# 23

## Composition Series

**Learning Objectives**

In this chapter we shall

- define subnormal, normal, and composition series for a group
- define maximal normal subgroup
- prove the Jordan–Hölder Theorem

In this chapter we shall establish the Jordan–Hölder Theorem. We start with the terminology.

### 23.1 SUBNORMAL AND NORMAL SERIES

**Definition 23.1** A **series** for a group $G$ is a finite sequence of subgroups $H_0, H_1, \ldots, H_n$ of $G$ with

$$\langle e \rangle = H_0 \leq \cdots \leq H_{n-1} \leq H_n = G$$

The subgroups $H_0, H_1, \ldots, H_n$ are called **terms** of the series and the number of strict inclusions is called the **length of the series**. ◆

**Example 23.1** $\langle (1) \rangle = H_0 \leq H_1 \leq H_2 = S_3$ is a series for $S_3$ of length 2, where $H_1 = \langle (1\ 2) \rangle$. ◇

**Example 23.2** $\langle (1) \rangle = H_0 \leq H_1 \leq H_2 \leq H_3 = S_3$ is a series for $S_3$ of length 2, where $H_1 = \langle (1\ 2\ 3) \rangle$ and $H_2 = \langle (1\ 3\ 2) \rangle$, as $H_1 = H_2$. ◇

**Definition 23.2** A **subnormal** series of a group $G$ is a series of subgroups $H_i$ of $G$, such that

$$\langle e \rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{n-1} \trianglelefteq H_n = G \tag{23.1}$$

The factor groups $H_{i+1}/H_i$ are called the **factors** of the subnormal series (23.1). ◆

**Example 23.3** $\langle e\rangle \le \{(1), (1\ 2)(3\ 4)\} \le \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \le S_4$ is a subnormal series of length 3 for $S_4$ as each $H_i$ is normal in $H_{i+1}$. ◇

**Definition 23.3** A subnormal series

$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{n-1} \trianglelefteq H_n = G$$

such that $H_i \trianglelefteq G$, for all $i = 0, \ldots, n-1$ is said to be a **normal** series for $G$. ♦

**Example 23.4** Consider the following three subnormal series for $S_4$

$$\langle e\rangle \trianglelefteq V_4 \trianglelefteq S_4 \tag{23.2}$$

$$\langle e\rangle \trianglelefteq V_4 \trianglelefteq A_4 \trianglelefteq S_4 \tag{23.3}$$

$$\langle e\rangle \trianglelefteq \{(1), (1\ 3)(2\ 4)\} \trianglelefteq V_4 \trianglelefteq A_4 \trianglelefteq S_4 \tag{23.4}$$

Subnormal series (23.2) and (23.3) are normal series for $S_4$, whereas the subnormal series (23.4) is *not* a normal series as $\{(1), (1\ 3)(2\ 4)\}$ is not a normal subgroup of $S_4$. ◇

## Refinement of a Subnormal Series

Consider Example 23.4. The subnormal series (23.3) is obtained from subnormal series (23.2) by inserting one term $A_4$ so that the resulting series (23.3) is also a subnormal series, and the subnormal series (23.4) is obtained from (23.3) by inserting one more term $\{(1), (1\ 3)(2\ 4)\}$ between $\langle e\rangle$ and $V_4$. This way each term of series (23.2) is a term of series (23.3), and each term of series (23.3) is also a term of series (23.4). Hence, series (23.3) is called a one-step refinement of series (23.2), and series (23.4) is a two-step refinement of series (23.2). Thus we have

**Definition 23.4** Consider a subnormal series

$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{r-1} \trianglelefteq H_r = G \tag{i}$$

Then a subnormal series

$$\langle e\rangle = K_0 \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq K_{s-1} \trianglelefteq K_s = G \tag{ii}$$

is called a **refinement of the series** (i) if every $H_i$ is some $K_j$ for $i = 1, 2, \ldots, r$ ; $j = 1, 2, \ldots, s$.

The series (ii) is called a **proper refinement** of (i) if the length of (ii) is greater than the length of (i). ♦

**Example 23.5** The series $\langle 0\rangle \trianglelefteq \langle 12\rangle \trianglelefteq \langle 2\rangle \trianglelefteq \mathbb{Z}_{24}$ has proper refinements, namely

$$\langle 0\rangle \trianglelefteq \langle 12\rangle \trianglelefteq \langle 6\rangle \trianglelefteq \langle 2\rangle \trianglelefteq \mathbb{Z}_{24} \tag{23.5}$$

and

$$\langle 0\rangle \trianglelefteq \langle 12\rangle \trianglelefteq \langle 4\rangle \trianglelefteq \langle 2\rangle \trianglelefteq \mathbb{Z}_{24} \tag{23.6}$$

But the series (23.5) and (23.6) have no proper refinements. ◇

## 23.2 COMPOSITION SERIES

**Definition 23.5** A series $\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{s-1} \trianglelefteq H_s = G$ is said to be a **composition series for** a group $G$ if $H_{i+1}/H_i$ is simple for every $i = 0, 1, 2, \ldots, s-1$. The factor groups $H_{i+1}/H_i$ are called the **composition factors** for $G$.

**Definition 23.6** Two composition series

$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{r-1} \trianglelefteq H_r = G$$

and

$$\langle e\rangle = K_0 \trianglelefteq K_1 \trianglelefteq \dots \trianglelefteq K_{s-1} \trianglelefteq K_s = G$$

for group $G$ are said to be **equivalent** if $r = s$ and there is a one-to-one correspondence between the composition factors of the two series such that the corresponding factors are isomorphic. ♦

The relation "is equivalent to" on the set of composition series for a group is an equivalence relation.

**Example 23.6** The series $\langle 0\rangle \trianglelefteq \langle 6\rangle \trianglelefteq \langle 3\rangle \trianglelefteq \mathbb{Z}_{12}$ is a normal series for $\mathbb{Z}_{12}$ and the factors $\mathbb{Z}_{12}/\langle 3\rangle, \langle 3\rangle/\langle 6\rangle, \langle 6\rangle/\langle 0\rangle$ are of order 3, 2, 2, respectively. Every factor group is of prime order and is therefore simple. Thus the normal series is a composition series, and the composition factors are isomorphic to $\mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_2$. ◇

**Example 23.7** $\langle e\rangle \trianglelefteq A_n \trianglelefteq S_n$ is a composition series for $S_n$, $n \geq 5$. We know that $A_n$ is simple for $n \geq 5$. Also $S_n/A_n$ is of order 2, a prime number and every group of prime order is simple. Hence this series is a composition series and the composition factors are $A_n$ and $\mathbb{Z}_2$. ◇

**Example 23.8** The series (23.5) and (23.6) are both composition series for $\mathbb{Z}_{24}$. The composition factors of series (23.5) are $\mathbb{Z}_{24}/\langle 2\rangle$; $\langle 2\rangle/\langle 6\rangle$; $\langle 6\rangle/\langle 12\rangle$; $\langle 12\rangle/\langle 0\rangle$. These factor groups are of order 2, 3, 2, 2, respectively. Thus they are cyclic groups and so these are isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2$, and $\mathbb{Z}_2$, respectively.

Similarly the composition factors of (23.6) are $\mathbb{Z}_{24}/\langle 2\rangle$, $\langle 2\rangle/\langle 4\rangle$, $\langle 4\rangle/\langle 12\rangle$, and $\langle 12\rangle/\langle 0\rangle$, which are isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_2$, respectively.

Hence the series (23.5) and (23.6) are equivalent. ◇

Examples 23.5 and 23.8 show that composition series for a group may not be unique. If a group has more than one composition series, then they are equivalent.

Even non-isomorphic groups can have equivalent composition series as is seen in the following example.

**Example 23.9** Let $G = \langle a\rangle$ be a cyclic group order 8. A subnormal series for $G$ is

$$\langle e\rangle \trianglelefteq \langle a^4\rangle \trianglelefteq \langle a^2\rangle \trianglelefteq \langle a\rangle = G \quad \text{(i)}$$

The factors are

$$\langle a^4\rangle/\langle e\rangle;\ \langle a^2\rangle/\langle a^4\rangle;\ \langle a\rangle/\langle a^2\rangle$$

Each of the three factors is of order 2, and therefore each factor is simple. Thus, series (i) is a composition series for $G$.

The series

$$\{e\} \trianglelefteq \{e, r^2\} \trianglelefteq \{e, r^2, sr, sr^3\} \trianglelefteq D_8 \quad \text{(ii)}$$

is a composition series for $D_8$ with three composition factors each of order 2. Hence, $G$ and $D_8$ have isomorphic composition factors, but $G$ is not isomorphic to $D_8$. ◇

### Maximal Normal Subgroup

**Definition 23.7** A normal subgroup $N$ of a group $G$ is said to be a **maximal normal subgroup** of $G$ if there is no normal subgroup $M$ of $G$ such that $N < M < G$. ♦

**Example 23.10** Let $G = \mathbb{Z}_{12}$. Let $M = 4\mathbb{Z}_{12}$, and $N = 6\mathbb{Z}_{12}$. Then $M$ and $N$ are normal subgroup of $G$ such that $|G/M| = 3$, $|G/N| = 2$. Subgroups of orders 2 and 3 are simple, and hence $M$ and $N$ are maximal normal subgroup of $G$. ◊

**Theorem 23.1** A normal subgroup $N$ of a group $G$ is maximal if and only if $G/N$ is a simple group.

*Proof* There is a one-to-one correspondence between the normal subgroups of $G/N$ and the normal subgroups of $G$ containing $N$. In this correspondence, $G/N \leftrightarrow G$ and the identity subgroup of $G/N$, $\{N\} \leftrightarrow N$. Thus there is no normal subgroup of $G$ strictly between $G$ and $N$ if and only if there is no normal subgroup of $G/N$ other than $G/N$ and the identity subgroup. That is to say that $N$ is a maximal normal subgroup of $G$ if and only if $G/N$ is a simple group. ♦

**Theorem 23.2** A series

$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{r-1} \trianglelefteq H_r = G$$

for group $G$ is a composition series if and only if each term $H_i$ is a maximal normal subgroup of its successor $H_{i+1}$.

*Proof* Let the given series be a composition series for $G$. Then for each $0 \le i \le r-1$, $H_i \trianglelefteq H_{i+1}$ and $H_{i+1}/H_i$ is simple. By Theorem 23.1, $H_i$ is a maximal normal subgroup of $H_{i+1}$.

Conversely suppose that each term $H_i$ is a maximal normal subgroup of its successor $H_{i+1}$. Then by Theorem 23.1, $H_{i+1}/H_i$ is simple, and hence

$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{r-1} \trianglelefteq H_r = G$$

is a composition series for $G$. ♦

**Example 23.11** A normal subgroup of prime index of a group $G$ is a maximal normal subgroup of $G$. ◊

## 23.3 JORDAN–HÖLDER THEOREM

The following theorem is analogous to the unique factorization theorem for integers. Here, the simple factors of the series are like the prime factors of $G$. If $H$ is a normal subgroup of a group $G$ we say that $G$ is an extension of $H$ by $G/H$. Hence, the factors of a composition series for a group are the building blocks, from which, by a succession of extensions, one can construct the original group. However, the resulting group is not determined solely by its blocks, but it also depends on the way they are placed on top of one another. For instance, there are two extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_3$ namely $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ and $S_3$.

We now prove the classic theorem and illustrate it with examples.

**Theorem 23.3 (Jordan–Hölder)** Let $G$ be a finite group of order greater than 1. Then

(a) $G$ has a composition series.

(b) If $\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{r-1} \trianglelefteq H_r = G$ and $\langle e\rangle = K_0 \trianglelefteq K_1 \trianglelefteq \dots \trianglelefteq K_s = G$ are two composition series for $G$, then the two series are equivalent.

*Proof* (a) Let $G$ be a finite group of order greater than 1.
We prove the result by induction on $|G|$. As $|G| \neq 1$ we start the induction from $|G| = 2$.
If $|G| = 2$, then

$$\langle e \rangle = G_0 \trianglelefteq G_1 = G \text{ is the only composition series for } G.$$

Further, if $G$ is simple then

$$\langle e \rangle = G_0 \trianglelefteq G_1 = G \text{ is the only composition series for } G.$$

Thus if $|G| = 2$ or $G$ is a simple group then $G$ has a composition series.

Suppose now that $|G| > 2$ and $G$ is not simple.

Assume that the result is true for all groups $T$ whose order is less than $|G|$. Since $G$ is not simple, $G$ has a non-identity normal subgroup other than $G$. Let $N$ be a proper normal subgroup of $G$ of largest order.

By induction hypothesis, $N$ has a composition series

$$\langle e \rangle = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_{r-1} \trianglelefteq N_r = N$$

We claim that $N$ is a maximal normal subgroup of $G$.

If $N$ is not maximal, then there exists a normal subgroup $M$ of $G$ such that $N \subset M \subset G$. Then

$$|N| < |M|.$$

This is contrary to our choice of $N$. Thus, $N$ is a maximal normal subgroup of $G$; consequently $G/N$ is simple by Theorem 23.1. Thus,

$$\langle e \rangle = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_{r-1} \trianglelefteq N_r = N \trianglelefteq N_{r+1} = G$$

is a composition series for $G$.

This proves that every finite group has a composition series.

(b) Suppose that

$$\langle e \rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{r-1} \trianglelefteq H_r = G \qquad \text{(i)}$$

and

$$\langle e \rangle = K_0 \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq K_{s-1} \trianglelefteq K_s = G \qquad \text{(ii)}$$

are two composition series for $G$.

We prove that the two series are equivalent. Again, we use induction on $|G|$.

If $|G| = 2$ or if $G$ is simple, then $G$ has only one composition series, namely, $\langle e \rangle \trianglelefteq G$. So $r = s = 1$ and there is only one composition factor and hence the desired conclusion holds in this case.

Now, assume that $|G| > 2$ and that $G$ is not simple, so that $r > 1$ and $s > 1$ (because if $r = 1$ or $s = 1$, then $G$ is simple). Assume that the result is true for all groups $T$ whose order is less than $|G|$. Since $G$ is not simple, $G$ has a non-identity normal subgroup other than $G$.

We consider two cases:

*Case 1:* $H_{r-1} = K_{s-1}$.

Then

$$\langle e \rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{r-1} \qquad \text{(iii)}$$

and

$$\langle e \rangle = K_0 \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq K_{s-1} = H_{r-1} \qquad \text{(iv)}$$

are composition series for $H_{r-1}$. Since $|H_{r-1}| < |G|$, by induction hypothesis, the composition series (iii) and (iv) have the same length, so that $r - 1 = s - 1$. Hence $r = s$.

Moreover, there is a one-to-one correspondence between composition factors of (iii) and (iv), so that the corresponding factors are isomorphic. But $G/H_{r-1} = G/K_{s-1}$, therefore they are isomorphic. The composition factors of (i) are those of (iii) together with $G/H_{r-1}$. Similarly, the composition factors of (ii) are those of (iv) together with $G/K_{s-1}$. Hence there is a one-to-one correspondence between composition factors of (i) and (ii), such that the corresponding factors are isomorphic. Consequently the series (i) and (ii) are equivalent.

*Case 2:* $H_{r-1} \neq K_{s-1}$.
We do the following steps.

1. Construct composition series (vi) and (vii) for $G$.
2. Show that (vi) and (vii) are equivalent.
3. Show that (i) is equivalent to (vi), and (ii) is equivalent to (vii).

*Step 1:* To construct two composition series for $G$, we first show that $H_{r-1}K_{s-1} = G$.

$H_{r-1}K_{s-1} \trianglelefteq G$ such that $H_{r-1}$ is a proper normal subgroup of $H_{r-1}K_{s-1}$, so that $H_{r-1}K_{s-1}/H_{r-1}$ is a proper normal subgroup of $G/H_{r-1}$. But,

$$G/H_{r-1} \text{ is simple}$$

$$\Rightarrow H_{r-1}K_{s-1}/H_{r-1} = G/H_{r-1}$$

$$\Rightarrow H_{r-1}K_{s-1} = G \qquad \text{(v)}$$

Let $L = H_{r-1} \cap K_{s-1}$. Then $L \trianglelefteq G$. By (a), $L$ has a composition series say

$$\langle e \rangle = L_0 \trianglelefteq L_1 \trianglelefteq \ldots \trianglelefteq L_t = L$$

We claim that

$$\langle e \rangle = L_0 \trianglelefteq L_1 \trianglelefteq \ldots \trianglelefteq L_t = L \trianglelefteq L_{t+1} = H_{r-1} \trianglelefteq L_{t+2} = G \qquad \text{(vi)}$$

and

$$\langle e \rangle = L_0 \trianglelefteq L_1 \trianglelefteq \ldots \trianglelefteq L_t = L \trianglelefteq L_{t+1} = K_{s-1} \trianglelefteq L_{t+2} = G \qquad \text{(vii)}$$

are two composition series for $G$. For this, we need only to show that $H_{r-1}/L_t$ and $K_{s-1}/L_t$ are simple.

Since $G/H_{r-1} = H_{r-1}K_{s-1}/H_{r-1} \cong K_{s-1}/(H_{r-1} \cap K_{s-1}) = K_{s-1}/L_t$, using Second Isomorphism Theorem Thus

$$G/H_{r-1} \cong K_{s-1}/L_t \qquad \text{(viii)}$$

As, $G/H_{r-1}$ is simple, it follows that $K_{s-1}/L_t$ is simple. Similarly,

$$G/K_{s-1} \cong H_{r-1}/L_t, \text{ so } H_{r-1}/L_t \text{ is simple.} \qquad \text{(ix)}$$

Hence, (vi) and (vii) are composition series for $G$.

*Step 2:* Let us compare (vi) and (vii).

The composition factors of (vi) are

$$L_1/L_0, L_2/L_1, \ldots, L_t/L_{t-1}, H_{r-1}/L_t, G/H_{r-1}$$

and of (vii) are

$$L_1/L_0, L_2/L_1, \ldots, L_t/L_{t-1}, K_{s-1}/L_t, G/K_{s-1}$$

From (viii), $G/H_{r-1} \cong K_{s-1}/L_t$, and from (ix), $G/K_{s-1} \cong H_{r-1}/L_t$

Thus, there is a one-to-one correspondence between composition factors of (vi) and (vii), such that the corresponding factors are isomorphic. Hence, (vi) and (vii) are equivalent.

*Step 3:* The composition series (i) and (vi) satisfy the condition of Case 1. Therefore, by Case 1 they are equivalent.

Similarly the composition series (ii) and (vii) are equivalent.

Consequently (i) and (ii) are equivalent.

This completes the proof of Jordan–Hölder Theorem. ♦

**Example 23.12** We now verify Jordan–Hölder Theorem for a given group.

Let $G = \langle a \rangle$ be a cyclic group of order 54. Then $|a| = 54$.

Consider the following two subnormal series:

$$\langle e \rangle \trianglelefteq \langle a^{18} \rangle \trianglelefteq \langle a^6 \rangle \trianglelefteq \langle a \rangle \tag{i}$$

$$\langle e \rangle \trianglelefteq \langle a^{18} \rangle \trianglelefteq \langle a^9 \rangle \trianglelefteq \langle a \rangle \tag{ii}$$

The factors in series (i) are $\langle a^{18} \rangle/\langle e \rangle$, $\langle a^6 \rangle/\langle a^{18} \rangle$, and $\langle a \rangle/\langle a^6 \rangle$, which are of orders 3, 3, and 6, respectively. The last factor $\langle a \rangle/\langle a^6 \rangle$ is not simple. We insert terms at last step to make the factors simple. Now consider

$$\langle e \rangle \trianglelefteq \langle a^{18} \rangle \trianglelefteq \langle a^6 \rangle \trianglelefteq \langle a^3 \rangle \trianglelefteq \langle a \rangle. \tag{iii}$$

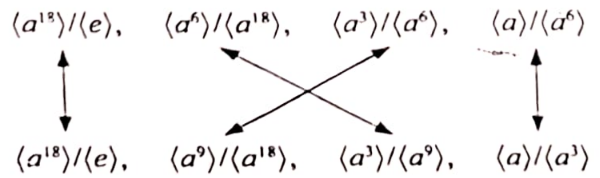
The factors $\langle a^{18} \rangle/\langle e \rangle$, $\langle a^6 \rangle/\langle a^{18} \rangle$, $\langle a^3 \rangle/\langle a^6 \rangle$, and $\langle a \rangle/\langle a^6 \rangle$ are of order 3, 3, 2, and 3, respectively. All factors are of prime order, therefore they are simple. Hence, the refined series (iii) is a composition series.

Similarly, the factors $\langle a^{18} \rangle/\langle e \rangle$, $\langle a^9 \rangle/\langle a^{18} \rangle$, and $\langle a \rangle/\langle a^9 \rangle$, in series (ii) are of orders 3, 2, and 9, respectively. Since $|\langle a \rangle/\langle a^9 \rangle| = 9$ is not prime, we insert a term $\langle a^3 \rangle$ between $\langle a^9 \rangle$ and $\langle a \rangle$ and the refined series

$$\langle e \rangle \trianglelefteq \langle a^{18} \rangle \trianglelefteq \langle a^9 \rangle \trianglelefteq \langle a^3 \rangle \trianglelefteq \langle a \rangle \tag{iv}$$

has factors $\langle a^{18} \rangle/\langle e \rangle$, $\langle a^9 \rangle/\langle a^{18} \rangle$, $\langle a^3 \rangle/\langle a^9 \rangle$, and $\langle a \rangle/\langle a^3 \rangle$ of orders 3, 2, 3, and 3, respectively. Since all the factors are of prime order, therefore they are simple. The refinement (iv) is thus a composition series.

The one-to-one correspondence between the factors of (iii) and (iv) is shown below:




Hence, Jordan–Hölder Theorem is verified. ◇

## SOLVED PROBLEMS 23A

**Problem 23.1** Find all the three composition series for $Q_8$ and list the composition factors.

*Solution* The lattice of subgroups of $Q_8$ is

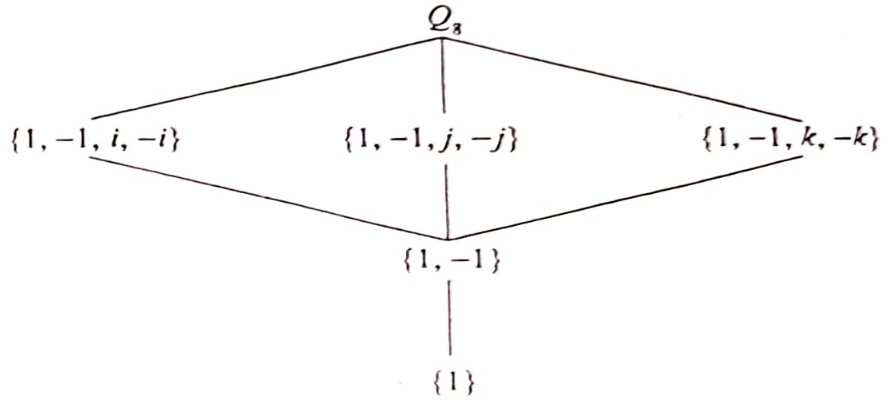



**Figure 23.1**

The three series of subgroups are

$$\{1\} \le \{1, -1\} \le \{1, -1, i, -i\} \le Q_8 \quad \text{(i)}$$

$$\{1\} \le \{1, -1\} \le \{1, -1, j, -j\} \le Q_8 \quad \text{(ii)}$$

$$\{1\} \le \{1, -1\} \le \{1, -1, k, -k\} \le Q_8 \quad \text{(iii)}$$

Consider the series (i):

Each term of the series is of index 2 in its successor. Thus $H_i \trianglelefteq H_{i+1}$ hold for all $0 \le i \le 2$.
Similarly series (ii) and (iii) are also composition series for $Q_8$. ◇

**Problem 23.2** Let $H$ be a normal subgroup of a finite group $G$. Prove that there is a composition series for $G$, one of whose terms is $H$.

*Solution* We consider two cases:

*Case 1:* $H = G$ or $H = \langle e \rangle$

In this case we have nothing to prove as every finite group has a composition series and $G$, $\langle e \rangle$ are the extreme terms of a composition series.

*Case 2:* $\langle e \rangle < H < G$.

By Jordan–Hölder Theorem, $H$ has a composition series, say

$$\langle e \rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{r-1} \trianglelefteq H_r = H$$

Since $G/H$ is finite, by Jordan–Hölder Theorem, $G/H$ has a composition series. Let

$$\langle H \rangle = \overline{K_0} \trianglelefteq \overline{K_1} \trianglelefteq \ldots \overline{K_s} = G/H$$

be a composition series for $G/H$. Each $\overline{K_i} = K_i/H$ for some subgroup $K_i$ of $G$, by Theorem 11.8. Moreover, $\overline{K_i} \trianglelefteq \overline{K_{i+1}}$ implies that $K_i \trianglelefteq K_{i+1}$, by Theorem 11.10.

Also $\overline{K_{i+1}}/\overline{K_i} = (K_{i+1}/H)/(K_i/H) \cong K_{i+1}/K_i$. Since $\overline{K_{i+1}}/\overline{K_i}$ is simple, therefore $K_{i+1}/K_i$ is also simple. This gives rise to a series of subgroups of $G$ from $H$ to $G$ as

$H \trianglelefteq K_1 \trianglelefteq K_2 \dots \trianglelefteq K_s = G$ such that $K_{i+1}/K_i$ is simple.

Hence

$$\langle e \rangle = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_{r-1} \trianglelefteq H_r = H \trianglelefteq K_1 \trianglelefteq K_2 \dots \trianglelefteq K_s = G$$

is a composition series for $G$, which has $H$ as one of the terms. ◇

**Problem 23.3** If $G$ is an Abelian simple group, prove that $G \cong \mathbb{Z}_p$ for some prime $p$.

*Solution* Let $G$ be an Abelian simple group. Now, we prove the following:

1. $G$ is cyclic.
2. $G$ must be finite.
3. $|G|$ is prime.
4. $G \cong \mathbb{Z}_p$.

*Step 1:* Let $e \neq a \in G$. $\langle a \rangle$ be a normal subgroup of $G$. But $G$ is simple, therefore $\langle a \rangle = G$. Hence, $G$ is cyclic.

*Step 2:* If $G$ is infinite, then $\langle e \rangle \trianglelefteq \langle a^2 \rangle \trianglelefteq \langle a \rangle = G$, which contradicts our assumption that $G$ is simple. lence, $G$ is finite.

*tep 3:* Let $p$ be a prime dividing $|G|$. Then $G$ has an element, say $b$, of order $p$. Moreover,

$$\langle e \rangle \trianglelefteq \langle b \rangle \trianglelefteq G.$$

As $G$ is simple, we must have $G = \langle b \rangle$. Thus $G$ is a cyclic group of prime order $p$.

*Step 4:* Since every cyclic group of order, $p$ is isomorphic to $\mathbb{Z}_p$. Hence $G \cong \mathbb{Z}_p$. ◇

**Problem 23.4** Find all seven composition series for $D_8$ and verify Jordan–Hölder Theorem.

*Solution* $D_8$ has five subgroups of order 2 and three subgroups of order 4. The relation of containment is shown by the following lattice diagram:

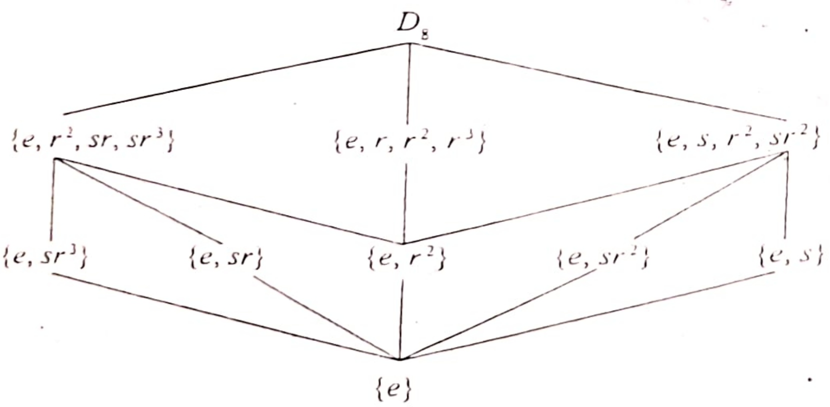



**Figure 23.2**

This diagram shows the seven subnormal series of $D_8$. The seven composition series are

$$\{e\} \trianglelefteq \{e, r^2\} \trianglelefteq \{e, r^2, sr, sr^3\} \trianglelefteq D_8 \quad \text{(i)}$$

$$\{e\} \leq \{e, sr^3\} \leq \{e, r^2, sr, sr^3\} \leq D_8 \quad \text{(ii)}$$

$$\{e\} \leq \{e, sr\} \leq \{e, r^2, sr, sr^3\} \leq D_8 \quad \text{(iii)}$$

$$\{e\} \leq \{e, r^2\} \leq \{e, r, r^2, r^3\} \leq D_8 \quad \text{(iv)}$$

$$\{e\} \leq \{e, r^2\} \leq \{e, s, r^2, sr^2\} \leq D_8 \quad \text{(v)}$$

$$\{e\} \leq \{e, s\} \leq \{e, s, r^2, sr^2\} \leq D_8 \quad \text{(vi)}$$

$$\{e\} \leq \{e, sr^2\} \leq \{e, s, r^2, sr^2\} \leq D_8 \quad \text{(vii)}$$

In each series, each subgroup is of index 2 in the next term of the series, so that each factor group is simple. Hence all the seven series are composition series.

Each series has three composition factors, each of which is a simple group of order 2 and is therefore isomorphic to $\mathbb{Z}_2$. Hence all the seven series are equivalent.

**Problem 23.5** Let a finite group $G$ have two composition series

$$\langle e \rangle = N_0 \trianglelefteq N_1 \trianglelefteq \ldots \trianglelefteq N_{r-1} \trianglelefteq N_r = G \quad \text{(i)}$$

and

$$\langle e \rangle = M_0 \trianglelefteq M_1 \trianglelefteq M_2 = G \quad \text{(ii)}$$

Show that $r = 2$ and the list of the composition factors is the same.

*Solution* Since (ii) is a composition series, $M_1/\langle e \rangle$ and $G/M_1$ are simple.

By Theorem 23.1, $M_1$ is a maximal normal subgroup of $G$. We consider two cases:

*Case 1:* $M_1 = N_{r-1}$

Since $M_1$ is simple, $N_{r-1}$ is also simple.

As $N_{r-2} \trianglelefteq N_{r-1}$ and $N_{r-1}$ is simple, therefore $N_{r-2} = \langle e \rangle$. Thus $r - 2 = 0$ so that $r = 2$.

The two series reduce to

$$\langle e \rangle = N_0 \trianglelefteq N_1 \trianglelefteq N_2 = G$$

$$\langle e \rangle = M_0 \trianglelefteq M_1 \trianglelefteq M_2 = G$$

The composition factors are $N_1/\langle e \rangle$, $G/N_1$, and $M_1/\langle e \rangle$, $G/M_1$. These are same as $M_1 = N_1$

*Case 2:* $M_1 \neq N_{r-1}$

We first show that $G = N_{r-1}M_1$

Since $M_1 < N_{r-1}M_1 \leq G$, and $M_1$ is a maximal normal subgroup of $G$, it follows that $N_{r-1}M_1 = G$.

Let $H = N_{r-1} \cap M_1$. Then $H \trianglelefteq M_1$. But $M_1$ is simple, therefore $H = M_1$ or $H = \langle e \rangle$.

Now
$$H = M_1$$
$$\Rightarrow N_{r-1} \cap M_1 = M_1$$
$$\Rightarrow M_1 \subseteq N_{r-1}.$$

which is not possible as $M_1$ is simple and $M_1 \neq N_{r-1}$.

Hence, $H = \langle e \rangle$. Then $N_{r-1} \cap M_1 = \langle e \rangle < N_{r-1} < N_{r-1} M_1$.

By Second Isomorphism Theorem $N_{r-1}/(N_{r-1} \cap M_1) \cong (N_{r-1} M_1)/M_1$.

Thus, $N_{r-1}/\langle e \rangle \cong G/M_1$. But $G/M_1$ is simple so that $N_{r-1}/\langle e \rangle$ ($\cong N_{r-1}$) is also simple, so that $N_{r-2} = \langle e \rangle$. This gives $r - 2 = 0$, i.e. $r = 2$.

Thus the two series reduce to

$$\langle e \rangle = N_0 \trianglelefteq N_1 \trianglelefteq N_2 = G$$

$$\langle e \rangle = M_0 \trianglelefteq M_1 \trianglelefteq M_2 = G$$

The composition factors are:

$$N_1/\langle e\rangle,\ G/N_1 \quad \text{and} \quad M_1/\langle e\rangle,\ G/M_1.$$

$G/N_1 = M_1N_1/N_1 \cong M_1/(N_1 \cap M_1) = M_1/\langle e\rangle$.

Similarly $G/M_1 \cong N_1/\langle e\rangle$. Thus there is a one to one correspondence between the composition factors of the two series such that the corresponding factors are isomorphic. ◇

**Problem 23.6** Find a composition series for a finite $p$-group. (Corollary 19.7).

*Solution* Let $G$ be a finite $p$-group of order $p^n$. $G$ has subgroups of order $p^i$ for every $1 \leq i \leq n-1$. By Sylow's First Theorem, every subgroup $H_i$ of order $p^i$ contains a normal subgroup $H_{i-1}$ of order $p^{i-1}$. Further, a subgroup of order $p^{i-1}$ is normal in a subgroup of order $p^i$. Thus we get a subnormal series

$$\langle e\rangle = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_{n-1}\ H_n = G \qquad \text{(i)}$$

The factors of the series are $H_{i+1}/H_i$, $i = 0, 1, \ldots, n-1$. But, $H_{i+1}/H_i$ is of order $p$ and every group of prime order is simple. Therefore all factors are simple.

Hence (i) is a composition series for $G$. ◇

## EXERCISES 23A

1. Find three different composition series for $Q_8$.
2. Find six different composition series for $\mathbb{Z}_{36}$.
3. Find a composition series for $D_{12}$ and find the composition factors.
4. Find a composition series and the composition factors for $D_6 \oplus \mathbb{Z}_2$.
5. Give three different refinements of the subnormal series.

$$\langle e\rangle \leq V_4 = \{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \leq S_4$$

to a composition series. Also find the composition factors for $S_4$ in each case and verify Jordan–Hölder Theorem.