24

Solvable Groups

Learning Objectives

In this chapter we shall

- define a solvable group
- prove that every subgroup, quotient group, and homomorphic image of a solvable group is solvable
- give equivalent conditions for a finite group to be solvable
- prove that S_n is not solvable for $n \geq 5$
- define higher commutator subgroups $G^{(i)}$
- find the condition of solvability in terms of $G^{(i)}$

Solvable groups arose in an attempt to find a formula for the roots of a polynomial $f(x)$, of degree n, over integers in terms of its coefficients using the operations of addition, subtraction, multiplication, division, and square roots on the coefficients of the polynomial. If such a formula can be obtained we can say that the polynomial is *solvable* by radicals. In Galois Theory, some group G is associated with every polynomial $f(x)$, called the Galois group of $f(x)$. It was proved by Galois that not all equations of degree $n \geq 5$ are solvable by radicals, because S_n is not solvable for $n \geq 5$. Thus solvable groups form an important class of groups in the theory of polynomial equations.

24.1 DEFINITION AND EXAMPLES

Definition 24.1 A group G is called a **solvable group** if G has a subnormal series

$$\langle e \rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{r-1} \trianglelefteq H_r = G$$

such that H_{i+1}/H_i is Abelian, for all $i = 0, \ldots, r-1$.

Such a subnormal series for a solvable group G is called a *solvable series* for G. ♦

Example 24.1 Every Abelian group G is trivially solvable as $\langle e\rangle \trianglelefteq G$ is a subnormal series for G with the factor $G/\langle e\rangle \cong G$ is Abelian. ◇

Example 24.2 S_n is solvable for $n = 1, 2, 3$.

$S_1 = \{(1)\}$, and $\langle(1)\rangle \trianglelefteq S_1$ is a solvable series for S_1.

$S_2 = \{(1), (1\ 2)\}$, and S_2 is an Abelian group, therefore $\langle(1)\rangle \trianglelefteq S_2$ is a solvable series for S_2.

$S_3 = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$.

Consider the series

$$\langle 1\rangle \trianglelefteq A_3 \trianglelefteq S_3 \tag{24.1}$$

(24.1) is a subnormal series for S_3. The factors of the series, S_3/A_3 and $A_3/\langle(1)\rangle$ are such that S_3/A_3 is a group of order 2, and $A_3/\langle(1)\rangle$ is a group of order 3. Since every group of prime order is Abelian, so every factor is Abelian.

Hence, (24.1) is a solvable series for S_3. ◇

Theorem 24.1 A subgroup of a solvable group is solvable.

Proof Let G be a solvable group and

$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{r-1} \trianglelefteq H_r = G$$

be a subnormal series for G such that H_{i+1}/H_i is Abelian for all $i = 0, \ldots, r-1$.

Let K be a subgroup of G. We show that

$$\langle e\rangle = (K \cap H_0) \leq (K \cap H_1) \leq \ldots \leq (K \cap H_{r-1}) \leq (K \cap H_r) = K$$

is a subnormal series for K such that $(K \cap H_{i+1})/(K \cap H_i)$ is Abelian, for all $i = 0, \ldots, r-1$.

We prove this in the following steps. For all $i = 0, \ldots, r-1$

1. $(K \cap H_i) \leq (K \cap H_{i+1})$.
2. $(K \cap H_i) \trianglelefteq (K \cap H_{i+1})$.
3. $(K \cap H_{i+1})/(K \cap H_i)$ is Abelian.

Step 1: Since $(K \cap H_i)$ and $(K \cap H_{i+1})$ are subgroups of G such that $(K \cap H_i) \subseteq (K \cap H_{i+1})$, therefore $(K \cap H_i) \leq (K \cap H_{i+1})$.

Step 2: Let $x \in (K \cap H_i)$ and $g \in (K \cap H_{i+1})$. Then $g, x \in K$; $x \in H_i$, and $g \in H_{i+1}$.

Now $g, x \in K$, $K \leq G \Rightarrow gxg^{-1} \in K$.

Also, $g \in H_{i+1}$, $x \in H_i$, $H_i \trianglelefteq H_{i+1} \Rightarrow gxg^{-1} \in H_i$.

Hence, $gxg^{-1} \in (K \cap H_i)$, so that $(K \cap H_i) \trianglelefteq (K \cap H_{i+1})$.

Step 3: By Theorem 11.16, it is sufficient to show that $(K \cap H_{i+1})' \subseteq (K \cap H_i)$.

Let $x, y \in (K \cap H_{i+1})$. Then $x, y \in K$, and $x, y \in H_{i+1}$. Therefore, $x^{-1}y^{-1}xy \in K$.

Further, H_{i+1}/H_i is Abelian $\Rightarrow H_{i+1}' \subseteq H_i \Rightarrow x^{-1}y^{-1}xy \in H_i$.

Thus, $x^{-1}y^{-1}xy \in K \cap H_i$. Hence, $(K \cap H_{i+1})' \subseteq (K \cap H_i)$, and so $(K \cap H_{i+1})/(K \cap H_i)$ is Abelian.

Steps 1–3 prove that K is solvable.

Factor Group of a Solvable Group

Theorem 24.2 Let N be a normal subgroup of a solvable group G. Then G/N is solvable.

Proof Since G is a solvable group, G has a solvable series.

Let
$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{r-1} \trianglelefteq H_r = G$$

be a solvable series of G. Then H_{i+1}/H_i is Abelian, for all $i = 0, \dots, r-1$.
Let $\eta : G \to G/N$ be the natural homomorphism, and $\eta(H_i)$ be denoted by $\overline{H}_i$.
Then

$$\langle e\rangle = \overline{H}_0 \trianglelefteq \overline{H}_1 \trianglelefteq \dots \trianglelefteq \overline{H}_{r-1} \trianglelefteq \overline{H}_r = G/N \tag{24.2}$$

is a series of subgroups of G/N. We prove that the series (24.2) is a solvable series for G/N.

By Problem 12.26, $H_i \trianglelefteq H_{i+1} \Rightarrow \overline{H}_i \trianglelefteq \overline{H}_{i+1}$. Now we show that $\dfrac{\overline{H}_{i+1}}{\overline{H}_i}$ is Abelian.

Let $\bar{x}, \bar{y} \in \overline{H}_{i+1}$. Then, $\bar{x} = \eta(x)$, $\bar{y} = \eta(y)$ for some $x, y \in H_{i+1}$.

H_{i+1}/H_i is Abelian

$$\Rightarrow xH_i\, yH_i = yH_i\, xH_i$$
$$\Rightarrow (xy)H_i = (yx)H_i$$
$$\Rightarrow (yx)^{-1}(xy) \in H_i$$
$$\Rightarrow x^{-1}y^{-1}xy \in H_i$$
$$\Rightarrow \eta(x^{-1}y^{-1}xy) \in \eta(H_i)$$
$$\Rightarrow \eta(x^{-1})\eta(y^{-1})\eta(x)\eta(y) \in \eta(H_i), \text{ as } \eta \text{ is a homomorphism}$$
$$\Rightarrow \eta(x)^{-1}\eta(y)^{-1}\eta(x)\eta(y) \in \eta(H_i)$$
$$\Rightarrow (\bar{x})^{-1}(\bar{y})^{-1}\,\bar{x}\,\bar{y} \in \overline{H}_i$$
$$\Rightarrow \bar{x}\,\bar{y}\,\overline{H}_i = \bar{y}\,\bar{x}\,\overline{H}_i$$
$$\Rightarrow \bar{x}\,\overline{H}_i\,\bar{y}\,\overline{H}_i = \bar{y}\,\overline{H}_i\,\bar{x}\,\overline{H}_i$$
$$\Rightarrow \overline{H}_{i+1}\ \overline{H}_i \text{ is Abelian.}$$

Thus $\langle e\rangle = \overline{H}_0 \trianglelefteq \overline{H}_1 \trianglelefteq \dots \trianglelefteq \overline{H}_{r-1} \trianglelefteq \overline{H}_r = G/N$ is a subnormal series of G/N such that every factors $\dfrac{\overline{H}_{i+1}}{\overline{H}_i}$ is Abelian. Hence G/N is solvable. ♦

Homomorphic Image of a Solvable Group

Theorem 24.3 Every homomorphic image of a solvable group is solvable.

Proof Let G be a solvable group and φ be a homomorphism from G onto a group T.
Let $\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{r-1} \trianglelefteq H_r = G$ be a solvable series for G. So that H_{i+1}/H_i is Abelian, for $i = 0$,

Since $H_i \leq G$, $\varphi(H_i)$ is a subgroup of T, let $H_i^* = \varphi(H_i)$.
Consider the series of subgroups of T:

$$\langle e \rangle = H_0^* \leq H_1^* \leq \ldots \leq H_{r-1}^* \leq H_r^* = T = \varphi(G) \quad (24.3)$$

We show that (24.3) is a solvable series for T.

$$H_i \trianglelefteq H_{i+1} \Rightarrow \varphi(H_i) \trianglelefteq \varphi(H_{i+1}),$$
$$\Rightarrow H_i^* \trianglelefteq H_{i+1}^*.$$

We prove that H_{i+1}^*/H_i^* is Abelian.
Let $x^*, y^* \in H_{i+1}^*$. Then $x^* = \varphi(x), y^* = \varphi(y)$ for some $x, y \in H_{i+1}$.
H_{i+1}/H_i is Abelian

$$\Rightarrow xH_i yH_i = yH_i xH_i$$
$$\Rightarrow x^{-1}y^{-1}xy \in H_i$$
$$\Rightarrow \varphi(x^{-1}y^{-1}xy) \in \varphi(H_i)$$
$$\Rightarrow (\varphi(x))^{-1}(\varphi(y))^{-1}\varphi(x)\varphi(y) \in \varphi(H_i) \text{ as } \varphi \text{ is a homomorphism}$$
$$\Rightarrow (x^*)^{-1}(y^*)^{-1}x^*y^* \in H_i^*$$
$$\Rightarrow x^*y^*H_i^* = y^*x^*H_i^*$$
$$\Rightarrow x^*H_i^*y^*H_i^* = y^*H_i^*x^*H_i^*$$
$$\Rightarrow H_{i+1}^*/H_i^* \text{ is Abelian.}$$

Thus $\langle e \rangle = H_0^* \trianglelefteq H_1^* \trianglelefteq \ldots \trianglelefteq H_{r-1}^* \trianglelefteq H_r^* = T$ is a subnormal series of T such that every factors H_{i+1}^*/H_i^* is Abelian. Hence T is solvable. ◆

Theorem 24.4 If N is a normal subgroup of a group G and N and G/N are solvable then G is solvable.

Proof Let

$$\langle e \rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{r-1} \trianglelefteq H_r = N \quad (24.4)$$

be a solvable series for N and

$$\langle N \rangle = \overline{K}_0 \trianglelefteq \overline{K}_1 \trianglelefteq \ldots \trianglelefteq \overline{K}_{s-1} \trianglelefteq \overline{K}_s = G/N$$

be solvable series for G/N. Let K_i be the pre-image of $\overline{K}_i$ under the natural homomorphism from G onto G/N. Then $K_i/N = \overline{K}_i$.
We get a series of subgroups of G,

$$N = K_0 \leq K_1 \leq \ldots \leq K_{s-1} \leq K_s = G \quad (24.5)$$

We prove that for all $i = 0, 1, \ldots, s-1$,

1. $K_i \trianglelefteq K_{i+1}$
2. K_{i+1}/K_i is Abelian.

Step 1: Let $x \in K_i$ and $g \in K_{i+1}$. Then $xN \in \overline{K}_i$ and $gN \in \overline{K}_{i+1}$.
Now $\overline{K}_i \trianglelefteq \overline{K}_{i+1}$

$$\Rightarrow (gN)(xN)(gN)^{-1} \in \overline{K}_i$$

$$\Rightarrow (gxg^{-1})N \in \overline{K}_i = K_i/N$$

$$\Rightarrow (gxg^{-1}) \in K_i$$

$$\Rightarrow K_i \trianglelefteq K_{i+1}$$

Step 2: $\overline{K}_{i+1}/\overline{K}_i = (K_{i+1}/N)/(K_i/N) \cong K_{i+1}/K_i$, by the Third Isomorphism Theorem.
Since $\overline{K}_{i+1}/\overline{K}_i$ is Abelian, therefore K_{i+1}/K_i is Abelian.
Hence, the series obtained by combining (24.4) and (24.5), namely

$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{r-1} \trianglelefteq H_r = N = K_0 \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq K_{s-1} \trianglelefteq K_s = G$$

is a solvable series for G. Hence G is solvable. ◆

Theorem 24.5 If G is a finite Abelian group, then G has subnormal series with each factor of prime order.

Proof Let G be an Abelian group of order n. We prove the result by induction on n. If $n = 2$, then $\langle e\rangle \trianglelefteq G$ is a subnormal series for G with only one factor and that is of prime order 2. So the result holds trivially. Assume that the result holds for all Abelian groups of order less than n.
Let $p|n$ for some prime p. Two cases arise:

Case 1: $p = n$. In this case, $\langle e\rangle \leq G$ is the required solvable series with every factor of prime order.

Case 2: $p < n$. By Theorem 11.14, G has a subgroup T of order (n/p) which is less than n. By induction hypothesis, T has subnormal series

$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{r-1} \trianglelefteq H_r = T \tag{24.6}$$

with every factor of prime order. Moreover, $|G/T| = p$. Consequently G has a subnormal series

$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots H_{r-1} \trianglelefteq \leq H_r = T \trianglelefteq G \tag{24.7}$$

whose factors are factors of series (24.6) together with G/T and $|G/T| = p$. Thus each factor of Eq. (24.7) is of prime order. ◆

24.2 EQUIVALENT CONDITIONS OF SOLVABILITY

We now study some equivalent conditions for solvability of a *finite* group G.

Theorem 24.6 Let G be a *finite* group. The following statements are equivalent:

(a) G is solvable.

(b) G has a subnormal series $\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{r-1} \trianglelefteq H_r = G$, such that H_{i+1}/H_i is a simple group of prime order.

(c) G has a subnormal series $\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{r-1} \trianglelefteq H_r = G$ such that H_{i+1}/H_i is cyclic.

Proof (*a*) ⟹ (*b*)
Suppose that G is solvable and G has a solvable series

$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{n-1} \trianglelefteq H_n = G \tag{24.8}$$

We prove the result by induction on n, the length of the series.

Let $n = 1$. Since $G \cong H_1/H_0$ which is Abelian, therefore G is Abelian. By Theorem 24.5, G has a subnormal series whose every factor is of prime order.

Now suppose that $n > 1$. Assume that every finite solvable group, which has a subnormal series of length less than n, has a subnormal series in which every factor is of prime order. The subgroup H_{n-1} of G is a finite solvable group which has a solvable series of length $(n - 1)$, namely

$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{n-1} \tag{24.9}$$

By induction hypothesis H_{n-1} has a subnormal series whose every factor is of order p. Now G/H_{n-1} is Abelian as (24.8) is a solvable series. By Theorem 24.5, G/H_{n-1} has a subnormal series

$$H_{n-1}/H_{n-1} = K_0/H_{n-1} \trianglelefteq K_1/H_{n-1} \trianglelefteq K_2/H_{n-1} \trianglelefteq \dots \trianglelefteq K_s/H_{n-1} = G/H_{n-1}$$

whose every factor is of prime order. Thus $(K_i/H_{i-1}) \trianglelefteq (K_{i+1}/H_{i-1})$ and $(K_{i+1}/H_{i-1})/(K_i/H_{i-1})$ is Abelian and $|(K_{i+1}/H_{i-1})/(K_i/H_{i-1})|$ is prime, say p. This gives rise to the series

$$H_{n-1} = K_0 \leq K_1 \leq K_2 \leq \dots \leq K_s = G \tag{24.10}$$

where $K_i \trianglelefteq K_{i+1}$ and $K_{i+1}/K_i \cong (K_{i+1}/H_{i-1})/(K_i/H_{i-1})$, which is Abelian, so that K_{i+1}/K_i is Abelian. Moreover, $|K_{i+1}/K_i| = |(K_{i+1}/H_{i-1})/(K_i/H_{i-1})| = p$.
Thus, combining series (24.9) and (24.10), we get the series

$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{n-1} = K_0 \trianglelefteq K_1 \trianglelefteq K_2 \trianglelefteq \dots \trianglelefteq K_s = G$$

which is a subnormal series for G whose every factor is of prime order. Further every factor group is a simple group as every group of prime order is simple.
Hence (*b*) holds.
(*b*) ⟹ (*c*)

Let
$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{n-1} \trianglelefteq H_n = G$$

be a subnormal series for G, whose every factor is of prime order. Since every group of prime order is cyclic, G has a subnormal series whose every factor is cyclic. Hence (*c*) holds.
(*c*) ⟹ (*a*)
Suppose G has a subnormal series whose every factor is cyclic. Since every cyclic group is Abelian, the subnormal series of G is a solvable series. Hence (*a*) holds. ♦

Theorem 24.7 Let G be a finite group. Then G is solvable if and only if G has a subnormal series

$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{r-1} \trianglelefteq H_r = G$$

such that $H_i \trianglelefteq G$ and H_{i+1}/H_i is Abelian, for all $i = 0, 1, \dots, r - 1$.

Proof Let G be a solvable group. Let M be a minimal normal subgroup of G. As subgroup of a solvable group is solvable, M is solvable. By Theorem 24.6(b), M has a normal subgroup N, such that N is of prime index in M. Hence N is a proper subgroup of M. We claim that

1. $L = \bigcap_{g \in G} gNg^{-1}$, is a normal subgroup of G.
2. $L \subseteq M$.
3. $L = \langle e \rangle$.
4. M is Abelian.

Step 1: Let $L = \bigcap_{g \in G} gNg^{-1}$.

For any $x \in G$,

$$xLx^{-1} = x\left(\bigcap_{g \in G} gNg^{-1}\right)x^{-1}$$

$$= \bigcap_{g \in G} (xg)\, N(xg)^{-1}$$

$$= \bigcap_{g \in G} gNg^{-1},$$

$$= \bigcap_{g \in G} gNg^{-1}, \text{ as } xG = G$$

$$= L$$

Thus L is normal subgroup of G.

Step 2: Further, $N \subseteq M$

$\Rightarrow gNg^{-1} \subseteq gMg^{-1}$, for all $g \in G$

$\Rightarrow gNg^{-1} \subseteq M$, for all $g \in G$, as M is normal subgroup of G

$\Rightarrow \bigcap_{g \in G} gNg^{-1} \subseteq M$

$\Rightarrow L \subseteq M$.

Step 3: By definition of L, $L \subseteq N$.
Since $L \subseteq N \subset$ M and M is minimal normal subgroup of G, $L = \langle e \rangle$.

Step 4: N is of prime index in M,

$\Rightarrow M/N$ is a cyclic

$\Rightarrow M/N$ is Abelian group

$\Rightarrow x^{-1}y^{-1}xy \in N$, for all $x, y \in M$ (by Theorem 11.16)

$$\langle M \rangle = \overline{K}_0 \trianglelefteq \overline{K}_1 \trianglelefteq \ldots \trianglelefteq \overline{K}_{s-1} \trianglelefteq \overline{K}_s = G/M \text{ with every factor Abelian.}$$

$$\text{Then } M = K_0 \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq K_{s-1} \trianglelefteq K_s = G \text{ where } \overline{K}_i = K_i/M$$

is a series of subgroups of G such that K_i is normal subgroup of G for all $i = 0, 1, \ldots, s-1$, as $K_{i+1}/K_i \cong \overline{K}_{i+1}/\overline{K}_i$ which is Abelian.

Then

$$\langle e \rangle \leq M = K_0 \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq K_{s-1} \trianglelefteq K_s = G$$

is the normal series for G in which every factor group is Abelian.
Converse is trivially true. ♦

Theorem 24.8 The symmetric group S_n is not solvable for $n \geq 5$.

Proof Suppose on contrary that S_n is solvable for $n \geq 5$.
S_n is solvable

$\Rightarrow A_n$ is solvable, as $A_n \leq S_n$ and a subgroup of a solvable group is solvable

$\Rightarrow A_n$ has a solvable series.

But $\langle(1)\rangle \trianglelefteq A_n$ is the only subnormal series for A_n, since A_n is simple for every $n \geq 5$.
Therefore $\langle(1)\rangle \trianglelefteq A_n$ is a solvable series for A_n.

$\Rightarrow A_n$ is Abelian, which is a contradiction.

Hence, our assumption is wrong so that S_n is not solvable for $n \geq 5$. ♦

24.3 HIGHER COMMUTATOR SUBGROUPS

We give now an equivalent definition of a solvable group. For this, we first define higher commutator subgroups.

Let G' be the commutator subgroup (or the derived subgroup) of G. Then G' is a characteristic subgroup, and therefore a normal subgroup of G such that G/G' is Abelian. Moreover, whenever N is a normal subgroup of G such that G/N is Abelian, then $G' \subseteq N$ (see Theorem 11.16). To define higher commutator subgroup, we denote G' by $G^{(1)}$ and the commutator subgroup of the group G' i.e. $(G')'$ we denote by $G^{(2)}$. In general $G^{(i+1)} = (G^{(i)})'$, for $i \geq 1$. Thus, $G^{(i+1)}$ is a normal subgroup of $G^{(i)}$ such that $G^{(i)}/G^{(i+1)}$ is Abelian. $G^{(n)}$ is called the nth commutator subgroup of G. By Theorems 13.11 and 13.14, $G^{(n)}$ is a normal subgroup of G for every n.

Thus $G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \cdots$. is called the derived series of G. This series need not reach $\langle e \rangle$ or even terminate.

Theorem 24.9 A group G is solvable if and only if $G^{(n)} = \langle e \rangle$ for some integer $n \geq 1$.

Proof Let G be solvable and let

$$\langle e \rangle = H_n \trianglelefteq H_{n-1} \trianglelefteq \ldots \trianglelefteq H_1 \trianglelefteq H_0 = G$$

be a subnormal series for G such that H_i/H_{i+1} is Abelian. By Theorem 11.16

$$(H_i)' \subseteq H_{i+1}, \text{ for all } i = 0, 1, 2, \ldots, n-1. \tag{24.11}$$

We prove by induction on i, that $G^{(i)} \subseteq H_i$ for each $i = 0, 1, 2, \ldots, n$.
G/H_1 is Abelian

$\Rightarrow G^{(1)} = G' \subseteq H_1$

$\Rightarrow H_0' \subseteq H_1$. So the assertion is true for $i = 1$.

Suppose that for some k, $G^{(k)} \subseteq H_k$. Then $G^{(k+1)} = (G^{(k)})' \subseteq H_k' \subseteq H_{k+1}$, using relation (24.11).

Therefore $G^{(i)} \subseteq H_i$ for all $i = 0, 1, 2, \ldots, n$. In particular for $i = n$, $G^{(n)} \subseteq H_n = \langle e \rangle$. Thus $G^{(n)} = \langle e \rangle$ and the result follows.

Conversely, suppose that $G^{(n)} = \langle e \rangle$ for some integer $n \geq 1$. Then

$$\langle e \rangle = G^{(n)} \trianglelefteq \ldots \trianglelefteq G' \trianglelefteq G$$

is a subnormal series for G and $G^{(i)}/G^{(i+1)}$ is Abelian. Hence G is solvable. ♦

Note that if G has a solvable series of length n, then $G^{(n)} = \langle e \rangle$. The length of the derived series is less than or equal to n, i.e., no solvable series can be shorter than the derived series.

We prove some of the results proved above by using this concept. We feel that the proofs are simpler using this technique.

Theorem 24.10 A group G is solvable if and only if G has a normal series

$$\langle e \rangle = N_k \trianglelefteq N_{k-1} \trianglelefteq \ldots \trianglelefteq N_1 \trianglelefteq N_0 = G$$

such that every factor N_{i-1}/N_i is Abelian, $i = 1, \ldots, k$.

Proof We show that the derived series is a normal series with every factor Abelian. We know that G' is a characteristic subgroup of G. Every characteristic subgroup is normal, so G' is normal in G. We prove by induction that $G^{(i)}$ is normal in G for all i.

As seen above the result is true for $i = 1$. Assume that $G^{(k)}$ is normal subgroup of G. Since $G^{(k+1)}$ is a characteristic subgroup of $G^{(k)}$ and $G^{(k)}$ is a normal subgroup of G, it follows that $G^{(k+1)}$ is normal in G. Hence, by induction, $G^{(i)}$ is normal in G for all i.

If G is a solvable group, then by Theorem 24.9, $G^{(n)} = \langle e \rangle$ for some n. Hence

$$\langle e \rangle = G^{(n)} \trianglelefteq \ldots \trianglelefteq G' \trianglelefteq G$$

is the required normal series.

Conversely, if G has a normal series with every factor group Abelian, then G is solvable as every normal series of G is a subnormal series. ♦

Theorem 24.11 A subgroup of a solvable group is solvable.

Proof Let H be any subgroup of a solvable group G. Since G is solvable, $G^{(n)} = \langle e \rangle$ for some integer $n \geq 1$. Clearly $H^{(n)} \subseteq G^{(n)}$. As $G^{(n)} = \langle e \rangle$, therefore $H^{(n)} = \langle e \rangle$.

Hence H is solvable. ♦

Theorem 24.12 Let N be a normal subgroup of a solvable group G. Then G/N is also solvable.

Proof Let $\eta: G \to G/N$ be the natural homomorphism from G onto G/N. Since η preserves product and inverses, $\eta(xyx^{-1}y^{-1}) = \eta(x)\eta(y)\eta(x)^{-1}\eta(y)^{-1}$. Therefore, for any subgroup H of G, $\eta(H') = \eta(H)'$, it follows that $\eta(G^{(n)}) = (\eta(G))^{(n)}$ holds. Since G is solvable $G^{(n)} = \langle e\rangle$ for some integer $n \geq 1$. $\eta(G^{(n)}) = \eta(e)$. This implies that $(G/N)^{(n)} = (\eta(G))^{(n)} = N$. Consequently, G/N is solvable. ♦

Theorem 24.13 Every homomorphic image of a solvable group is solvable.

Proof Let G be a solvable group and T be a homomorphic image of G under a homomorphism φ from G onto T. Since φ preserves product and inverses, $\varphi(xyx^{-1}y^{-1}) = \varphi(x)\varphi(y)\varphi(x)^{-1}\varphi(y)^{-1}$ holds for all $x, y \in G$. Thus, for any subgroup H of G, $\varphi(H') = \varphi(H)'$. Whence, $\varphi(G^{(k)}) = \varphi(G)^{(k)}$ for all $k \geq 1$.

Since G is solvable, $G^{(n)} = \langle e\rangle$. Now $eT = \varphi(e) = \varphi(G^{(n)}) = \varphi(G)^{(n)} = T^{(n)}$. This proves that T is solvable. ♦

SOLVED PROBLEMS 24A

Problem 24.1 Show that S_4 is solvable.

Solution $(1) \trianglelefteq H = \{(1), (1\ 3)(2\ 4)\} \trianglelefteq V_4 \trianglelefteq A_4 \trianglelefteq S_4$, (i)

where $V_4 = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. We show that (i) is a subnormal series. $A_4 \trianglelefteq S_4$, V_4 is the unique subgroup of order 4 in A_4, $V_4 \trianglelefteq A_4$. Also H is subgroup of index 2 in V_4, $H \trianglelefteq V_4$.

Thus (i) is subnormal series whose every factors S_4/A_4; A_4/V_4; V_4/H; $H/\langle(1)\rangle$ are of order 2, 3, 2, 2 which are all prime. Therefore S_4 is solvable by Theorem 24.6. ◇

Problem 24.2 Exhibit all solvable series for Q_8.

Solution The lattice of subgroups of Q_8 is

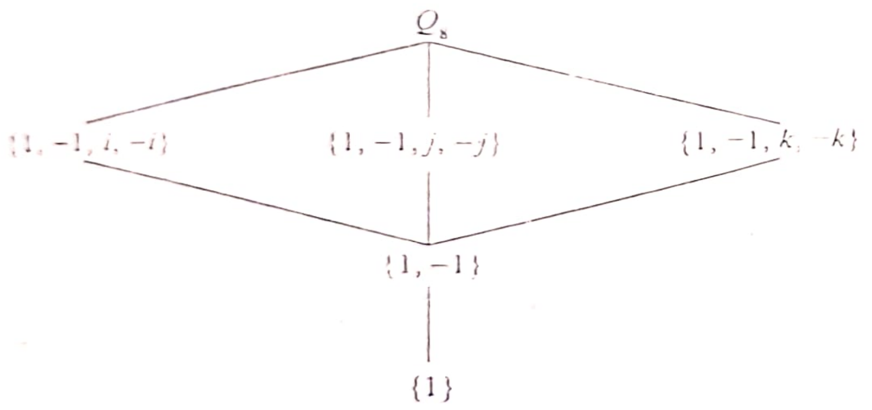

Figure 24.1

The three series of subgroups are as follows:

$$\{1\} \trianglelefteq \{1, -1\} \trianglelefteq \{1, -1, i, -i\} \trianglelefteq Q_8$$

$$\{1\} \trianglelefteq \{1, -1\} \trianglelefteq \{1, -1, j, -j\} \trianglelefteq Q_8$$

$$\{1\} \trianglelefteq \{1, -1\} \trianglelefteq \{1, -1, k, -k\} \trianglelefteq Q_8$$

Each subgroup is of index 2 in the next higher term of the series; hence, each factor group is simple. Therefore, all the series are solvable. ◇

Problem 24.3 Find all seven solvable series for D_8.

Solution D_8 has five subgroups of order 2 and 3 subgroups of order 4. The relation of containment is shown by the following lattice diagram:

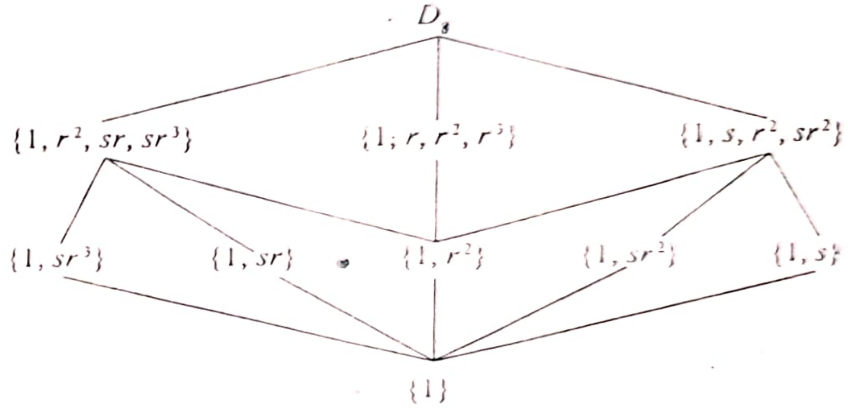

Figure 24.2

This diagram shows the seven subnormal series of D_8. Each subgroup is of index 2 in the next higher term of the series; hence, each factor group is simple. ◇

Problem 24.4 Prove that every finite p-group is solvable.

Solution Let G be a finite p-group. Then $|G| = p^n$, for some $n \geq 1$. We prove the result by induction on n. For $n = 1$, $|G| = p$. Then G is cyclic and hence Abelian. The series $\langle e \rangle \leq G$ is a solvable series, so that G is solvable.

Assume the result to be true for $m < n$.

Since G is a finite p-group, $Z = Z(G) \neq \langle e \rangle$. Then $|G/Z| < |G|$, so that by the induction hypothesis, G/Z is solvable. Since Z is Abelian and every Abelian group is solvable, Z is solvable. By Theorem 24.4, G is solvable. ◇

EXERCISES 24A

1. Show that every group of order p^2 is solvable.
2. Prove that every finite p group is solvable.
3. Prove that every group of order pq is solvable, where p and q are distinct primes.
4. If G is a solvable group, $G \neq \langle e \rangle$, then show that G contains a non-identity normal Abelian subgroup.
5. Write a solvable series for a cyclic group of order 8 in which every factor is of prime order.
6. Prove that a group of order $4p$, where p is a prime different from 2, is solvable.
7. Show that every group of order p^2q is solvable, where p and q are distinct primes.
8. Prove that D_{2n} is solvable for all integers n.