D'Alembert's Principle

The principle of virtual work deals only with statics. We are tempted at this point to find a principle that involves the general motion of the system. Such a principle was first suggested by Bernoulli and then developed by D'Alembert. We first write Newton's second law of motion in the following form

$$\mathbf{F}_i - \frac{d\mathbf{p}_i}{dt} = 0$$

If we regard $-d\mathbf{p}_i/dt$ as a force, an inertial or reversed effective force as named by Bernoulli and D'Alembert, which added to $\mathbf{F}_i$ produces equilibrium, then dynamics reduces to statics. Now instead of Eq 4.10 we have

$$\sum_{i=1}^{N} (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta\mathbf{r}_i = 0 \tag{4.21}$$

We now resolve $\mathbf{F}_i$ into applied force $\mathbf{F}_i^{(e)}$ and forces of constraint $\mathbf{f}_i$, and if we again restrict ourselves to a system for which the virtual work of the forces of constraint vanishes, we obtain

$$\sum_{i=1}^{N} (\mathbf{F}_i^{(e)} - \dot{\mathbf{p}}_i) \cdot \delta\mathbf{r}_i = 0 \tag{4.22}$$

This equation is the Lagrangian form of D'Alembert's principle. The superscript e in Eq. 4.22 can now be dropped without ambiguity.

Lagrange's Equations: From D'Alembert's Principle

Lagrange selected D'Alembert's principle as the starting point of his "Mecanique Analytique" and obtained the equations of motion, now known as Lagrange's equations, from it. This is what we now proceed to do. We shall first transform Eq. 4.22 into an equation involving virtual displacements of the generalized coordinates δq_i, which are independent of each other. In terms of the generalized coordinates the virtual work done by the force $\mathbf{F}_i$ (the external applied force) becomes $\Sigma_j Q_j\, \delta q_j$ as shown in Eq. 4.12:

$$\delta W' = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta\mathbf{r}_i = \sum_{j=1} Q_j\, \delta q_j \tag{4.23}$$

and Q_j is given by Eq. 4.13. We now write the inertial force term in Eq. 4.22 as

$$\begin{aligned}\sum_{i=1}^{N} \dot{\mathbf{p}}_i \cdot \delta\mathbf{r}_i &= \sum_{i}^{N}\sum_{j}^{n} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\,\delta q_j \\ &= \sum_{j}^{n} \delta q_j \sum_{i}^{N} m_i \left[\frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\right) - \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right)\right]\end{aligned} \tag{4.24}$$

and from Eq. 4.1 we find that

$$d\mathbf{r}_i = \sum_{j}^{n} \frac{\partial \mathbf{r}_i}{\partial q_j}\, dq_j + \frac{\partial \mathbf{r}_i}{\partial t}\, dt \quad \text{and} \quad \dot{\mathbf{r}}_i = \sum_{j}^{n} \frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}}{\partial t} \tag{4.25}$$

The partial derivatives in Eq. 4.25 are themselves functions of the generalized coordinates q_i and the time. As a result, the particle velocities have the following functional form

$$\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_i(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n), \quad i = 1, \ldots, N$$

Moreover, Eq 4.25 provides an explicit function of the indicated variables and shows that $\dot{\mathbf{r}}_i$ in fact depends linearly on the generalized velocities $\dot{q}_j$. Thus we can readily evaluate the partial derivative $\partial\dot{\mathbf{r}}_i/\partial\dot{q}_j$ to obtain

$$\partial\dot{\mathbf{r}}_i/\partial\dot{q}_j = \partial\mathbf{r}_i/\partial q_j \tag{4.26}$$

Note that now the independent variables appearing in parentheses in Eqs. 4.24 are physically independent, in the sense that each can be specified independently at a given instant of time. The subsequent motion of the system is then, of course, determined by the equations of motion. We now substitute Eq. 4.26 into the first term on the right-hand side of Eq. 4.24, with the result that

$$\begin{aligned}\sum_{i=1}^{N} m_i \frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}\right) &= \sum_{i=1}^{N} m_i \frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}\right) \\ &= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\sum_{i=1}^{N} m_i |\dot{\mathbf{r}}_i|^2\right) = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}\end{aligned} \tag{4.27}$$

where $T = \frac{1}{2}\sum_{i}^{N} m_i \dot{r}_i^2$ is the total kinetic energy of the system.

We can also rewrite the second term on the right side of Eq. 4.24 as

$$\sum_i^N m_i\dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right) = \sum_i^N m_i\dot{\mathbf{r}}_i \cdot \left[\sum_k \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_j}\dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial q}\right] = \sum_i^N m_i\dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}$$
$$= \frac{\partial}{\partial q_j}\left(\frac{1}{2}\sum_i^N m_i|\dot{\mathbf{r}}_i|^2\right) = \frac{\partial T}{\partial q_i} \tag{4.28}$$

With Eqs. 4.27 and 4.28, Eq. 4.24 becomes

$$\sum_i^N \dot{\mathbf{p}}_i \cdot \delta\mathbf{r}_i = \sum_j \delta q_j\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}\right) \tag{4.29}$$

From Eqs. 4.23 and 4.29, D'Alembert's principle gives

$$\sum_j^n \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j\right)\delta q_j = 0$$

The δq_j are all independent for a holonomic system, and each of the coefficients must separately vanish. From which it follows that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \tag{4.30}$$

there being n such equations in all, $j = 1, 2, \ldots, n$, n being the number of degrees of freedom of the system.

Equation 4.30 can be simplified further if the external forces $\mathbf{F}_i$ are conservative: $\mathbf{F}_i = -\nabla_i V$. Then

$$Q_j = \sum_{i=1}^N \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_{i=1}^N \nabla_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

which is exactly the same expression for the partial derivative of a function $-V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t)$ with respect to q

$$Q_j = -\frac{\partial V}{\partial q_j} \tag{4.31}$$

and Eq. 4.30 becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T - V)}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n$$

Now if the potential V is a function of position only, then it is independent of the generalized velocities $\dot{q}_j$. We can now include a term in V in the first term on the right side of the preceding equation:

$$\frac{d}{dt}\frac{\partial (T - V)}{\partial \dot{q}_j} - \frac{\partial (T - V)}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, n$$

We now introduce a new function L defined by

$$L(q_i, \dot{q}_i, t) = T(q_i, \dot{q}_i, t) - V(q_i) \tag{4.32}$$

This function is called the Lagrangian function of the system. In terms of this function, the preceding equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad (j = 1, 2, \ldots, n) \tag{4.33}$$

where n is the number of degrees of freedom of the system. These n second-order differential equations are called Lagrange's equations for a conservative, holonomic dynamical system. If some of the forces acting on the system are not conservative, the Lagrange's equation can be written in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j', \qquad i = 1, 2, \ldots, n \tag{4.34}$$

where L contains the potential of the conservative forces as before, and Q_j' represents the force not arising from a potential. Examples of typical nonconservative forces Q' are frictional forces and time-varying force functions.