T in terms of Cartesian coordinates and then write T in terms of generalized coordinates. (Note that we are using T instead of K for kinetic energy.) Let x, y, and z be Cartesian coordinates, while $q_1, q_2, \ldots, q_n$ are generalized coordinates. The kinetic energy of the particle in Cartesian coordinates is

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{12.15}$$

since

$$x = x(q_1, q_2, \ldots, q_n) = x(q) \tag{12.16}$$

similarly

$$y = y(q), \quad z = z(q) \tag{12.17}$$

we can evaluate $\dot{x}$ in terms of q_k by the following procedure:

$$\dot{x} = \frac{\partial x}{\partial q_1}\frac{\partial q_1}{\partial t} + \frac{\partial x}{\partial q_2}\frac{\partial q_2}{\partial t} + \cdots + \frac{\partial x}{\partial q_n}\frac{\partial q_n}{\partial t}$$

$$= \sum_{k=1}^{n}\frac{\partial x}{\partial q_k}\frac{\partial q_k}{\partial t} = \sum_{k=1}^{n}\frac{\partial x}{\partial q_k}\dot{q}_k = \dot{x}(q, \dot{q}) \tag{12.18}$$

Thus we can describe the different components of velocity in terms of the generalized coordinates q_k and generalized velocities $\dot{q}_k$; that is,

$$\dot{x} = \dot{x}(q, \dot{q}), \quad \dot{y} = \dot{y}(q, \dot{q}), \quad \dot{z} = \dot{z}(q, \dot{q}) \tag{12.19}$$

We may now write Eq. (12.15) for kinetic energy as

$$T = \tfrac{1}{2}m[\dot{x}^2(q, \dot{q}) + \dot{y}^2(q, \dot{q}) + \dot{z}^2(q, \dot{q})] \tag{12.20}$$

Taking the derivative with respect to the generalized velocity $\dot{q}_k$,

$$\frac{\partial T}{\partial \dot{q}_k} = m\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{q}_k} + \dot{y}\frac{\partial \dot{y}}{\partial \dot{q}_k} + \dot{z}\frac{\partial \dot{z}}{\partial \dot{q}_k}\right) \tag{12.21}$$

Using Eq. (12.18), we may write

diff. eq (12.18) w.r.t. $\dot{q}_K$

$$\frac{\partial \dot{x}}{\partial \dot{q}_k} = \frac{\partial x}{\partial q_k} \tag{12.22}$$

Note that $\partial x/\partial q_k$ is the coefficient of $\dot{q}_k$ in the expression of $\dot{x}$ in Eq. (12.18). Substituting this and similar expressions for other terms in Eq. (12.21),

$$\frac{\partial T}{\partial \dot{q}_k} = m\left(\dot{x}\frac{\partial x}{\partial q_k} + \dot{y}\frac{\partial x}{\partial q_k} + \dot{z}\frac{\partial z}{\partial q_k}\right) \tag{12.23}$$

Now differentiate both sides of this equation with respect to t:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = m\ddot{x}\frac{\partial x}{\partial q_k} + m\ddot{y}\frac{\partial y}{\partial q_k} + m\ddot{z}\frac{\partial z}{\partial q_k} + m\dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right) + m\dot{y}\frac{d}{dt}\left(\frac{\partial y}{\partial q_k}\right) + m\dot{z}\frac{d}{dt}\left(\frac{\partial z}{\partial q_k}\right) \tag{12.24}$$

To simplify the last three terms on the right side, we use the fact that d/dt and $\partial/\partial q_k$ are interchangeable.

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right) = \frac{\partial}{\partial q_k}\left(\frac{dx}{dt}\right) = \frac{\partial \dot{x}}{\partial q_k} \tag{12.25}$$

Thus the fourth term on the right of Eq. (12.24) may be written as

$$m\dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial q_k}\right) = m\dot{x}\frac{\partial \dot{x}}{\partial q_k} = \frac{\partial}{\partial q_k}\left(\frac{1}{2}m\dot{x}^2\right) \tag{12.26}$$

with similar expressions for other terms. Also note that

$$F_x = m\ddot{x}, \quad F_y = m\ddot{y}, \quad F_z = m\ddot{z} \tag{12.27}$$

Combining Eqs. (12.25) and (12.26) with Eq. (12.24), we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = F_x\frac{\partial x}{\partial q_k} + F_y\frac{\partial y}{\partial q_k} + F_z\frac{\partial z}{\partial q_k} + \frac{\partial}{\partial q_k}\left[\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right] \tag{12.28}$$

Using the definition of generalized force and kinetic energy given by Eqs. (12.8) and (12.20),

$$Q_k = F_x\frac{\partial x}{\partial q_k} + F_y\frac{\partial y}{\partial q_k} + F_z\frac{\partial z}{\partial q_k} \tag{12.8}$$

$$T = \tfrac{1}{2}m[\dot{x}^2(q,\dot{q}) + \dot{y}^2(q,\dot{q}) + \dot{z}^2(q,\dot{q})] \tag{12.20}$$

in Eq. (12.28) gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = Q_k + \frac{\partial T}{\partial q_k} \tag{12.29}$$

These differential equations in generalized coordinates describe the motion of a particle and are known as *Lagrange's equations* of motion.

Lagrange's equations take a much simpler form if the motion is in a conservative force field so that

$$Q_k = -\frac{\partial V}{\partial q_k} \tag{12.30}$$

which on substituting in Eq. (12.29) yields

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{\partial T}{\partial q_k} - \frac{\partial V}{\partial q_k} \tag{12.31}$$

Let us define a *Lagrangian function* L as the difference between the kinetic energy and potential energy; that is,

$$L = T - V \quad \text{or} \quad L(q,\dot{q}) = T(q,\dot{q}) - V(q) \tag{12.32}$$

It is important to know that, if V is a function of the generalized coordinates and not of the generalized velocities, then

$$V = V(q) \quad \text{and} \quad \frac{\partial V}{\partial \dot{q}_k} = 0 \tag{12.33}$$

[If V is not independent of velocity $\dot{q}$, then $V = V(q, \dot{q})$ will lead to a tensor force, which we will not discuss here.] Thus we may write

$$\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}(T - V) = \frac{\partial T}{\partial \dot{q}_k}$$

$$\frac{\partial L}{\partial q_k} = \frac{\partial}{\partial q_k}(T - V) = \frac{\partial T}{\partial q_k} - \frac{\partial V}{\partial q_k}$$

Substituting these results in Eq. (12.31) yields

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \tag{12.34}$$

which are *Lagrange's equations describing the motion of a particle in a conservative force field.* To solve these equations, we must know the Lagrangian function L in the appropriate generalized coordinates. Since energy is a scalar quantity, the Lagrangian L is a scalar function. Thus the Lagrangian L will be invariant with respect to coordinate transformations. This means that *the Lagrangian gives the same description of the system under given conditions no matter which generalized coordinates are used.* Thus Eq. (12.34) describes the motion of a particle moving in a conservative force field in terms of any generalized coordinates.