{p_k} \in T\right).$$

If there exists a natural number m such that $T^m = \{0\}$, then T is termed a *nilpotent subring* of R.

Let R be as in Definition 2.6, and suppose that S is a nilpotent subring of R with $S^n = \{0\}$. Define

$$G = 1 + S = \{1 + x \mid x \in S\}.$$

Clearly, G is closed under multiplication since

$$(1 + x)(1 + y) = 1 + y + x + xy \in G$$

for all $x, y \in S$, and it is closed under inverses because

$$\left(1 + x\right)\left(1 - x + x^2 - x^3 + \cdots + (-1)^{n-1}x^{n-1}\right) = 1$$

for all $x \in S$. Thus, G is a subgroup of the group of units of R.

We claim that G is nilpotent of class at most $n - 1$. Let

$$G_i = 1 + S^i = \left\{1 + x \mid x \in S^i\right\} \text{ for } i = 1,\ 2,\ \ldots,\ n.$$

By the same argument as before, we find that G_i is a subgroup of G.

Consider the (descending) series

$$G = G_1 \geq G_2 \geq \cdots \geq G_n = 1. \tag{2.10}$$

We claim that (2.10) is a central series for G. By Lemma 2.1, it suffices to show that $[G_i,\ G] \leq G_{i+1}$ for $i = 1,\ 2,\ \ldots,\ n - 1$. Let $g = 1 + x \in G_i$ and $h = 1 + y \in G$, where $x \in S^i$ and $y \in S$. A straightforward computation gives

$$\begin{aligned} gh - hg &= (1 + x)(1 + y) - (1 + y)(1 + x) \\ &= xy - yx \in S^{(i+1)}. \end{aligned}$$

Thus,

$$[g,\ h] = g^{-1}h^{-1}(gh - hg) + 1 \in S^{(i+1)} + 1 = G_{i+1},$$

and consequently, G is nilpotent of class at most $n - 1$ as claimed.

Example 2.17 Let R be a commutative ring with unity, and let T be the ring of $n \times n$ matrices over R. Let S be the subring of T consisting of all $n \times n$ matrices over R whose entries on and below the main diagonal are equal to zero. Thus,

$$S = \left\{ \begin{pmatrix} 0 & b_{12} & \ldots & b_{1n} \\ 0 & 0 & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \,\middle|\, b_{ij} \in R \right\}.$$

A direct computation shows that S^p consists of all elements of S whose first $p-1$ superdiagonals have zero entries. Thus, a typical matrix in S^p has the form

$$\begin{pmatrix} 0 \cdots\cdots 0 & c_{1\,p+1} & c_{1\,p+2} & \cdots & c_{1n} \\ 0 \cdots\cdots 0 & 0 & c_{2\,p+2} & \cdots & c_{2n} \\ 0 \cdots\cdots 0 & 0 & \ddots & \ddots & \vdots \\ \vdots \cdots\cdots\cdots & \ddots & \ddots & \ddots & c_{p+1\,n} \\ \vdots \cdots\cdots\cdots & \ddots & \ddots & 0 & 0 \\ \vdots \cdots\cdots\cdots & \ddots & \ddots & \vdots & \vdots \\ 0 \cdots\cdots\cdots & \cdots & \cdots & 0 & 0 \end{pmatrix},$$

where $c_{ij} \in R$. In particular, S^n is the $n \times n$ zero matrix. Let

$$UT_n(R) = \{I_n + M \mid M \in S\},$$

where I_n is the $n \times n$ identity matrix (we use this notation throughout the book). It follows from the above that $UT_n(R)$ is a nilpotent group of class less than n, called the *(upper) unitriangular group of degree n over R*. A typical element of $UT_n(R)$ is an $n \times n$ upper unitriangular matrix of the form

$$\begin{pmatrix} 1 & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ 0 & 1 & a_{23} & \cdots & \cdots & a_{2n} \\ 0 & 0 & 1 & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & a_{n-1\,n} \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix},$$

where $a_{ij} \in R$. In particular, $UT_3(\mathbb{Z})$ is the Heisenberg group $\mathscr{H}$.

For more on nilpotent rings and nilpotent groups which arise from them, see [6].

2.3 Elementary Properties of Nilpotent Groups

In this section, we take a look at some fundamental results on nilpotent groups. The first one deals with subgroups and homomorphic images of nilpotent groups.

Theorem 2.4 *If G is a nilpotent group of class c, then every subgroup and homomorphic image of G is nilpotent of class at most c.*

Proof Suppose that H is a subgroup of G. By Lemma 2.4, $\gamma_i H \leq \gamma_i G$ for each $i \in \mathbb{N}$. Since G has nilpotency class c, $\gamma_{c+1}G = 1$ by Theorem 2.2. Thus, $\gamma_{c+1}H = 1$ and H is nilpotent of class at most c by Corollary 2.3.

Let K be any group and $\varphi \in Hom(G, K)$. By Lemma 2.5, $\varphi(\gamma_i G) = \gamma_i(\varphi(G))$ for each $i \in \mathbb{N}$. Since $\gamma_{c+1}G = 1$ and φ is a homomorphism,

$$1 = \varphi(\gamma_{c+1}G) = \gamma_{c+1}(\varphi(G)).$$

It follows from Corollary 2.3 that $\varphi(G)$ is nilpotent of class at most c. □

Corollary 2.5 *If G is a nilpotent group of class c and $N \trianglelefteq G$, then G/N is nilpotent of class at most c.*

This is immediate from Theorem 2.4 since G/N is a homomorphic image of G. Note that Corollary 2.5 is also a consequence of Corollaries 2.1 and 2.3.

2.3.1 Establishing Nilpotency by Induction

Many of the theorems on nilpotent groups are proven using induction on the nilpotency class. The next few results are commonly used.

Lemma 2.8 *If G is a nilpotent group of class $c \geq 1$, then $G/\gamma_c G$ is nilpotent of class $c - 1$.*

Proof Let $\pi : G \to G/\gamma_c G$ be the natural homomorphism. By Corollary 2.5, $G/\gamma_c G$ is a nilpotent group of class at most c. Furthermore, for any $n \in \mathbb{N}$,

$$\gamma_n(G/\gamma_c G) = \pi(\gamma_n G) = \gamma_n G/\gamma_c G$$

by Lemma 2.5. In particular, $\gamma_{c-1}(G/\gamma_c G) = \gamma_{c-1}G/\gamma_c G \neq 1$ and $\gamma_c(G/\gamma_c G) = 1$. Thus, $G/\gamma_c G$ has nilpotency class $c - 1$ by Theorem 2.2. □

Lemma 2.9 *Let G be a nilpotent group of class $c \geq 2$. For any element $g \in G$, the subgroup $H = gp(g, \gamma_2 G)$ is nilpotent of class less than c.*

Proof We prove that $\gamma_i H \leq \gamma_{i+1}G$ for $i \geq 2$ by induction on i. If $i = 2$, then

$$\gamma_2 H = gp\left([g^m h, g^n k] \mid h, k \in \gamma_2 G \text{ and } m, n \in \mathbb{Z}\right).$$

By Lemmas 1.1 and 1.4 (iv), (v), and (vi),

$$\begin{aligned}
[g^m h, g^n k] &= [g^m, g^n k]^h [h, g^n k] \\
&= \left([g^m, k] [g^m, g^n]^k\right)^h [h, g^n k] \\
&= [g^m, k]^h [g^m, g^n]^{kh} [h, g^n k].
\end{aligned}$$

Now, $[g^m, k]^h$ and $[h, g^n k]$ are contained in $\gamma_3 G$ and $[g^m, g^n] = 1$. Therefore, $[g^m h, g^n k] \in \gamma_3 G$, and consequently, $\gamma_2 H \le \gamma_3 G$.

If we assume that $\gamma_{i-1} H \le \gamma_i G$ for $i > 2$, then

$$\gamma_i H = [\gamma_{i-1} H, H] \le [\gamma_i G, H] \le [\gamma_i G, G] = \gamma_{i+1} G.$$

Thus, $\gamma_i H \le \gamma_{i+1} G$. In particular, $\gamma_c H \le \gamma_{c+1} G = 1$. By Corollary 2.3, H has nilpotency class less than c. □

Lemma 2.10 *If G is any group, then* $\zeta_n G/Z(G) \cong \zeta_{n-1}(G/Z(G))$ *for any* $n \in \mathbb{N}$.

Proof The proof is done by induction on n. If $n = 1$, then the result is obviously true. Suppose that $\zeta_i G/Z(G) \cong \zeta_{i-1}(G/Z(G))$ for $2 \le i \le n-1$. We claim that $\zeta_n G/Z(G) \cong \zeta_{n-1}(G/Z(G))$. By definition, $\zeta_n G/\zeta_{n-1} G = Z(G/\zeta_{n-1} G)$. By the Third Isomorphism Theorem,

$$\frac{\zeta_n G/Z(G)}{\zeta_{n-1} G/Z(G)} \cong Z\left(\frac{G/Z(G)}{\zeta_{n-1} G/Z(G)}\right). \tag{2.11}$$

By induction, $\zeta_{n-1} G/Z(G) \cong \zeta_{n-2}(G/Z(G))$. Substituting this in (2.11) yields

$$\frac{\zeta_n G/Z(G)}{\zeta_{n-2}(G/Z(G))} \cong Z\left(\frac{G/Z(G)}{\zeta_{n-2}(G/Z(G))}\right) = \frac{\zeta_{n-1}(G/Z(G))}{\zeta_{n-2}(G/Z(G))}.$$

The result follows. □

More generally, we have the next result of P. Hall.

Lemma 2.11 *If G is any group, then* $\zeta_i(G/\zeta_j G) \cong \zeta_{i+j} G/\zeta_j G$ *for* $i, j \ge 0$.

Proof The proof is done by induction on j. Lemma 2.10 settles the case for $j = 1$. Suppose that the lemma is true for $j > 1$. By the Third Isomorphism Theorem,

$$\frac{\zeta_{i+j+1} G}{\zeta_{j+1} G} \cong \frac{\zeta_{(i+1)+j} G/\zeta_j G}{\zeta_{j+1} G/\zeta_j G}.$$

By induction, $\zeta_{(i+1)+j} G/\zeta_j G \cong \zeta_{i+1}\big(G/\zeta_j G\big)$. Since $\zeta_{j+1} G/\zeta_j G$ is just $Z\big(G/\zeta_j G\big)$, we have

$$\frac{\zeta_{i+j+1} G}{\zeta_{j+1} G} \cong \frac{\zeta_{i+1}\big(G/\zeta_j G\big)}{Z\big(G/\zeta_j G\big)} \cong \zeta_i\left(\frac{G/\zeta_j G}{Z\big(G/\zeta_j G\big)}\right)$$

by Lemma 2.10. However,

$$\zeta_i\left(\frac{G/\zeta_j G}{Z(G/\zeta_j G)}\right) \cong \zeta_i\left(\frac{G}{\zeta_{j+1} G}\right)$$

by the Third Isomorphism Theorem. This completes the proof. □

Theorem 2.5 *Let G be a group, and suppose that $N \trianglelefteq G$. If $N \leq \zeta_i G$ for some $i \in \mathbb{N}$ and G/N is nilpotent, then G is nilpotent.*

Proof Consider the upper central series

$$1 = \zeta_0(G/\zeta_i G) \leq \zeta_1(G/\zeta_i G) \leq \cdots \tag{2.12}$$

for $G/\zeta_i G$. By Lemma 2.11, $\zeta_k(G/\zeta_i G) \cong \zeta_{k+i}G/\zeta_i G$ for $k \geq 0$. Thus, (2.12) becomes

$$1 = \zeta_i G/\zeta_i G \leq \zeta_{i+1}G/\zeta_i G \leq \cdots . \tag{2.13}$$

Since $G/\zeta_i G \cong (G/N)/(\zeta_i G/N)$ by the Third Isomorphism Theorem and G/N is nilpotent, then $G/\zeta_i G$ is nilpotent by Corollary 2.5. Thus, the series (2.13) terminates at $G/\zeta_i G$. Therefore, there exists an integer $n \geq i$ such that $\zeta_n G/\zeta_i G = G/\zeta_i G$, and hence, $\zeta_n G = G$. By Theorem 2.3, G is nilpotent. □

If $N \leq Z(G)$ in Theorem 2.5, then the next theorem gives information about the nilpotency class of G.

Theorem 2.6 *Let G be a group, and suppose that $N \leq Z(G)$. If G/N is nilpotent of class c, then G is nilpotent of class either c or $c + 1$.*

Proof We first prove that if $gN \in \zeta_n(G/N)$ for any $g \in G$ and $n \geq 0$, then $g \in \zeta_{n+1}G$. If $n = 0$, then $\zeta_0(G/N) = N$. In this case, $gN \in \zeta_0(G/N) = N$, and thus $g \in N$. And so, g is central because $N \leq Z(G)$ by the hypothesis. Assume that $hN \in \zeta_{k-1}(G/N)$ implies $h \in \zeta_k G$ for $2 \leq k \leq n$, and let $gN \in \zeta_n(G/N)$. Since

$$[\zeta_n(G/N),\ G/N] \leq \zeta_{n-1}(G/N),$$

we have $[gN,\ hN] \in \zeta_{n-1}(G/N)$ for all $h \in G$. Thus, $[g,\ h] \in \zeta_n G$ by the induction hypothesis. Consequently, $g \in \zeta_{n+1}G$ as claimed.

Next, we prove that $G = \zeta_{c+1}G$. If $g \in G$, then $gN \in G/N = \zeta_c(G/N)$ by Theorem 2.2. This implies that $g \in \zeta_{c+1}G$ by our discussion above. Hence $G = \zeta_{c+1}G$. Now, if $\zeta_c G \neq G$, then G has nilpotency class $c + 1$ by Theorem 2.2. Suppose that $\zeta_c G = G$. If $\zeta_{c-1}G = G$, then G is of class $d \leq c - 1$ by Theorem 2.3. By Corollary 2.5, G/N is of class at most d. However, G/N is of class c by hypothesis. Thus, $c \leq d \leq c - 1$, which is false. It follows from Theorem 2.2 that $\zeta_{c-1}G \neq G$, and thus G is of nilpotency class c. □

If $N = Z(G)$ in Theorem 2.6, then the nilpotency class can be determined.

Lemma 2.12 *A group G is nilpotent of class $c \geq 1$ if and only if $G/Z(G)$ is nilpotent of class $c - 1$.*

Proof We invoke Theorem 2.2 (iii). If G is nilpotent of class c, then $\zeta_c G = G$ and $\zeta_{c-1}G \neq G$. Thus,

$$\zeta_{c-1}(G/Z(G)) \cong \zeta_c G/Z(G) = G/Z(G)$$

and

$$\zeta_{c-2}(G/Z(G)) \cong \zeta_{c-1}G/Z(G) \neq G/Z(G)$$

by Lemma 2.10. Therefore, $G/Z(G)$ is of class $c-1$. The converse is similar. □

2.3.2 A Theorem on Root Extraction

We illustrate how Lemma 2.9 is used to prove a theorem on the extraction of roots in nilpotent groups by induction on the nilpotency class.

Definition 2.7 Let G be a group, and let P be a set of primes. A natural number n is called a *P-number* if every prime divisor of n belongs to P.

By convention, 1 is a P-number for any set of primes P. If P happens to be the empty set, then the only P-number is 1.

Definition 2.8 Let G be a group, and let P be a set of primes.

1. An element of G is called a *P-torsion element* if its order is a P-number. The set of P-torsion elements of G is denoted by $\tau_P(G)$. Thus,

$$\tau_P(G) = \{g \in G \mid g^n = 1 \text{ for some } P\text{-number } n\}.$$

2. If every element of G is P-torsion, then G is called a *P-torsion group*.
3. If G has no P-torsion elements other than the identity, then G is *P-torsion-free*.

If $P = \{p\}$, then a P-torsion group is just a p-group by Definition 1.7. If P is the set of all primes, then $\tau_P(G)$ is the set of all elements of finite order of G and is written as $\tau(G)$. Note that G is P-torsion-free whenever P is empty.

An element of $\tau(G)$ is called a *torsion element* of G, and G is a *torsion* (or *periodic*) group if $\tau(G) = G$. We say that G is *torsion-free* if it has no torsion elements other than the identity element.

The group properties "P-torsion" and "P-torsion-free" are preserved under extensions.

Definition 2.9 Let G, H, and N be groups.

(i) If $N \trianglelefteq G$ and $G/N \cong H$, then G is called an *extension* of H by N. Thus, there exists a short exact sequence

$$1 \to N \to G \to H \to 1.$$

(ii) An extension G of H by N is called *central* if $N \leq Z(G)$.
(iii) Let $N \trianglelefteq G$, and suppose that G is an extension of H by N. A property $\mathscr{Q}$ of groups is said to be *preserved under extensions* if G has property $\mathscr{Q}$ whenever both N and H have property $\mathscr{Q}$.

Lemma 2.13 *If P is a set of primes, then "P-torsion" and "P-torsion-free" are preserved under extensions.*

Proof Let G be a group with $N \trianglelefteq G$.

- Suppose that N and G/N are P-torsion, and let $g \in G$. Since G/N is P-torsion, the element $gN \in G/N$ has order a P-number n. Thus, $(gN)^n = N$, or equivalently, $g^n \in N$. Since N is also P-torsion, there exists a P-number m such that $(g^n)^m = 1$; that is, $g^{nm} = 1$. Since nm is a P-number, G is P-torsion.
- Suppose that N and G/N are P-torsion-free. Let $g \in G$ such that $g^n = 1$ for some P-number n. Then $(gN)^n = N$ in G/N. Since G/N is P-torsion-free, $gN = N$; that is, $g \in N$. Therefore, $g = 1$ because N is P-torsion-free. □

We now prove a classical result on extraction of roots in nilpotent groups. If G is any group and $g \in G$, then $h \in G$ is an *nth root* of g if $h^n = g$ for some natural number $n > 1$.

Theorem 2.7 (S. N. Černikov, A. I. Mal'cev) *Let P be a nonempty set of primes. A nilpotent group G is P-torsion-free if and only if the following condition holds:*

$$\textit{if } g,\ h \in G \textit{ and } g^n = h^n \textit{ for some } P\textit{-number } n,\ \textit{then } g = h. \tag{2.14}$$

Equation (2.14) is equivalent to the condition that every element of G has at most one nth root for every P-number n.

Proof Suppose that G is P-torsion-free, and assume that $g^n = h^n$ for some $g,\ h \in G$ and P-number n. We prove that $g = h$ by induction on the class c of G. If $c = 1$, then G is abelian. In this case, $g^n = h^n$ for some P-number n implies that $\left(gh^{-1}\right)^n = 1$. Since G is P-torsion-free, $gh^{-1} = 1$ and $g = h$.

Suppose that $c > 1$, and assume that the result holds for all P-torsion-free nilpotent groups of class less than c. By Lemma 2.9, $H = gp(g,\ \gamma_2 G)$ is nilpotent of class less than c. It is clear that $h^{-1}gh \in H$ because $h^{-1}gh = g[g,\ h]$. Now, $g^n = h^n$ is the same as $g^n = h^{-1}h^n h$ which, after replacing h^n by g^n, becomes

$$g^n = h^{-1}g^n h = \left(h^{-1}gh\right)^n.$$

By induction, $g = h^{-1}gh$, so g and h commute. Hence, the equality $g^n = h^n$ can be expressed as $\left(gh^{-1}\right)^n = 1$. Since G is P-torsion-free, $gh^{-1} = 1$, and thus $g = h$.

Conversely, suppose that G is *any* group such that (2.14) is satisfied for any elements g and h in G. If we take $h = 1$, then $g^n = 1^n = 1$ implies $g = 1$. And so, G is P-torsion-free. □

Example 2.18 The Heisenberg group is torsion-free. To see this, suppose that

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.15}$$

for some $a,\ b,\ c \in \mathbb{Z}$ and $n \in \mathbb{N}$. We use the Binomial Theorem to compute the left-hand side of (2.15):

$$\begin{aligned}
\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}^n &= \left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + \begin{pmatrix}0 & a & b\\ 0 & 0 & c\\ 0 & 0 & 0\end{pmatrix}\right)^n \\
&= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + n\begin{pmatrix}0 & a & b\\ 0 & 0 & c\\ 0 & 0 & 0\end{pmatrix} + \binom{n}{2}\begin{pmatrix}0 & a & b\\ 0 & 0 & c\\ 0 & 0 & 0\end{pmatrix}^2 + \cdots \\
&= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + n\begin{pmatrix}0 & a & b\\ 0 & 0 & c\\ 0 & 0 & 0\end{pmatrix} + \binom{n}{2}\begin{pmatrix}0 & 0 & ac\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} + \cdots \\
&= \begin{pmatrix}1 & na & nb + \binom{n}{2}ac\\ 0 & 1 & nc\\ 0 & 0 & 1\end{pmatrix}.
\end{aligned}$$

Therefore,

$$\begin{pmatrix}1 & na & nb + \binom{n}{2}ac\\ 0 & 1 & nc\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},$$

and thus $a = b = c = 0$.

Since $\mathscr{H}$ is torsion-free, (2.14) must hold in $\mathscr{H}$. Indeed, suppose that

$$\begin{pmatrix}1 & a_1 & b_1\\ 0 & 1 & c_1\\ 0 & 0 & 1\end{pmatrix}^n = \begin{pmatrix}1 & a_2 & b_2\\ 0 & 1 & c_2\\ 0 & 0 & 1\end{pmatrix}^n$$

for some $a_1,\ a_2,\ b_1,\ b_2,\ c_1,\ c_2 \in \mathbb{Z}$ and $n \in \mathbb{N}$. The same computation used above gives

$$\begin{pmatrix}1 & na_1 & nb_1 + \binom{n}{2}a_1c_1\\ 0 & 1 & nc_1\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & na_2 & nb_2 + \binom{n}{2}a_2c_2\\ 0 & 1 & nc_2\\ 0 & 0 & 1\end{pmatrix}.$$

Therefore, $a_1 = a_2,\ b_1 = b_2$, and $c_1 = c_2$. Hence, $\begin{pmatrix}1 & a_1 & b_1\\ 0 & 1 & c_1\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & a_2 & b_2\\ 0 & 1 & c_2\\ 0 & 0 & 1\end{pmatrix}$ as claimed.

2.3.3 The Direct Product of Nilpotent Groups

The direct product of finitely many nilpotent groups is again nilpotent. This is the point behind the next theorem.

Theorem 2.8 *If* $\{H_1, \ldots, H_n\}$ *is a set of nilpotent groups of class* $c_1, \ldots, c_n$ *respectively, then the direct product* $H_1 \times \cdots \times H_n$ *is nilpotent of class* $max\{c_1, \ldots, c_n\}$.

Proof We prove the theorem for $n = 2$. Assume that H_1 and H_2 are nontrivial groups of nilpotency classes c_1 and c_2 respectively, and suppose that $c_1 \geq c_2 > 0$. The proof is done by induction on c_1. If $c_1 = 1$, then H_1 and H_2 are abelian, and thus $H_1 \times H_2$ is abelian.

Suppose that $c_1 > 1$. By Lemma 1.3,

$$\frac{H_1 \times H_2}{Z(H_1 \times H_2)} \cong \frac{H_1}{Z(H_1)} \times \frac{H_2}{Z(H_2)}. \tag{2.16}$$

Note that the right side of (2.16) is a direct product of nilpotent groups of classes less than c_1. By Lemma 2.12, the class of $H_1/Z(H_1)$ is $c_1 - 1$. By induction, $(H_1 \times H_2)/Z(H_1 \times H_2)$ is a nilpotent group of class $c_1 - 1$. The result follows from Lemma 2.12. □

Remark 2.9 It is not always the case that the direct product of an arbitrary number of nilpotent groups is nilpotent. For example, suppose that $\{G_1, G_2, \ldots\}$ is an infinite set of nilpotent groups, and assume that G_i has nilpotency class at least i for each $i = 1, 2, \ldots$. We claim that the infinite direct product of the groups $G_1, G_2, \ldots$ is not nilpotent. Assume, on the contrary, that this direct product is nilpotent of class c. By Theorem 2.4, each of its subgroups is of class at most c. Consequently, every G_i is of class at most c. This contradicts the fact that G_j is of class at least j whenever $j > c$.

On the other hand, if the nilpotency class of each G_i is bounded above, then their direct product is nilpotent. The proof of this is analogous to that of Theorem 2.8.

2.3.4 Subnormal Subgroups

Subgroups of nilpotent groups enjoy several noteworthy properties, one of which is subnormality.

Definition 2.10 A subgroup H of a group G is called *subnormal* if there is a subnormal series of subgroups of G beginning at H and terminating at G.

Theorem 2.9 *Every subgroup of a nilpotent group is subnormal.*

Proof Let H be a subgroup of a nilpotent group G of class c, and consider the subgroups $H\zeta_i G$ of G for $i = 1, 2, \dots, c$. Since the upper central series of G is normal, we have

$$H = H\zeta_0 G \leq H\zeta_1 G \leq \cdots \leq H\zeta_c G = G. \tag{2.17}$$

We claim that (2.17) is a subnormal series. If $h \in H$ and $z \in \zeta_{i+1}G$, then

$$z^{-1}hz = h[h, z] \in H[H, \zeta_{i+1}G] = H\zeta_i G.$$

Therefore, $z \in N_G(H\zeta_i G)$, and thus $\zeta_{i+1}G < N_G(H\zeta_i G)$. Since $H < N_G(H\zeta_i G)$ as well, $H\zeta_{i+1}G < N_G(H\zeta_i G)$ and the claim is proved. Thus, (2.17) is a subnormal series from H to G in c steps. □

Remark 2.10 Another subnormal series from H to G can be constructed using successive normalizers. Put $H_0 = H$, and recursively define $H_{i+1} = N(H_i)$. It is simple to verify that the series

$$H = H_0 < H_1 < \cdots < H_c = G$$

is, indeed, subnormal.

Corollary 2.6 *If G is a nilpotent group and $H < G$ with $[G : H] = n$, then $g^n \in H$ for all $g \in G$.*

Proof Suppose that G has nilpotency class c. If H is a normal subgroup of G, then $|G/H| = [G : H] = n$. Hence, $(gH)^n = H$ for all $g \in G$, and thus $g^n \in H$.

Assume that H is any subgroup of G. By Theorem 2.9, there is a subnormal series

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_c = G.$$

Furthermore, each H_i is nilpotent by Theorem 2.4. If we put $[H_{i+1} : H_i] = m_i$, so that $n = m_{c-1}m_{c-2}\cdots m_0$, then we obtain

$$g^n = ((g^{m_{c-1}})^{m_{c-2}})^{\cdots m_0}.$$

Since each H_i is normal in G, we have

$$g^{m_{c-1}} \in H_{c-1},\ (g^{m_{c-1}})^{m_{c-2}} \in H_{c-2},\ \dots.$$

Continuing in this way leads to $g^n \in H_0 = H$. □

2.3.5 The Normalizer Condition

An important feature of nilpotent groups is that all of their maximal subgroups are normal. In fact, this property leads to a structure theorem for finite nilpotent groups which will be proven in the next section. Groups whose maximal subgroups are normal satisfy the so-called *normalizer condition*.

Definition 2.11 A group G satisfies the *normalizer condition* if H is a proper subgroup of $N_G(H)$ whenever H is a proper subgroup of G.

Lemma 2.14 *If a group G satisfies the normalizer condition, then every maximal subgroup of G is normal.*

Proof Let M be a maximal subgroup of G. By hypothesis, M is a proper subgroup of $N_G(M)$. Thus, $N_G(M) = G$ because M is maximal. And so, $M \lhd G$. □

Lemma 2.15 *If every subgroup of a group G is subnormal, then G satisfies the normalizer condition.*

Proof Suppose that H is a proper subgroup of G. Since H is subnormal, there exists a subnormal series

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G$$

for some $n \in \mathbb{N}$. Clearly, H_1 properly contains and normalizes H since $H \lhd H_1$. □

Theorem 2.10 *Every nilpotent group satisfies the normalizer condition.*

Proof This is a consequence of Theorem 2.9 and Lemma 2.15. □

Corollary 2.7 *Every maximal subgroup of a nilpotent group is normal.*

Proof The result follows at once from Theorem 2.10 and Lemma 2.14. □

2.3.6 Products of Normal Nilpotent Subgroups

We prove a theorem pertaining to the product of normal nilpotent subgroups of an arbitrary group.

Theorem 2.11 (H. Fitting) *Let G be any group, and suppose that H and K are normal nilpotent subgroups of G of classes c and d respectively. Then HK is a normal nilpotent subgroup of G of class at most $c + d$.*

Proof By Theorem 2.2, $\gamma_{c+1}H = 1$ and $\gamma_{d+1}K = 1$. The result will follow at once from Theorem 2.3 once we prove that $\gamma_{c+d+1}(HK) = 1$. By repeatedly applying Lemma 1.11, we get

$$\begin{aligned}\gamma_{c+d+1}(HK) &= [\underbrace{HK,\ HK,\ \cdots,\ HK}_{c+d+1}] \\ &= [\underbrace{H,\ HK,\ \cdots,\ HK}_{c+d+1}][\underbrace{K,\ HK,\ \cdots,\ HK}_{c+d+1}] \\ &= \cdots\end{aligned}$$

Thus, $\gamma_{c+d+1}(HK)$ is a product of commutators of the form

$$[X_1,\ X_2,\ \ldots,\ X_{c+d+1}],$$

where X_j is either H or K for $1 \leq j \leq c+d+1$. Let $Y = [X_1,\ X_2,\ \ldots,\ X_{c+d+1}]$ be one of the commutators arising in this product. Since Y contains $(c+d+1)$ X_j's, either H appears at least $(c+1)$ times in Y or K appears at least $(d+1)$ times in Y. Now, $\gamma_m H \trianglelefteq G$ and $\gamma_n K \trianglelefteq G$ for each $m,\ n > 0$ by Corollary 1.3 because both H and K are normal in G. By Theorem 1.4,

$$[\gamma_m H,\ K] \leq \gamma_m H \ \text{ and } \ [\gamma_n K,\ H] \leq \gamma_n K. \tag{2.18}$$

Hence, if s of the X_j's in the commutator Y equal H, then $Y \leq \gamma_{s+1}H$ by (2.18). Similarly, if t of the X_j's in the commutator Y equal K, then $Y \leq \gamma_{t+1}K$. It follows that if H occurs at least $(c+1)$ times in Y, then $Y \leq \gamma_{c+1}H$. However, if K occurs at least $(d+1)$ times in Y, then $Y \leq \gamma_{d+1}K$. In either case, we obtain $Y = 1$. Therefore, $\gamma_{c+d+1}(HK) = 1$. □

2.4 Finite Nilpotent Groups

In this section, we give a characterization of finite nilpotent groups. We begin by mentioning some of the well-known Sylow theorems and consequences of them. These play a fundamental role in the study of finite groups, and their proofs can be found in various places in the literature (see [3, 9], or [10] for instance).

Definition 2.12 Let G be a finite group of order $p^n k$, where p is a prime, $k \in \mathbb{N}$, and p doesn't divide k. A subgroup of G whose order is exactly p^n is called a *Sylow p-subgroup* of G.

A subgroup H of a finite group G is called a *Sylow subgroup* of G if it is a Sylow p-subgroup of G for some prime p. The fact that a finite group has Sylow subgroups is contained in the next fundamental theorem.

Theorem 2.12 (Sylow) *Let G be a finite group of order $p^n k$, where p is a prime, $k \in \mathbb{N}$, and p doesn't divide k.*

(i) G has at least one subgroup of order p^i for each $i = 1, 2, \ldots, n$.
(ii) If $H \leq G$ and $|H| = p^n$, then H is contained in some Sylow p-subgroup.
(iii) Any two Sylow p-subgroups of G are conjugate.

A consequence of Theorem 2.12 (iii) is:

Corollary 2.8 *Let p be a prime, and suppose that P is a Sylow p-subgroup of a finite group G. Then $P \trianglelefteq G$ if and only if P is the unique Sylow p-subgroup of G.*

Another result which will be needed later is:

Lemma 2.16 *Let P be a Sylow p-subgroup of a finite group G.*

(i) If $K \leq G$ and K contains $N_G(P)$, then $K = N_G(K)$.
(ii) If $N \lhd G$, then $P \cap N$ is a Sylow p-subgroup of N and PN/N is a Sylow p-subgroup of G/N.

The proof of Lemma 2.16 (i) relies on the so-called Frattini Argument.

Lemma 2.17 (Frattini Argument) *Let G be a finite group and $H \trianglelefteq G$. If P is a Sylow p-subgroup of H for some prime p, then $G = HN_G(P)$.*

We now prove the main theorem of this section.

Theorem 2.13 *Let G be a finite group. The following are equivalent:*

(i) G is nilpotent.
(ii) Every subgroup of G is subnormal.
(iii) G satisfies the normalizer condition.
(iv) Every maximal subgroup of G is normal.
(v) Every Sylow subgroup of G is normal.
(vi) G is a direct product of its Sylow subgroups.
(vii) Elements of coprime order commute.

Proof (i) $\Rightarrow$ (ii) by Theorem 2.9, (ii) $\Rightarrow$ (iii) by Lemma 2.15, and (iii) $\Rightarrow$ (iv) by Lemma 2.14.

We prove (iv) $\Rightarrow$ (v) by contradiction. Let P be a Sylow subgroup of G, and assume that P is not normal in G. Then $N_G(P) < G$, and consequently, $N_G(P) < M$ for some maximal subgroup M of G. Since $M \lhd G$, we have $N_G(M) = G$. This contradicts Lemma 2.16 (i).

Next, we prove (v) $\Rightarrow$ (vi). Suppose that G has order $p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$, where the p_i's are distinct primes and $r_i \in \mathbb{N}$. Assume that each Sylow subgroup of G is normal.

By Corollary 2.8, there is a unique Sylow p_i-subgroup P_i of order $p_i^{r_i}$ for each p_i. We claim that G is the direct product of the P_i's. Observe that if $g_i \in P_i$ and $g_j \in P_j$ for $i \neq j$, then

$$\left[g_i,\ g_j\right] \in P_i \cap P_j = 1$$

by Lagrange's Theorem and normality of P_i and P_j. Thus, the elements of P_i commute with the elements of P_j whenever $i \neq j$. Now, define the map

$$\varphi : P_1 \times \cdots \times P_n \to G \text{ by } \varphi(g_1, \ldots, g_n) = g_1 \cdots g_n.$$

By the observation above, we have that φ is a homomorphism. We claim that φ is injective. Suppose that

$$\varphi(h_1, \ldots, h_n) = h_1 \cdots h_n = 1$$

for some $h_i \in P_i$. Since the h_i and h_j commute and have coprime order when $i \neq j$, we have

$$|h_1 h_2 \cdots h_n| = |h_1||h_2| \cdots |h_n| = 1.$$

This means that $|h_1| = |h_2| = \cdots = |h_n| = 1$, and thus $h_1 = h_2 = \cdots = h_n = 1$. And so, *ker* φ is trivial. This proves the claim. Since φ is an injective map between finite groups of equal order, it is an isomorphism. Therefore, G is a direct product of its Sylow subgroups.

Next, we prove (vi) $\Leftrightarrow$ (vii). Suppose that $G = P_1 \times \cdots \times P_n$ for Sylow p_i-subgroups P_i (here, of course, the p_i are distinct primes). Let $g = g_1 \cdots g_n$ and $h = h_1 \cdots h_n$ be elements of coprime order in G, where $g_i,\ h_i \in P_i$. Since

$$\left[g_i,\ g_j\right] = \left[h_i,\ h_j\right] = 1$$

when $i \neq j$, we have $|g| = |g_1| \cdots |g_n|$ and $|h| = |h_1| \cdots |h_n|$. Now, $|g|$ and $|h|$ are coprime only if one of the g_i or h_i equals 1 for each $i = 1, 2, \ldots, n$. We conclude that $gh = hg$.

Conversely, suppose that the elements of coprime order commute. Let $p_1, \ldots, p_n$ be the distinct prime divisors of $|G|$, and let $P_1, \ldots, P_n$ be corresponding Sylow subgroups associated with these primes. We assert that $G \cong P_1 \times \cdots \times P_n$. Let $g \in G$ and $h \in P_i$ for some $1 \leq i \leq n$. Clearly, $h^g \in P_i$ if $g \in P_i$. If $g \notin P_i$, then $|g|$ is coprime to $|h|$. By assumption, $[g,\ h] = 1$, and thus $h^g = h \in P_i$. And so, $P_i \trianglelefteq G$. Furthermore, $G = P_1 P_2 \cdots P_n$ because P_i and P_j are commuting subgroups for $i \neq j$. Finally, we find that

$$gp(P_1, \ldots, \widehat{P_i}, \ldots, P_n) \cap P_i = 1$$

for any $1 \leq i \leq n$ by Lagrange's Theorem. Here, $\widehat{P_i}$ means that P_i is omitted from the collection $P_1, \ldots, P_n$. This proves the assertion.

It remains to prove that (vii) ⇒ (i). Suppose that the elements of coprime order in G commute. By (vii) ⇒ (vi), G is a direct product of its Sylow subgroups. Since the Sylow subgroups have prime power order, each of them is nilpotent by Theorem 2.3. The result follows from Theorem 2.8. □

2.5 The Tensor Product of the Abelianization

Tensor products serve as a useful tool in the study of nilpotent groups. In this section, we discuss the connection between the factors $\gamma_i G/\gamma_{i+1} G$ of the lower central series of a group G and the i-fold tensor product of $Ab(G)$, the abelianization of G. In particular, we demonstrate that certain properties of a nilpotent group are inherited from its abelianization.

2.5.1 The Three Subgroup Lemma

We begin with a result of P. Hall and L. Kalužnin.

Lemma 2.18 (Three Subgroup Lemma) *Let G be a group with subgroups H, K, and L. If $N \trianglelefteq G$ and any two of the following subgroups $[H, K, L]$, $[K, L, H]$, $[L, H, K]$ are subgroups of N, then the third subgroup is also a subgroup of N.*

Proof Let h, k, and l be any elements of the subgroups H, K, and L respectively. By Corollary 1.5, the groups $[H, K, L]$, $[K, L, H]$, and $[L, H, K]$ are generated by conjugates of commutators of the forms $\left[h, k^{-1}, l\right]$, $\left[k, l^{-1}, h\right]$, and $\left[l, h^{-1}, k\right]$ respectively. By Lemma 1.5,

$$\left[h, k^{-1}, l\right]^k \left[k, l^{-1}, h\right]^l \left[l, h^{-1}, k\right]^h = 1.$$

Without loss of generality, suppose that $[H, K, L]$ and $[K, L, H]$ are contained in N. Since $N \trianglelefteq G$, we have $\left[h, k^{-1}, l\right]^k \in N$ and $\left[k, l^{-1}, h\right]^l \in N$. Hence,

$$\left[l, h^{-1}, k\right] = \left[k, l^{-1}, h\right]^{-l} \left(\left[h, k^{-1}, l\right]^{-k}\right)^{h^{-1}}$$

belongs to N, and consequently, $[L, H, K]$ is contained in N. □

Corollary 2.9 *If H, K, and L are normal subgroups of a group G, then*

$$[H, K, L] \leq [K, L, H][L, H, K].$$

Proof The result follows from Corollary 1.3 by putting $N = [K, L, H][L, H, K]$ in Lemma 2.18. □

The Three Subgroup Lemma plays a fundamental role in establishing certain connections between the commutators of the upper and lower central subgroups.

Theorem 2.14 (P. Hall) *Let G be any group and $i, j \in \mathbb{N}$.*

(i) $\left[\gamma_i G, \gamma_j G\right] \leq \gamma_{i+j} G$;
(ii) $\gamma_i\left(\gamma_j G\right) \leq \gamma_{ij} G$;
(iii) *If $j \geq i$, then* $\left[\gamma_i G, \zeta_j G\right] \leq \zeta_{j-i} G$.

Proof The proofs of (i), (ii), and (iii) are done by induction on i.

(i) If $i = 1$, then $\left[\gamma_1 G, \gamma_j G\right] = \gamma_{1+j} G$ by Definition 2.4. Assume that $i > 1$ and the result holds for $i - 1$. By definition,

$$\left[\gamma_i G, \gamma_j G\right] = \left[\left[\gamma_1 G, \gamma_{i-1} G\right], \gamma_j G\right] = \left[\gamma_1 G, \gamma_{i-1} G, \gamma_j G\right].$$

We examine the subgroups obtained by permuting the entries of $\left[\gamma_1 G, \gamma_{i-1} G, \gamma_j G\right]$. Observe that

$$\left[\gamma_{i-1} G, \gamma_j G, \gamma_1 G\right] = \left[\left[\gamma_{i-1} G, \gamma_j G\right], \gamma_1 G\right] \leq \left[\gamma_{i-1+j} G, \gamma_1 G\right] = \gamma_{i+j} G$$

and

$$\left[\gamma_j G, \gamma_1 G, \gamma_{i-1} G\right] = \left[\left[\gamma_j G, \gamma_1 G\right], \gamma_{i-1} G\right] = \left[\gamma_{j+1} G, \gamma_{i-1} G\right] \leq \gamma_{i+j} G.$$

Setting $N = \gamma_{i+j} G$ in Lemma 2.18 gives $\left[\gamma_i G, \gamma_j G\right] = \left[\gamma_1 G, \gamma_{i-1} G, \gamma_j G\right] \leq \gamma_{i+j} G$.

(ii) The result is obvious when $i = 1$. Suppose that $i > 1$, and assume that the result holds for $i - 1$. By (i), we have

$$\gamma_i\left(\gamma_j G\right) = \left[\gamma_{i-1}\left(\gamma_j G\right), \gamma_j G\right] \leq \left[\gamma_{(i-1)j} G, \gamma_j G\right] \leq \gamma_{(i-1)j+j} G = \gamma_{ij} G.$$

(iii) If $i = 1$, then $\left[\gamma_1 G, \zeta_j G\right] = \left[G, \zeta_j G\right] \leq \zeta_{j-1} G$ and the result holds by Lemma 2.1. Let $j \geq i > 1$, and suppose that the result is true for $i - 1$. By induction and Lemma 2.1, we have

$$\left[G, \zeta_j G, \gamma_{i-1} G\right] = \left[\left[G, \zeta_j G\right], \gamma_{i-1} G\right] \leq \left[\zeta_{j-1} G, \gamma_{i-1} G\right] \leq \zeta_{j-i} G$$

and

$$\left[\zeta_j G,\ \gamma_{i-1}G,\ G\right] = \left[\left[\zeta_j G,\ \gamma_{i-1}G\right],\ G\right] = \left[\zeta_{j-i+1}G,\ G\right] \le \zeta_{j-i}G.$$

Lemma 2.18 ultimately gives

$$\left[\gamma_i G,\ \zeta_j G\right] = \left[\gamma_{i-1}G,\ G,\ \zeta_j G\right] \le \zeta_{j-i}G.$$

This completes the proof. □

2.5.2 *The Epimorphism* $\bigotimes_{\mathbb{Z}}^{n} Ab(G) \to \gamma_n G/\gamma_{n+1}G$

We illustrate how the abelianization of a group influences the factors of its lower central series.

Definition 2.13 Suppose that $A,\ B$, and M are R-modules. A function $\varphi : A \times B \to M$ is called *bilinear* if, for all $a,\ a_1,\ a_2 \in A,\ b,\ b_1,\ b_2 \in B$, and $r \in R$, we have:

$$\begin{aligned}\varphi(a_1 + a_2,\ b) &= \varphi(a_1,\ b) + \varphi(a_2,\ b);\\ \varphi(a,\ b_1 + b_2) &= \varphi(a,\ b_1) + \varphi(a,\ b_2);\\ \varphi(ra,\ b) &= \varphi(a,\ rb) = r\varphi(a,\ b).\end{aligned}$$

If the R-modules are written using multiplicative notation, then the conditions above become:

$$\begin{aligned}\varphi(a_1 a_2,\ b) &= \varphi(a_1,\ b)\varphi(a_2,\ b);\\ \varphi(a,\ b_1 b_2) &= \varphi(a,\ b_1)\varphi(a,\ b_2);\\ \varphi(a^r,\ b) &= \varphi(a,\ b^r) = (\varphi(a,\ b))^r.\end{aligned}$$

In this case, φ is said to be *multiplicative in each variable*. In what follows, all $\mathbb{Z}$-modules (equivalently, abelian groups) are written multiplicatively.

Theorem 2.15 (D. J. S. Robinson) *Let G be any group. For each integer $n > 1$, the mapping*

$$\psi : \gamma_{n-1}G/\gamma_n G \bigotimes_{\mathbb{Z}} Ab(G) \to \gamma_n G/\gamma_{n+1}G$$

defined by

$$\psi(x\gamma_n G \otimes y\gamma_2 G) = [x,\ y]\gamma_{n+1}G \quad (x \in \gamma_{n-1}G,\ y \in G)$$

is a well-defined $\mathbb{Z}$-module epimorphism.

Proof Consider the function

$$\varphi_n : \gamma_{n-1}G/\gamma_n G \times Ab(G) \to \gamma_n G/\gamma_{n+1}G$$

defined by

$$(x\gamma_n G,\ y\gamma_2 G) \mapsto [x,\ y]\gamma_{n+1}G \quad (x \in \gamma_{n-1}G,\ y \in G).$$

We claim that φ_n is well defined and multiplicative in each variable.

- φ_n is well defined.
  (i) Let $g \in G$, $g_{n-1} \in \gamma_{n-1}G$, and $g_n \in \gamma_n G$. By Theorem 2.14 (i), the commutators $[g_n,\ g]$ and $[g_{n-1},\ g,\ g_n]$ are contained in $\gamma_{n+1}G$. By Lemma 1.4 (v), we have

$$\begin{aligned}\varphi_n(g_{n-1}g_n\gamma_n G,\ g\gamma_2 G) &= [g_{n-1}g_n,\ g]\gamma_{n+1}G\\ &= [g_{n-1},\ g][g_{n-1},\ g,\ g_n][g_n,\ g]\gamma_{n+1}G\\ &= [g_{n-1},\ g]\gamma_{n+1}G\\ &= \varphi_n(g_{n-1}\gamma_n G,\ g\gamma_2 G).\end{aligned}$$

  (ii) Let $g \in G$, $g_{n-1} \in \gamma_{n-1}G$, and $g_2 \in \gamma_2 G$. The commutators $[g_{n-1},\ g_2]$ and $[g_{n-1},\ g,\ g_2]$ are elements of $\gamma_{n+1}G$ by Theorem 2.14 (i). An application of Lemma 1.4 (vi) gives

$$\begin{aligned}\varphi_n(g_{n-1}\gamma_n G,\ gg_2\gamma_2 G) &= [g_{n-1},\ gg_2]\gamma_{n+1}G\\ &= [g_{n-1},\ g_2][g_{n-1},\ g][g_{n-1},\ g,\ g_2]\gamma_{n+1}G\\ &= [g_{n-1},\ g]\gamma_{n+1}G\\ &= \varphi_n(g_{n-1}\gamma_n G,\ g\gamma_2 G).\end{aligned}$$

  Hence, φ_n is well defined. Consequently, φ_n naturally extends to a $\mathbb{Z}$-module homomorphism from the free $\mathbb{Z}$-module on $\gamma_{n-1}G/\gamma_n G \times Ab(G)$ to $\gamma_n G/\gamma_{n+1}G$.

- φ_n is multiplicative in each variable.
  (i) Let $a_1,\ a_2 \in \gamma_{n-1}G$. By Theorem 2.14 (i), $[a_1,\ g,\ a_2] \in \gamma_{n+1}G$. Thus,

$$\begin{aligned}\varphi_n(a_1a_2\gamma_n G,\ g\gamma_2 G) &= [a_1a_2,\ g]\gamma_{n+1}G\\ &= [a_1,\ g][a_1,\ g,\ a_2][a_2,\ g]\gamma_{n+1}G\\ &= [a_1,\ g][a_2,\ g]\gamma_{n+1}G\\ &= \varphi_n(a_1\gamma_n G,\ g\gamma_2 G)\varphi_n(a_2\gamma_n G,\ g\gamma_2 G).\end{aligned}$$

(ii) Let $b_1,\ b_2 \in G$. Since $[g_{n-1},\ b_1,\ b_2] \in \gamma_{n+1}G$ by Theorem 2.14 (i), we have

$$\begin{aligned}\varphi_n(g_{n-1}\gamma_n G,\ b_1 b_2\gamma_2 G) &= [g_{n-1},\ b_1 b_2]\gamma_{n+1}G\\ &= [g_{n-1},\ b_2][g_{n-1},\ b_1][g_{n-1},\ b_1,\ b_2]\gamma_{n+1}G\\ &= [g_{n-1},\ b_2][g_{n-1},\ b_1]\gamma_{n+1}G\\ &= [g_{n-1},\ b_1][g_{n-1},\ b_2]\gamma_{n+1}G\\ &= \varphi_n(g_{n-1}\gamma_n G,\ b_1\gamma_2 G)\varphi_n(g_{n-1}\gamma_n G,\ b_2\gamma_2 G).\end{aligned}$$

This shows that φ_n is multiplicative in each variable.

By the Universal Mapping Property of the Tensor Product, there is an induced $\mathbb{Z}$-module homomorphism from the tensor product $\gamma_{n-1}G/\gamma_n G \bigotimes_{\mathbb{Z}} Ab(G)$ to $\gamma_n G/\gamma_{n+1}G$ given by

$$x\gamma_n G \otimes y\gamma_2 G \mapsto [x,\ y]\gamma_{n+1}G \quad (x \in \gamma_{n-1}G,\ y \in G).$$

This map is an epimorphism since $\gamma_n G = [\gamma_{n-1}G,\ G]$. □

Remark 2.11 Theorem 2.15 also holds for groups which come equipped with operator domains. See [7].

In the next few results, we exploit Theorem 2.15. Some notation is needed. If M is an R-module, then the n-fold tensor product of M is written as

$$\bigotimes_R^n M = \underbrace{M \otimes_R \cdots \otimes_R M}_{n}.$$

By convention, we set $\bigotimes_R^1 M = M$.

Corollary 2.10 *Let G be any group. For each $n \in \mathbb{N}$, the mapping*

$$\varphi_n : \bigotimes_{\mathbb{Z}}^n Ab(G) \to \gamma_n G/\gamma_{n+1}G$$

defined by

$$\varphi_n(x_1\gamma_2 G \otimes \cdots \otimes x_n\gamma_2 G) = [x_1,\ \ldots,\ x_n]\gamma_{n+1}G$$

is a $\mathbb{Z}$-module epimorphism.

Proof This easily follows by induction on n. □

Corollary 2.11 *Suppose that G is a finitely generated group with generating set $X = \{x_1,\ \ldots,\ x_k\}$. For each $n \in \mathbb{N}$, the factor group $\gamma_n G/\gamma_{n+1}G$ is finitely* *generated, modulo $\gamma_{n+1}G$, by the simple commutators of weight n of the form $[x_{i_1}, \ldots, x_{i_n}]$, where the x_{i_j}'s vary over all elements of X and are not necessarily distinct.*

Proof Since G is finitely generated by X, $Ab(G)$ is finitely generated by the elements $x_1\gamma_2G, \ldots, x_k\gamma_2G$. Hence, $\bigotimes_{\mathbb{Z}}^n Ab(G)$ is finitely generated by the k^n n-fold tensor products of the form

$$x_{i_1}\gamma_2G \otimes \cdots \otimes x_{i_n}\gamma_2G,$$

where the x_{i_j} vary over X. It follows from Corollary 2.10 that $\gamma_nG/\gamma_{n+1}G$ is finitely generated by the simple commutators, modulo $\gamma_{n+1}G$, of the form $[x_{i_1}, \ldots, x_{i_n}]$, where the x_{i_j}'s vary over all elements of X. □

Remark 2.12 Corollary 2.11 could also be proven using Lemma 2.6. Notice however, that Lemma 2.6 allows inverses of elements of the generating set in the simple commutators, whereas the corollary does not. This issue can be resolved by a repeated application of Lemmas 1.4 and 1.13.

Example 2.19 Let G be a group generated by $X = \{x_1, x_2, x_3\}$. If $g = x_2^3x_1^{-1}$ and $h = x_1x_2^{-4}x_3^2$ are elements of G, then $[g, h]\gamma_3G \in \gamma_2G/\gamma_3G$. Using Lemmas 1.4 and 1.13, together with the fact that all simple commutators of weight 2 are central, modulo γ_3G, we have

$$\begin{aligned}
[g, h]\gamma_3G &= \left[x_2^3x_1^{-1}, x_1x_2^{-4}x_3^2\right]\gamma_3G \\
&= \left[x_2^3, x_1x_2^{-4}x_3^2\right]\left[x_1^{-1}, x_1x_2^{-4}x_3^2\right]\gamma_3G \\
&= \left[x_2^3, x_1\right]\left[x_2^3, x_2^{-4}\right]\left[x_2^3, x_3^2\right]\left[x_1^{-1}, x_1\right]\left[x_1^{-1}, x_2^{-4}\right]\left[x_1^{-1}, x_3^2\right]\gamma_3G \\
&= [x_2, x_1]^3[x_2, x_2]^{-12}[x_2, x_3]^6[x_1, x_1]^{-1}[x_1, x_2]^4[x_1, x_3]^{-2}\gamma_3G \\
&= [x_2, x_1]^3[x_2, x_3]^6[x_1, x_2]^4[x_3, x_1]^2\gamma_3G,
\end{aligned}$$

which illustrates that $[g, h]$ modulo γ_3G is expressible as a product of commutators of weight 2 in the elements of X.

Corollary 2.10 can be used to prove that a nilpotent group is finitely generated whenever its abelianization is finitely generated. We need some preliminary material.

Definition 2.14 A group G is said to satisfy condition *Max* (the *maximal condition on subgroups*) if every subgroup of G is finitely generated.

A group in which every ascending series of subgroups stabilizes is said to satisfy the *Noetherian condition*.

Theorem 2.16 *A group G satisfies Max if and only if it satisfies the Noetherian condition.*

Proof Suppose that G satisfies Max, and let

$$H_1 < H_2 < H_3 < \cdots$$

be an ascending series of subgroups of G. We assert that this series stabilizes. Put $H = \bigcup_{i=1}^{\infty} H_i$. Clearly, H is a subgroup of G and is finitely generated by hypothesis. Let $X = \{h_1, \ldots, h_k\}$ be a set of generators of H. It is evident that each element of X is contained in some H_i since X generates H. Thus, there exists $n \in \mathbb{N}$ such that $X \subset H_n$. It follows that $H \leq H_n$. Since $H_n \leq H$, we have $H = H_n$ and the series stabilizes.

Conversely, suppose that every ascending series of subgroups stabilizes. Let H be a subgroup of G, and choose an element $h_1 \in H$. If $H = gp(h_1)$, then H is finitely generated. Otherwise, there exists an element $h_2 \in H$ such that $h_2 \notin gp(h_1)$. Now, if $H = gp(h_1, h_2)$, then H is finitely generated. If $H \neq gp(h_1, h_2)$, then we continue this argument to obtain an ascending series of subgroups

$$gp(h_1) \leq gp(h_1, h_2) \leq \cdots$$

which stabilizes by assumption. Hence, $H = gp(h_1, h_2, \ldots, h_n)$ for some $n \in \mathbb{N}$. And so, H is finitely generated. □

Groups which satisfy Max must be finitely generated. There are finitely generated groups, however, which do not satisfy Max. For example, let $F = \langle x, y \rangle$ be the free group of rank two, and let

$$G_i = gp\left(x, yxy^{-1}, \ldots, y^i x y^{-i}\right).$$

Every element of G_i can be written as

$$y^{m_1} x^{n_1} y^{m_2 - m_1} x^{n_2} y^{m_3 - m_2} \cdots y^{-m_k} \quad (0 \leq m_r \leq i).$$

Thus, $y^{i+1} x y^{-(i+1)}$ is not an element of G_i. This implies that the ascending sequence of subgroups

$$G_1 < G_2 < G_3 < \cdots$$

does not stabilize. By Theorem 2.16, F does not satisfy Max.

Lemma 2.19 *Max is preserved under extensions.*

Proof Let G be a group with $N \trianglelefteq G$, and suppose that G/N and N satisfy Max. Let H be any subgroup of G. Clearly, $H \cap N$ is finitely generated since $H \cap N < N$ and N satisfies Max. By the Second Isomorphism Theorem,

$$H/(H \cap N) \cong HN/N < G/N.$$

This implies that $H/(H \cap N)$ is finitely generated because G/N satisfies Max. It follows that H is finitely generated. □

Theorem 2.17 *Every finitely generated abelian group satisfies Max.*

Proof Let G be a finitely generated abelian group with generating set $\{x_1, \ldots, x_k\}$. The proof is done by induction on k. If $k = 1$, then G is cyclic. In this case, it is easy to show that $[G : H] < \infty$ for every nontrivial subgroup H of G. Hence, H must be finitely generated.

Suppose that the theorem is true for $1 \leq i \leq k - 1$, and consider the subgroup $H = gp(x_1, \ldots, x_{k-1})$ of G. Since H is finitely generated and abelian, H satisfies Max by induction. Furthermore, $G/H \cong gp(x_k)$ is cyclic, and thus satisfies Max. The result follows from Lemma 2.19. □

Theorem 2.18 (R. Baer) *Every finitely generated nilpotent group satisfies Max.*

Proof Let G be a finitely generated nilpotent group of class c, and let $H \leq G$. Set $H_i = H \cap \gamma_i G$ for $1 \leq i \leq c$. It follows from Lemma 2.1 that the series

$$H = H_1 \geq H_2 \geq \cdots \geq H_c \geq H_{c+1} = 1$$

is a central series for H. Furthermore, the Second Isomorphism Theorem gives

$$\frac{H_i}{H_{i+1}} = \frac{H \cap \gamma_i G}{H \cap \gamma_{i+1} G} = \frac{H \cap \gamma_i G}{(H \cap \gamma_i G) \cap \gamma_{i+1} G} \cong \frac{\gamma_{i+1} G (H \cap \gamma_i G)}{\gamma_{i+1} G}$$

for $1 \leq i \leq c$. Therefore, each H_i/H_{i+1} is isomorphic to a subgroup of $\gamma_i G/\gamma_{i+1} G$. Since $\gamma_i G/\gamma_{i+1} G$ is finitely generated and abelian by Corollary 2.11, so is H_i/H_{i+1} by Theorem 2.17. In particular, $H_c = H_c/H_{c+1}$ is finitely generated. Thus, H_{c-1} is finitely generated since both H_{c-1}/H_c and H_c are finitely generated. Repeating this argument gives that H_i is finitely generated for $1 \leq i \leq c - 2$. In particular, $H_1 = H$ is finitely generated. □

We now prove that nilpotent groups with finitely generated abelianization must be finitely generated.

Corollary 2.12 *If G is a nilpotent group and $Ab(G)$ is finitely generated, then G satisfies Max. Hence, G is finitely generated.*

Proof The proof is done by induction on the class c of G. Theorem 2.17 takes care of the case $c = 1$. Assume that the corollary is true for nilpotent groups of class less than c, and let $n \in \{1, \ldots, c\}$. The tensor product $\bigotimes_{\mathbb{Z}}^n Ab(G)$ is finitely generated because it involves a finite number of finitely generated abelian groups. By Corollary 2.10, each $\gamma_n G/\gamma_{n+1} G$ is finitely generated abelian, and thus satisfies Max by Theorem 2.17. In particular, $\gamma_c G$ satisfies Max. By the induction hypothesis, $G/\gamma_c G$ also satisfies Max. The result now follows from Lemma 2.19. □

2.5.3 Property $\mathscr{P}$

The proof of Corollary 2.12 shows that certain properties of the abelianization of a nilpotent group can be passed on to the group itself. This is the substance of the next result.

Definition 2.15 A group-theoretical property is called *property* $\mathscr{P}$ if it satisfies the following criteria:

1. Property $\mathscr{P}$ is preserved under extensions.
2. If G is an abelian group having property $\mathscr{P}$ and $k \in \mathbb{N}$, then any homomorphic image of the k-fold tensor product $\bigotimes_{\mathbb{Z}}^{k} G$ has property $\mathscr{P}$.

It is clear that finiteness is a property $\mathscr{P}$. Other possibilities for property $\mathscr{P}$ include finite generation, P-torsion for a set of primes P (see Lemma 2.13), and Max (see Lemma 2.19 and the proof of Corollary 2.12).

Theorem 2.19 (D. J. S. Robinson) *If G is nilpotent and $Ab(G)$ has property $\mathscr{P}$, then G has property $\mathscr{P}$.*

Proof Suppose that G is of class c, and let $k > 0$. By Corollary 2.10, $\gamma_k G/\gamma_{k+1} G$ is an image of the k-fold tensor product $\bigotimes_{\mathbb{Z}}^{k} Ab(G)$. Thus, each $\gamma_k G/\gamma_{k+1} G$ has property $\mathscr{P}$ because $Ab(G)$ does. Now, $\gamma_{c+1} G = 1$ by Theorem 2.3. This means that $\gamma_c G$ has property $\mathscr{P}$. Since $\gamma_{c-1} G/\gamma_c G$ has property $\mathscr{P}$ and $\gamma_{c-1} G$ is an extension of $\gamma_{c-1} G/\gamma_c G$ by $\gamma_c G$, we have that $\gamma_{c-1} G$ also has property $\mathscr{P}$. We continue this argument to conclude that G has property $\mathscr{P}$. □

Definition 2.16 The *exponent* of a torsion group G is the smallest natural number m, if it exists, satisfying $g^m = 1$ for every $g \in G$. If no such m exists, then G has *infinite exponent*.

Every finite group has finite exponent dividing the order of the group. For any prime p, both the infinite direct product

$$\mathbb{Z}_p \times \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^3} \times \cdots$$

and the p-quasicyclic group are infinite torsion groups with infinite exponent. Thus, torsion groups need not be finite nor have finite exponent. A group with infinite exponent is necessarily infinite. However, the infinite direct product of cyclic groups of order p is an example of an infinite group with finite exponent.

Theorem 2.20 (S. Dixmier) *Let G be a nilpotent group of class c. If $Ab(G)$ has finite exponent m, then G has finite exponent dividing m^c.*

Proof The exponent of $\bigotimes_{\mathbb{Z}}^{i} Ab(G)$ divides m for $1 \le i \le c$ because $Ab(G)$ has exponent m. Thus, $\gamma_i G/\gamma_{i+1} G$ also has exponent dividing m by Corollary 2.10. In particular, $\gamma_c G = \gamma_c G/\gamma_{c+1} G$ has exponent dividing m. This, combined with the fact that $\gamma_{c-1} G/\gamma_c G$ also has exponent dividing m, gives that the exponent of $\gamma_{c-1} G$ divides m^2. We iterate this process to finally obtain that the exponent of $G = \gamma_{c-(c-1)} G$ divides m^c. □

2.5.4 The Hirsch-Plotkin Radical

We end this section with an important result whose proof depends on Theorem 2.18. Motivated by Theorem 2.11, it is natural to ask whether or not a group has a *maximal* normal nilpotent subgroup.

Definition 2.17 A maximal normal nilpotent subgroup of a group is called a *nilpotent radical* of the group.

One attempt to construct a nilpotent radical is by trying to use Zorn's Lemma. Suppose that

$$N_1 < N_2 < N_3 < \cdots$$

is an ascending chain of normal nilpotent subgroups of a group G, where N_i is of class c_i for $i = 1, 2, \ldots$. A nilpotent radical would exist if $\cup_{i=1}^{k} N_i$ were normal and nilpotent for all $k \geq 1$. However, it is not nilpotent since the class of

$$\bigcup_{i=1}^{k} N_i = N_1 \cdots N_k,$$

which is $c_1 + \cdots + c_k$ according to Theorem 2.11, becomes unbounded as k approaches infinity. Hence, Zorn's Lemma does not apply.

Even though the nilpotent radical doesn't always exist, one can always find a *locally nilpotent radical*. This is the basis of our next discussion.

Definition 2.18 A group G is called *locally nilpotent* if every finitely generated subgroup of G is nilpotent.

Clearly, every nilpotent group is locally nilpotent. If $G = \prod_{i=1}^{\infty} G_i$, where each G_i is nilpotent of class c_i and $c_k < c_{k+1}$ for $k \geq 1$, then G is locally nilpotent. In particular, $\prod_{i=1}^{\infty} \mathbb{Z}_{p^i}$ and $\prod_{i=1}^{\infty} UT_i(\mathbb{Z})$ are locally nilpotent.

Lemma 2.20 *(i) Every nilpotent group is locally nilpotent.*
(ii) Every subgroup of a locally nilpotent group is locally nilpotent.
(iii) Every homomorphic image of a locally nilpotent group is locally nilpotent.

Proof

(i) This is immediate from Theorem 2.4.
(ii) Let G be a locally nilpotent group, and suppose that $H < G$. If K is a finitely generated subgroup of H, then it is also a finitely generated subgroup G. Since G is locally nilpotent, K is nilpotent, and thus K is a nilpotent subgroup of H. This means that H is locally nilpotent.
(iii) Let G be a locally nilpotent group, and suppose that $\varphi \in Hom(G, H)$ for some group H. Let K be a finitely generated subgroup of $\varphi(G)$ with finite generating set $\{x_1, \ldots, x_m\}$. There exist elements $g_1, \ldots, g_m$ in G such that $\varphi(g_i) = x_i$

for $1 \leq i \leq m$. Consider the subgroup $L = gp(g_1, \ldots, g_m)$ of G. It is finitely generated, and thus nilpotent since G is locally nilpotent. By Theorem 2.4, $\varphi(L) = K$ is also nilpotent. And so, $\varphi(G)$ is locally nilpotent. □

Theorem 2.21 (K. Hirsch, B. Plotkin) *If H and K are normal locally nilpotent subgroups of a group G, then HK is a normal locally nilpotent subgroup of G.*

Proof We adopt the proof given by D.J.S. Robinson in [8]. Clearly, $HK \trianglelefteq G$ since $H \trianglelefteq G$ and $K \trianglelefteq G$. We claim that HK is locally nilpotent. Let

$$\{h_1, \ldots, h_m\} \subset H \text{ and } \{k_1, \ldots, k_m\} \subset K.$$

Then $\{h_1k_1, \ldots, h_mk_m\} \subset HK$. Define the subgroups

$$A = gp(h_1, \ldots, h_m) \leq H \text{ and } B = gp(k_1, \ldots, k_m) \leq K,$$

and set $C = gp(A, B)$ and $S = gp(h_1k_1, \ldots, h_mk_m)$. In order to prove the claim, we need to establish that S is nilpotent. Since $S \leq C$, it suffices to show that C is nilpotent.

Define the set $T = \{[h_i, k_j] \mid i, j = 1, \ldots, m\}$, and observe that $T \subseteq H \cap K$ since $H \trianglelefteq G$ and $K \trianglelefteq G$. Clearly, both A and T are finitely generated and contained in H. Thus, $gp(A, T)$ is a finitely generated subgroup of H. Since H is locally nilpotent, $gp(A, T)$ is also nilpotent. By Theorems 2.4 and 2.18, the normal closure T^A of T in $gp(A, T)$ is finitely generated and nilpotent. Furthermore, $T^A \leq H \cap K$, and consequently, $gp(B, T^A) \leq K$. Therefore, $gp(B, T^A)$ is finitely generated and nilpotent. By Corollary 1.5, we have $[A, B] = (T^A)^B$. Hence,

$$gp(B, T^A) = gp\left(B, (T^A)^B\right) = gp(B, [A, B]) = B^A.$$

It follows that B^A is nilpotent, and similarly, A^B is nilpotent. By Theorem 2.11, $A^BB^A = C$ is nilpotent. □

Corollary 2.13 *Every group G has a unique maximal normal locally nilpotent subgroup containing all normal locally nilpotent subgroups of G.*

This subgroup is called the *Hirsch-Plotkin radical* of G.

Proof If $N_1 < N_2 < \cdots$ is a chain of locally nilpotent subgroups of G, then $\cup_{i=1}^{\infty} N_i$ is locally nilpotent. By Zorn's Lemma, each normal locally nilpotent subgroup of G is contained in a maximal normal locally nilpotent subgroup of G.

We establish uniqueness. Suppose that M_1 and M_2 are both maximal normal locally nilpotent subgroups of G. By Theorem 2.21, the product M_1M_2 is locally nilpotent. The maximality of M_1 and M_2 implies that $M_1 = M_1M_2 = M_2$. □

The Hirsch-Plotkin radical is a valuable tool for studying various generalized nilpotent groups. We refer the reader to [8] for a discussion of such groups.

2.5.5 An Extension Theorem for Nilpotent Groups

The symmetric group S_3 is an extension of S_3/A_3 by A_3, groups of order 2 and 3 respectively. Both of these groups are cyclic (hence, nilpotent). However, S_3 is not nilpotent. This illustrates that nilpotency is not preserved under extensions. The next theorem addresses the following question: when is an extension of a nilpotent group by another group again nilpotent?

Theorem 2.22 (P. Hall, A. G. R. Stewart) *Let G be any group, and suppose that $N \lhd G$. If N is nilpotent of class c and $G/\gamma_2 N$ is nilpotent of class d, then G is nilpotent of class at most $cd + (c-1)(d-1)$.*

In [5], P. Hall initially found the bound on the class of G to be at most

$$\binom{c+1}{2}d - \binom{c}{2}.$$

A. G. R. Stewart improved on this in [12] and obtained the bound to be at most

$$cd + (c-1)(d-1).$$

In the same paper, he provided an example to illustrate that this bound cannot be improved. We give A. G. R. Stewart's proof below. In what follows, we define

$$[N, \underbrace{G, \ldots, G}_{0}] = N.$$

Lemma 2.21 *Let G be any group. If $N \trianglelefteq G$, then*

$$[\gamma_2 N, \underbrace{G, G, \ldots, G}_{s}] \leq \prod_{k=1}^{m} S_k$$

for some $m \in \mathbb{N}$, where

$$S_k = [[N, \underbrace{G, \ldots, G}_{i}], [N, \underbrace{G, \ldots, G}_{s-i}]]$$

for some $i \in \{1, 2, \ldots, s\}$.

Proof The proof is done by induction on s. Suppose that $s = 1$. By Proposition 1.1 (i) and Corollary 2.9, we have

$$\begin{aligned}[\gamma_2 N,\ G] &\leq [N,\ G,\ N][G,\ N,\ N] \\ &= [N,\ G,\ N][N,\ G,\ N] \\ &= [N,\ G,\ N] \\ &= [[N,\ G],\ [N,\ \underbrace{G,\ \ldots,\ G}_{0}]]\end{aligned}$$

and the lemma holds. Next, assume that the lemma is true for $s - 1$:

$$[\gamma_2 N, \underbrace{G,\ \ldots,\ G}_{s-1}] \leq \prod_{k=1}^{n} T_k$$

for some $n \in \mathbb{N}$, where

$$T_k = [[N,\ \underbrace{G,\ \ldots,\ G}_{i}],\ [N,\ \underbrace{G,\ \ldots,\ G}_{s-i-1}]]$$

for some $i \in \{1,\ 2,\ \ldots,\ s-1\}$. Notice that

$$[\gamma_2 N, \underbrace{G,\ \ldots,\ G}_{s}] = [\gamma_2 N, \underbrace{G,\ \ldots,\ G}_{s-1},\ G] \leq \left[\prod_{k=1}^{n} T_k,\ G\right] = \prod_{k=1}^{n} [T_k,\ G]\,,$$

where the last equality follows from Lemma 1.10. By applying Proposition 1.1 (i) and Corollary 2.9, we get

$$\begin{aligned}[T_k,\ G] &= [[N,\ \underbrace{G,\ \ldots,\ G}_{i}],\ [N,\ \underbrace{G,\ \ldots,\ G}_{s-i-1}],\ G] \\ &\leq [[N,\ \underbrace{G,\ \ldots,\ G}_{(s-1)-i}],\ G,\ [N,\ \underbrace{G,\ \ldots,\ G}_{i}]][G,\ [N,\ \underbrace{G,\ \ldots,\ G}_{i}],\ [N,\ \underbrace{G,\ \ldots,\ G}_{s-i-1}]] \\ &= [[N,\ \underbrace{G,\ \ldots,\ G}_{s-i}],\ [N,\ \underbrace{G,\ \ldots,\ G}_{i}]][[N,\ \underbrace{G,\ \ldots,\ G}_{i+1}],\ [N,\ \underbrace{G,\ \ldots,\ G}_{s-(i+1)}]] \\ &= [[N,\ \underbrace{G,\ \ldots,\ G}_{i}],\ [N,\ \underbrace{G,\ \ldots,\ G}_{s-i}]][[N,\ \underbrace{G,\ \ldots,\ G}_{i+1}],\ [N,\ \underbrace{G,\ \ldots,\ G}_{s-(i+1)}]]\end{aligned}$$

and the result follows. □

We now prove Theorem 2.22. First, note that $\gamma_{c+1} N = 1$ and $\gamma_{d+1} G \leq \gamma_2 N$ by Theorem 2.2 because the classes of N and $G/\gamma_2 N$ are c and d respectively. The proof is done by induction on c. If $c = 1$, then N is abelian. In this case, $\gamma_2 N = 1$, and thus $G/\gamma_2 N \cong G$ is nilpotent of class d.

Next, suppose that $c > 1$, and assume that the theorem is true for $c - 1$. For any $r \in \{1, 2, \ldots, c\}$, $M_r = N/\gamma_{r+1}N$ is a normal subgroup of $H_r = G/\gamma_{r+1}N$, where M_r is of class r and $H_r/\gamma_2 M_r$ is of class d by the Third Isomorphism Theorem. Thus, we may assume by induction that

$$\gamma_{2rd-r-d+2}G \leq \gamma_{r+1}N \tag{2.19}$$

for all $r \in \{1, 2, \ldots, c-1\}$. We invoke Lemma 2.21 to find that

$$\gamma_{2cd-c-d+2}G = [\gamma_{d+1}G, \underbrace{G, \ldots, G}_{2cd-2d-c+1}] \leq [\gamma_2 N, \underbrace{G, \ldots, G}_{2cd-2d-c+1}] \leq \prod_{k=1}^{m} S_k$$

for some $m \in \mathbb{N}$, where

$$S_k = [[N, \underbrace{G, \ldots, G}_{i}], [N, \underbrace{G, \ldots, G}_{2cd-2d-c+1-i}]]$$

for some $i \in \{1, 2, \ldots, (2cd - 2d - c + 1)\}$. Now, each

$$i \in \{1, 2, \ldots, (2cd - 2d - c + 1)\}$$

is contained in one of the following sets:

$$2(j-1)d - d - (j-1) + 1 \leq i \leq 2jd - d - j + 1, \quad \text{where } j \in \{1, 2, \ldots, c\}.$$

For arbitrary j,

$$[[N, \underbrace{G, \ldots, G}_{i}], [N, \underbrace{G, \ldots, G}_{2cd-2d-c+1-i}]] \leq \left[\gamma_j N, \gamma_w G\right], \tag{2.20}$$

where

$$w = 2d(c-j) - d - (c-j) + 2 + 2dj - d - j - i.$$

The result follows from the fact that $[N, \underbrace{G, \ldots, G}_{t}] \leq \gamma_{t+1}G$. Since $2dj - d - j \geq 1$ and $\gamma_{r+s}G \leq \gamma_r G$ for all $s \geq 0$, we find that

$$\left[\gamma_j N, \gamma_w G\right] \leq \left[\gamma_j N, \gamma_{2d(c-j)-d-(c-j)+2}G\right]. \tag{2.21}$$

Substituting r by $(c - j)$ in (2.19) shows that

$$\left[\gamma_j N, \gamma_{2d(c-j)-d-(c-j)+2}G\right] \leq \left[\gamma_j N, \gamma_{c-j+1}N\right].$$

By Theorem 2.14 (i), $\left[\gamma_j N,\ \gamma_{c-j+1} N\right] \leq \gamma_{c+1} N$. We conclude that for all possible k, $S_k \leq \gamma_{c+1} N = 1$, and thus $\prod_{k=1}^{m} S_k = 1$. This completes the proof of Theorem 2.22.

2.6 Finitely Generated Torsion Nilpotent Groups

In [1], R. Baer proved that every finitely generated torsion nilpotent group is finite. This allows one to answer certain questions involving torsion in a nilpotent group by passing to a finite group. In this section, we focus on some of these questions. We begin with a result due to A. I. Mal'cev which contains R. Baer's theorem as a special case.

Theorem 2.23 (A. I. Mal'cev) *Let G be a finitely generated nilpotent group, and let $H \leq G$. If G has a finite set of generators X such that some positive power of each element of X is contained in H, then a positive power of every element of G is contained in H. Furthermore, H is of finite index in G.*

Proof The proof is done by induction on the class c of G. If $c = 1$, then G is a finitely generated abelian group and the result is clear.

Suppose that $c > 1$, and assume that the lemma is true for all finitely generated nilpotent groups of class less than c. By Lemma 2.8, $G/\gamma_c G$ is finitely generated nilpotent of class $c - 1$. By induction, $H\gamma_c G$ has finite index in G and a positive power of every element of G is contained in $H\gamma_c G$. We claim that a positive power of every element of G is contained in H and $[G : H] < \infty$.

Let $G = gp(g_1,\ g_2,\ \ldots,\ g_s)$ such that $g_i^{m_i} \in H$, where $m_i > 0$ and $1 \leq i \leq s$. By Theorem 2.18, $\gamma_{c-1} G$ is finitely generated. Suppose that $\gamma_{c-1} G = gp(x_1,\ x_2, \ldots, x_t)$ such that $x_j^{n_j} \in H\gamma_c G$, where $n_j > 0$ and $1 \leq j \leq t$. By Lemmas 1.4 and 1.13, together with Remark 2.5, we have

$$\gamma_c G = gp\left(\left[x_j,\ g_i\right] \mid 1 \leq i \leq s,\ 1 \leq j \leq t\right)$$

and

$$\left[x_j,\ g_i\right]^{n_j m_i} = \left[x_j^{n_j},\ g_i^{m_i}\right] \in [H\gamma_c G,\ H] = [H,\ H] \leq H$$

for $1 \leq i \leq s$ and $1 \leq j \leq t$. Since $\gamma_c G \leq Z(G)$, a positive power of every element of $\gamma_c G$ lies in H. If $g \in G$, then there exists $m \in \mathbb{N}$ such that $g^m = hz$, where $h \in H$ and $z \in \gamma_c G$. Furthermore, there exists $n \in \mathbb{N}$ such that $z^n \in H$. Thus,

$$g^{mn} = (hz)^n = h^n z^n \in H$$

since z is central. This means that a positive power of every element of G is contained in H.

Next, we show that $H\gamma_c G/H$ is a finite abelian group. This, together with the fact that $[G : H\gamma_c G] < \infty$, will give $[G : H] < \infty$ as claimed. By the Second Isomorphism Theorem, $H\gamma_c G/H$ is abelian since it is isomorphic to $\gamma_c G/(H \cap \gamma_c G)$, a quotient of the abelian group $\gamma_c G$. It is finite because it has a finite set of generators, each having finite order. More precisely, $\gamma_c G/(H \cap \gamma_c G)$ is finitely generated because $\gamma_c G$ is finitely generated, and each generator $\left[x_j,\ g_i\right](H \cap \gamma_c G)$ of $\gamma_c G/(H \cap \gamma_c G)$ has finite order since $\left[x_j,\ g_i\right]^{n_j m_i} \in H \cap \gamma_c G$. □

An analogue of Theorem 2.23 for a given nonempty set of primes is:

Theorem 2.24 *Let P be a nonempty set of primes. Suppose that G is a finitely generated nilpotent group and $H \leq G$. If G has a finite set of generators X such that some P-number power of each element of X is contained in H, then each element of G has a P-number power contained in H. Furthermore, $[G : H]$ is a P-number.*

The proof is the same as for Theorem 2.23.

Theorem 2.25 (R. Baer) *Let P be a nonempty set of primes. If there is a finite set of generators X of a finitely generated nilpotent group G for which each element of X has order a P-number, then G is a finite P-torsion group. In particular, finitely generated torsion nilpotent groups are finite.*

Proof Set $H = 1$ in Theorem 2.24. □

We point out that the finiteness of G in Theorem 2.25 is a consequence of the fact that the trivial subgroup $H = 1$ must be of finite index in G according to Theorem 2.24.

Corollary 2.14 *The elements of coprime order in any locally nilpotent group commute.*

Proof Let G be a locally nilpotent group, and suppose that g and h are elements of coprime order in G. The subgroup $H = gp(g,\ h)$ of G is finitely generated, and thus nilpotent. Since each generator g and h has finite order, H must be finite by Theorem 2.25. Therefore, g and h commute by Theorem 2.13. □

2.6.1 The Torsion Subgroup of a Nilpotent Group

If P is a nonempty set of primes and G is a group, then the set $\tau_P(G)$ of P-torsion elements of G is not necessarily a subgroup of G. For example, consider the (non-nilpotent) *infinite dihedral group*

$$D_\infty = \left\langle x,\ y \,\middle|\, x^2 = 1,\ y^2 = 1 \right\rangle.$$

Clearly, xy is not a torsion element, even though x and y are torsion elements. For nilpotent groups, however, we have:

Theorem 2.26 (R. Baer, K. A. Hirsch) *If G is a nilpotent group and P is any nonempty set of primes, then $\tau_P(G)$ is a normal subgroup of G. Furthermore, if $\mathbb{P}$ denotes the set of all prime numbers, then*

$$\tau(G) = \prod_{p \in \mathbb{P}} \tau_p(G).$$

This coincides with Theorem 2.13 in the case when G is finite.

Proof Let g and h be P-torsion elements. By Theorem 2.25, $gp(g,\ h)$ is a finite P-torsion group. Hence, $g^{-1}h$ is a P-torsion element, and thus $\tau_P(G)$ is a subgroup of G. It is easy to see that $\tau_P(G)$ is, in fact, normal in G. In particular, $\tau_p(G)$ is a normal p-subgroup of G for any prime p. Moreover, if q is a prime different from p, then $\left[\tau_p(G),\ \tau_q(G)\right] = 1$ by Corollary 2.14. Thus,

$$\prod_{p \in \mathbb{P}} \tau_p(G) = gp\big(\tau_p(G) \bigm| p \text{ varies over all of } \mathbb{P}\big). \tag{2.22}$$

We claim that the right-hand side of (2.22) is just $\tau(G)$. It is clearly contained in $\tau(G)$ by the previous discussion. We establish the reverse inclusion. Let $g \in \tau(G)$ be a torsion element of order $d = p_1^{m_1} \cdots p_n^{m_n}$ for some $m_1,\ \ldots,\ m_n \in \mathbb{N}$ and distinct primes $p_1,\ \ldots,\ p_n$. Define $a_i = d/p_i^{m_i}$ for $i = 1,\ \ldots,\ n$. Since $(g^{a_i})^{p_i^{m_i}} = 1$, we have $g^{a_i} \in \tau_{p_i}(G)$. Furthermore, the greatest common divisor of $a_1,\ \ldots,\ a_n$ is 1 because they are pairwise relatively prime. Thus, there are integers $s_1,\ \ldots,\ s_n$ such that $\sum_{i=1}^{n} a_i s_i = 1$. Hence,

$$\begin{aligned} g &= g^{a_1 s_1 + \cdots + a_n s_n} \\ &= (g^{a_1})^{s_1} \cdots (g^{a_n})^{s_n}, \end{aligned}$$

which is contained in $\tau_{p_1}(G)\tau_{p_2}(G) \cdots \tau_{p_n}(G)$. This proves the claim. □

Corollary 2.15 *Let P be a nonempty set of primes. If G is a nilpotent group, then $G/\tau_P(G)$ is P-torsion-free.*

Proof By Theorem 2.26, $\tau_P(G) \trianglelefteq G$. Suppose that $(g\tau_P(G))^n = \tau_P(G)$ for some $g\tau_P(G) \in G/\tau_P(G)$ and P-number n. We need to show that $g\tau_P(G) = \tau_P(G)$. Since $(g\tau_P(G))^n = \tau_P(G)$, we have $g^n \in \tau_P(G)$. Thus, there is a P-number m such that $g^{nm} = (g^n)^m = 1$. Since mn is a P-number, $g \in \tau_P(G)$; that is, $g\tau_P(G) = \tau_P(G)$. □

Corollary 2.16 *Let P be a nonempty set of primes. If G is a finitely generated nilpotent group, then $\tau_P(G)$ is a finite P-torsion group.*

Proof By Theorems 2.18 and 2.26, $\tau_P(G)$ is a finitely generated P-torsion nilpotent group. The result follows from Theorem 2.25. □

Theorem 2.26 holds for locally nilpotent groups as well.

Theorem 2.27 *If G is a locally nilpotent group and P is any nonempty set of primes, then $\tau_P(G) \trianglelefteq G$. If $\mathbb{P}$ denotes the set of all prime numbers, then*

$$\tau(G) = \prod_{p \in \mathbb{P}} \tau_p(G).$$

Proof Let $g, h \in \tau_P(G)$, and put $H = gp(g, h)$. Since H is a finitely generated subgroup of G, it is nilpotent. By Theorem 2.26, $\tau_P(H) \trianglelefteq H$. Therefore, $gh \in \tau_P(H)$, and thus $gh \in \tau_P(G)$. The rest of the proof is the same as for Theorem 2.26. □

An analogue of Corollary 2.15 clearly holds for locally nilpotent groups.

Corollary 2.17 *If P is a nonempty set of primes and G is a locally nilpotent group, then $G/\tau_P(G)$ is P-torsion-free.*

2.7 The Upper Central Subgroups and Their Factors

In this section, we focus our attention on some properties of the upper central subgroups and their factors.

2.7.1 Intersection of the Center and a Normal Subgroup

We begin by proving that every nontrivial normal subgroup of a nilpotent group contains a nonidentity central element.

Theorem 2.28 (K. A. Hirsch) *If G is a nilpotent group and N is a nontrivial normal subgroup of G, then $N \cap Z(G) \neq 1$.*

Proof If $N \leq Z(G)$, then the result is immediate. Suppose that $N \not\leq Z(G)$. Since G is nilpotent, there exists $i \in \mathbb{N}$ such that $N \cap \zeta_i G \neq 1$. If $i = 1$, then the result is immediate. Assume that $i > 1$, and let $n \in N \cap \zeta_i G$ for some $n \neq 1$. If $n \in Z(G)$, then we have the result. If $n \notin Z(G)$, then there exists $g \in G$ such that $[n, g] \neq 1$. Observe that $[n, g] \in [\zeta_i G, G] \leq \zeta_{i-1} G$ and $[n, g] \in N$ since $N \trianglelefteq G$. Thus, if $N \cap \zeta_i G \neq 1$, then $N \cap \zeta_{i-1} G \neq 1$ for $i > 1$. It follows that $N \cap Z(G) \neq 1$. □

Theorem 2.28 has several consequences.

Lemma 2.22 *Every maximal normal abelian subgroup of a nilpotent group G coincides with its centralizer in G.*

Proof The proof is done by contradiction. Let M be a maximal normal abelian subgroup of G, and assume that $M \neq C_G(M)$. Clearly, $M \leq C_G(M)$ and $C_G(M)/M$ is a nontrivial normal subgroup of G/M. By Theorem 2.28, there exists an element

$$gM \in Z(G/M) \cap (C_G(M)/M)$$

such that $g \notin M$. Now, $gp(g,\ M)$ is abelian because $g \in C_G(M)$. Moreover, $gp(g,\ M)$ is normal in G. To see this, let $g^k m \in gp(g,\ M)$ for some $k \in \mathbb{Z}$ and $m \in M$, and let $h \in G$. Since $g^k M \in Z(G/M)$, we have

$$h^{-1} g^k m h = g^k m_1 \in gp(g,\ M)$$

for some $m_1 \in M$. By the maximality of M, we have $g \in M$, a contradiction. □

Corollary 2.18 *Let G be a nilpotent group, and let K be any group. A homomorphism $\varphi \in Hom(G,\ K)$ is a monomorphism if and only if $\varphi|_{Z(G)}$, the restriction of φ to $Z(G)$, is a monomorphism.*

Proof Suppose that $\varphi|_{Z(G)}$ is a monomorphism. Assume, on the contrary, that φ is not a monomorphism. Then $ker\ \varphi$ is a nontrivial normal subgroup of G. By Theorem 2.28, $ker\ \varphi \cap Z(G) \neq 1$, and thus $\varphi|_{Z(G)}$ also has a nontrivial kernel. Thus, $\varphi|_{Z(G)}$ is not a monomorphism, a contradiction. The converse is clear. □

Definition 2.19 A nontrivial normal subgroup N of a group G is termed a *minimal normal subgroup* if there is no normal subgroup M of G such that $1 < M < N$.

Thus, if N is a minimal normal subgroup of G and $M \trianglelefteq N$, then either $M = 1$ or $M = N$.

Corollary 2.19 *If G is a nilpotent group, then every minimal normal subgroup of G is contained in $Z(G)$.*

Proof Let N be a minimal normal subgroup of G. Clearly, $N \cap Z(G) \trianglelefteq N$. By minimality, either $N \cap Z(G) = 1$ or $N \cap Z(G) = N$. However, $N \cap Z(G) \neq 1$ by Theorem 2.28. Thus, $N \cap Z(G) = N$ and the result follows. □

Corollary 2.19 allows us to characterize a finite nilpotent group in terms of a certain type of series. Our discussion that follows is based on [9].

Definition 2.20 Let G be a group. A normal series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

of G is called a *chief series* if each factor group G_{i+1}/G_i for $i = 0,\ 1,\ \ldots,\ n-1$ is a minimal normal subgroup of G/G_i. The factor groups G_{i+1}/G_i are called the *chief factors* of G.

Every finite group has a chief series. By the Correspondence Theorem, the condition that G_{i+1}/G_i is a minimal normal subgroup of G/G_i is equivalent to the condition that if $N \triangleleft G$ and $G_i \leq N \leq G_{i+1}$, then either $N = G_i$ or $N = G_{i+1}$.

Lemma 2.23 *Let G be a group with normal subgroups M and N, and suppose that $N < M$. Further suppose that G has a chief series. The factor M/N is a minimal normal subgroup of G/N if and only if it is a chief factor of G.*

Proof Suppose that M/N is a minimal normal subgroup of G/N. Since G has a chief series, every proper normal series of G can be refined to a chief series of G. In particular, G has a chief series containing M and N as two of its terms. We conclude that M/N is a chief factor of G. The converse is trivial. □

Theorem 2.29 *A finite group G is nilpotent if and only if every chief factor of G is central.*

Proof If G is nilpotent, then so is any factor group of G by Corollary 2.5. In view of Lemma 2.23, it suffices to show that every minimal normal subgroup of G is in $Z(G)$. This was done in Corollary 2.19.

Conversely, suppose that every chief factor of G is central. This implies that every chief series of G is also a central series of G. Therefore, G is nilpotent. □

Lemma 2.24 *If G is any group with a chief series, then any central factor of the series is finite and has prime order.*

Proof In light of Lemma 2.23, it suffices to consider a minimal normal subgroup N of G such that $N \leq Z(G)$ and to prove that $|N| = p$ for some prime p. Clearly, every subgroup of N is normal in G because $N \leq Z(G)$. By the minimality of N, the only normal subgroups of N are 1 and N. It follows that $|N| = p$ for some prime p. □

Remark 2.13 By Theorem 2.29 and Lemma 2.24, every factor of a chief series in a finite nilpotent group is central and has prime order. The converse need not be true (consider S_3).

2.7.2 Separating Points in a Group

Certain properties of the upper central subgroups of a group, as well as their factors, are inherited from the center of the group. These properties allow one to understand the structure of the group, especially when it is nilpotent. The next definition can be found in [13] for abelian groups.

Definition 2.21 Let G and H be any pair of nontrivial groups. We say that H *separates* G if for each element $g \neq 1$ in G, there exists $\varphi \in Hom(G, H)$ such that $\varphi(g) \neq 1$. Such elements of $Hom(G, H)$ are said to *separate points* in G.

Lemma 2.25 *Let P be a nonempty set of primes. Suppose that G and H are groups and H separates G.*

(i) If H is P-torsion-free, then G is P-torsion-free.
(ii) If H has finite exponent m, then G has finite exponent dividing m.

In particular, if H is torsion-free and H separates G, then G is torsion-free.

Proof Both results are proven by contradiction.

(i) Suppose that $1 \neq g \in G$ is a P-torsion element. There exists $\varphi \in Hom(G, H)$ such that $\varphi(g) \neq 1$. If $g^n = 1$ for some P-number n, then $\varphi(g^n) = (\varphi(g))^n = 1$; that is, $\varphi(g)$ is a P-torsion element of H. This contradicts the P-torsion-freeness of H. Hence, G is P-torsion-free.

(ii) Assume that there exists $g \in G$ such that $g^m \neq 1$. There exists $\varphi \in Hom(G, H)$ such that $\varphi(g^m) \neq 1$; that is, $(\varphi(g))^m \neq 1$. However, $\varphi(g) \in H$ and H has exponent m. Therefore, $g^m = 1$ for every $g \in G$. Thus, G has exponent dividing m. □

Theorem 2.30 *If G is any group, then $Z(G)$ separates $\zeta_i G/\zeta_{i-1}G$.*

Here of course, we are assuming that $Z(G)$ and $\zeta_i G/\zeta_{i-1}G$ are nontrivial. In particular, if both $Z(G)$ and $Z(G/Z(G))$ are nontrivial, then there exists a homomorphism of G onto a nontrivial subgroup of $Z(G)$. This is the case for $i = 2$ and it is due to O. Grün.

Proof The proof is done by induction on i. The case for $i = 1$ is obviously true. Suppose $i = 2$. We prove that $Z(G)$ separates $\zeta_2 G/Z(G)$. For any element $g \in G$, consider the map

$$\psi_g : \zeta_2 G \to Z(G) \ \text{ defined by } \ \psi_g(x) = [x, g].$$

This map makes sense since $[\zeta_2 G, G] \leq Z(G)$. By Lemma 1.12, ψ_g is a homomorphism whose kernel clearly contains $Z(G)$. Thus, ψ_g induces a well-defined homomorphism

$$\overline{\psi}_g : \zeta_2 G/Z(G) \to Z(G) \ \text{ given by } \ \overline{\psi}_g(xZ(G)) = [x, g].$$

Let $hZ(G)$ be a nonidentity element of $\zeta_2 G/Z(G)$, so that $h \in \zeta_2 G$ and $h \notin Z(G)$. There exists some element $g \in G$ such that $[h, g] \neq 1$. This means that $\psi_g(h) \neq 1$, and consequently, $\overline{\psi}_g(hZ(G)) \neq 1$. Therefore, $Z(G)$ separates $\zeta_2 G/Z(G)$.

Assume that $Z(G)$ separates $\zeta_i G/\zeta_{i-1}G$ for $i > 2$. In order to prove that $Z(G)$ separates $\zeta_{i+1}G/\zeta_i G$, it is enough to show that $\zeta_i G/\zeta_{i-1}G$ separates $\zeta_{i+1}G/\zeta_i G$. By Lemma 2.11 and the Third Isomorphism Theorem,

$$\frac{\zeta_{i+1}G}{\zeta_i G} \cong \frac{\zeta_{i+1}G/\zeta_{i-1}G}{\zeta_i G/\zeta_{i-1}G} \cong \frac{\zeta_2(G/\zeta_{i-1}G)}{Z(G/\zeta_{i-1}G)}. \tag{2.23}$$

It follows from the previous case that $\zeta_i G/\zeta_{i-1}G$ separates $\zeta_{i+1}G/\zeta_i G$. □

By Lemma 2.25 (i) and Theorem 2.30, we have:

Corollary 2.20 (D. H. McLain) *Let G be any group, and let P be a nonempty set of primes. If $Z(G)$ is P-torsion-free, then $\zeta_{i+1}G/\zeta_i G$ is P-torsion-free for each integer $i \geq 0$.*

We mention that A. I. Mal'cev and S. N. Černikov proved Corollary 2.20 for the case when P is the set of all primes.

We offer another proof of Corollary 2.20 which uses Lemma 1.13. Let $g \neq 1$ be an element of $\zeta_2 G$ such that $(gZ(G))^n = Z(G)$ in $\zeta_2 G/Z(G)$, where n is a P-number. This means that $g^n \in Z(G)$. If $h \in G$, then

$$[g,\ h]^n = [g^n,\ h] = 1$$

by Lemma 1.13 because $[g,\ h] \in Z(G)$. Since $Z(G)$ is P-torsion-free, $[g,\ h] = 1$. Therefore, $g \in Z(G)$ and $\zeta_2 G/Z(G)$ is P-torsion-free. The rest now follows by induction on i.

Corollary 2.21 *Let P be a nonempty set of primes. A nilpotent group is P-torsion-free if and only if its center is P-torsion-free.*

Proof Suppose that G is nilpotent of class c and $Z(G)$ is P-torsion-free. By Corollary 2.20, $\zeta_{i+1}G/\zeta_i G$ is P-torsion-free for $0 \leq i \leq c-1$. Let $1 \neq g \in G$, and let n be any P-number. Since $g \neq 1$, there exists an integer $i \in \{0,\ \ldots,\ c-1\}$ such that $g \in \zeta_{i+1}G \setminus \zeta_i G$. Now, $(g\zeta_i G)^n \neq \zeta_i G$ because $\zeta_{i+1}G/\zeta_i G$ is P-torsion-free. Hence, $g^n \notin \zeta_i G$. This means that $g^n \neq 1$, and thus G is P-torsion-free. The converse is obvious. □

By Corollaries 2.20 and 2.21, we see that each upper central factor of a torsion-free nilpotent group must be torsion-free abelian.

Corollary 2.22 *Let P be a nonempty set of primes. If G is a P-torsion-free nilpotent group, then so is $G/Z(G)$.*

Proof The center of G is P-torsion-free since G is. By Corollary 2.20, $\zeta_2 G/Z(G)$ is P-torsion-free as well. The result follows from Corollary 2.21 since $\zeta_2 G/Z(G)$ is the center of $G/Z(G)$. □

Remark 2.14 The lower central factors of a torsion-free nilpotent group are not necessarily torsion-free. For example, fix a positive integer $n > 1$, and let

$$G = \left\{ \begin{pmatrix} 1 & xn & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, x,\ y,\ z \in \mathbb{Z} \right\}.$$

It is easy to see that G is a subgroup of the Heisenberg group, which is torsion-free and nilpotent (see Example 2.18). Thus, G is also torsion-free and nilpotent. Now,

$$\gamma_2 G = \left\{ \begin{pmatrix} 1 & 0 & wn \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, w \in \mathbb{Z} \right\}.$$

It follows that $G/\gamma_2 G$ is isomorphic to the direct sum $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_n$, and this group has torsion.

Corollary 2.23 (S. Dixmier) *Let G be a nilpotent group of class c. If $Z(G)$ has finite exponent m, then $\zeta_{i+1}G/\zeta_i G$ has exponent dividing m for $0 \leq i \leq c$. Consequently, G has exponent dividing m^c.*

Proof By Theorem 2.30 and Lemma 2.25 (ii), each $\zeta_{i+1}G/\zeta_i G$ has exponent dividing m. Let $1 \neq g \in G$. For some $i \in \{0, \ldots, c-1\}$, we have $g \in \zeta_{i+1}G \setminus \zeta_i G$. Since every upper central quotient has exponent dividing m, we have

$$g^m \in \zeta_i G,\ g^{m^2} \in \zeta_{i-1}G,\ \ldots,\ g^{m^{i+1}} \in \zeta_0 G = 1.$$

Thus, $g^{m^c} = 1$. □

Lemma 2.26 *Let P be a nonempty set of primes. If G is a finitely generated nilpotent group and $Z(G)$ is a P-torsion group, then G is a finite P-torsion group.*

In particular, every finitely generated nilpotent group with finite center is finite.

Proof By Theorem 2.18, $Z(G)$ is finitely generated. Since $Z(G)$ is also P-torsion and abelian, it must be finite with exponent a P-number. Thus, G has finite exponent which is a P-number by Corollary 2.23. The result follows from Theorem 2.25. □

The center of any torsion group is obviously a torsion group. There are nilpotent groups which are torsion-free, yet their center is a torsion group. This is illustrated in the next example.

Example 2.20 Suppose that A is an additive abelian torsion group with infinite exponent, and let $\vartheta \in Aut(A \oplus A)$ be defined by $\vartheta(x,\ y) = (x + y,\ y)$. For each $m \in \mathbb{N}$, set

$$\vartheta^{\circ m} = \underbrace{\vartheta \circ \cdots \circ \vartheta}_{m}.$$

Since $\vartheta^{\circ m}(x,\ y) = (x + my,\ y)$ for every $m \in \mathbb{N}$, ϑ has infinite order. Define a mapping

$$\varphi : \mathbb{Z} \to Aut(A \oplus A) \ \text{ by } \ \varphi(k) = \vartheta^{\circ k},$$

and let $G = (A \oplus A) \rtimes_\varphi \mathbb{Z}$. Observe that

$$\begin{aligned}(i,\ (x,\ y))(j,\ (\tilde{x},\ \tilde{y})) &= (i + j,\ (\varphi(j))(x,\ y) + (\tilde{x},\ \tilde{y})) \\ &= (i + j,\ (x + jy,\ y) + (\tilde{x},\ \tilde{y})) \\ &= (i + j,\ (x + \tilde{x} + jy,\ y + \tilde{y})).\end{aligned}$$

It is easy to check that G is torsion-free and

$$Z(G) = \{(0,\ (x,\ 0)) \mid x \in A\} \cong A.$$

It follows that G is nilpotent of class 2.

We end this section with a lemma which will be useful later.

Lemma 2.27 *Every infinite finitely generated nilpotent group contains a central element of infinite order.*

Proof Let G be an infinite finitely generated nilpotent group. If G has no central elements of infinite order, then $Z(G)$ is a torsion group. By Lemma 2.26, G must be finite, a contradiction. Therefore, G has a central element of infinite order. □

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