12.2 GENERALIZED COORDINATES AND CONSTRAINTS

To locate the position of a particle, we need three coordinates. These coordinates could be Cartesian coordinates x, y, and z, cylindrical coordinates r, θ, and z, spherical coordinates r, θ, and ϕ, or any other three suitable coordinates. If there are some restrictions or constraints on the motion of the particle, we need less than three coordinates. For example, if a particle is constrained to move on a plane surface, only two coordinates are sufficient, while if the particle is constrained to move in a straight line, only one coordinate is sufficient to describe the motion of the particle.

Let us consider a mechanical system consisting of N particles. To specify the position of such a system at any given time, we need N vectors, while each vector can be described by three coordinates. Thus, in general, we need $3N$ coordinates to describe a given mechanical system. If there are constraints, the total number of coordinates needed to specify the system will be reduced. As an example, suppose the system is a rigid body, and as we know, the distances between different particles are fixed. These fixed distances can be expressed in the form of equations. As we explained in Chapter 9, a rigid body can be completely described by only six coordinates; that is, only six coordinates are needed to specify the configuration of a rigid body system. Of these six, three coordinates give the position of some convenient reference point in the body, usually the center of mass with respect to the origin of some chosen coordinate system, and the remaining three coordinates describe the orientation of the body in space.

We are interested in finding the minimum number of coordinates needed to describe a system of N particles. Usually, the constraints on any given system are described by means of equations. Suppose there are m number of such equations that describe the constraints. The minimum number of coordinates, n, needed to completely describe the motion or the configuration of such a system at any given time is given by

$$n = 3N - m \tag{12.1}$$

where n is the number of *degrees of freedom* of the system. It is not necessary that these n coordinates should be rectangular, cylindrical, or any other curvilinear coordinates. As a matter of fact, n could be any parameter, such as length, (length)^2, angle, energy, a dimensionless quantity, or any other quantity, as long as it completely describes the configuration of the system. The name *generalized coordinates* is given to *any set of quantities that completely describes the state or configuration of a system*. These n generalized coordinates are customarily written as

$$q_1, q_2, q_3, \ldots, q_n \tag{12.2a}$$

or

$$q_k, \quad \text{where} \quad k = 1, 2, 3, \ldots, n \tag{12.2b}$$

These n generalized coordinates are not restricted by any constraints. If each coordinate can vary independently of the other, the system is said to be *holonomic*. In a *nonholonomic* system, the coordinates *cannot* vary independently. Hence in such systems the number of degrees of freedom is less than the minimum number of coordinates needed to specify the configuration of the system. As an example, a sphere constrained to roll on a perfectly rough plane surface needs only five coordinates to specify its configuration, two for the position of its center of mass and three for its orientation. But these five coordinates cannot all vary independently. When the sphere rolls, at least two coordinates must change. Hence this is a nonholonomic system. The investigation and description of nonholonomic systems are involved and will not be considered here. We shall limit ourselves to the discussion of holonomic systems for the time being.

A suitable set of generalized coordinates of a system is that which results in equations of motion leading to any easy interpretation of the motion. These q_n generalized coordinates form a configuration space, with each dimension represented by a coordinate q_k. The path of the system is represented by a curve in this configuration space. The path in the configuration space does not lend itself to the same interpretation as a path in ordinary three-dimensional space. In analogy with Cartesian coordinates, we may define the derivatives of q_k, that is $\dot{q}_1, \dot{q}_2, \ldots$, or $\dot{q}_k$ as *generalized velocities*.

Let us consider a single particle whose rectangular coordinates x, y, and z are a function of the generalized coordinates q_1, q_2, and q_3; that is

$$x = x(q_1, q_2, q_3) = x(\mathbf{q}_k)$$

$$y = y(q_1, q_2, q_3) = y(\mathbf{q}_k)$$

$$z = z(q_1, q_2, q_3) = z(\mathbf{q}_k) \tag{12.3}$$

Suppose the system changes from an initial configuration given by (q_1, q_2, q_3) to a neighborhood configuration given by $(q_1 + \delta q_1, q_2 + \delta q_2, q_3 + \delta q_3)$. We can express the corresponding changes in the Cartesian coordinates by the following relations:

$$\delta x = \frac{\partial x}{\partial q_1}\delta q_1 + \frac{\partial x}{\partial q_2}\delta q_2 + \frac{\partial x}{\partial q_3}\delta q_3 = \sum_{k=1}^{n} \frac{\partial x}{\partial q_k}\delta q_k \tag{12.4}$$

with similar expression for δy and δz, where n is equal to three and the partial derivatives $\partial x/\partial q_k, \ldots$, are functions of q's. The value of n depends on the degrees of freedom. For example, if there were no constraints, $m = 0$, and from Eq. (12.1) for $N = 1$, $n = 3$, as we have used above, n would be less than 3 if there were constraints on the system.

Let us consider a more general case in which a mechanical system consists of a large number of particles having n degrees of freedom. The configuration of the system is specified by the generalized coordinates $q_1, q_2, \ldots, q_n$. Suppose the configuration of the system changes from $(q_1, q_2, \ldots, q_n)$ to a new configuration $(q_1 + \delta q_1, q_2 + \delta q_2, \ldots, q_n + \delta q_n)$. The Cartesian coordinates of a particle i change from (x_i, y_i, z_i) to $(x_i + \delta x_i, y_i + \delta y_i, z_i + \delta z_i)$. This displacements δx_i, δy_i, and δz_i can be expressed in terms of the generalized coordinates q_k as

$$\delta x_i = \frac{\partial x_i}{\partial q_1}\delta q_1 + \frac{\partial x_i}{\partial q_2}\delta q_2 + \cdots + \frac{\partial x_i}{\partial q_k}\delta q_k = \sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k}\delta q_k, \tag{12.5}$$

with similar expression for δy_i and δz_i. Once again the partial derivatives are functions of the generalized coordinates q_k.

It is essential at this point to distinguish between two types of displacements: an actual displacement $d\mathbf{r}_i$ and a virtual (not in actual fact or name) displacement $\delta\mathbf{r}_i$. Suppose a mass m_i is acted on by an external force $\mathbf{F}_i$ and causes the mass m_i to move from $\mathbf{r}_i$ to $\mathbf{r}_i + d\mathbf{r}_i$ in a time interval dt. This displacement must be consistent with both the equations of motion and the equations of constraints that describe this mass system; hence such displacements are *actual displacements*. On the other hand, *virtual displacements* are consistent with the equations of the constraints but do not satisfy the equations of motion or time. For example, the bob of a pendulum of length l may be moved from (l, θ) to $(l, \theta + \delta\theta)$ in any arbitrary time interval as long as the bob remains on the arc of a circle of radius l. Thus $\delta\mathbf{r}_i$ and δq_i are the virtual displacements. We shall make use of the principle of virtual work in the following. We shall cause a virtual displacement $\delta\mathbf{r}$, resulting in virtual work δW. Basically, in such displacements, the relative orientations and distances between the particles remain unchanged.

2.3 GENERALIZED FORCES

Single Particle

Consider a force $\mathbf{F}$ that is acting on a single particle of mass m and produces a virtual displacement $\delta\mathbf{r}$ of the particle. The work done δW by this force is given by

$$\delta W = \mathbf{F} \cdot \delta\mathbf{r} = F_x\,\delta x + F_y\,\delta y + F_z\,\delta z \qquad (12.6)$$

where F_x, F_y, and F_z are the rectangular components of $\mathbf{F}$. We can express the displacements δx, δy, and δz in terms of the generalized coordinates q_k. Making use of Eqs. (12.4) and (12.6), we may write

$$\delta W = \sum_{k=1}^{n}\left(F_x\frac{\partial x}{\partial q_k} + F_y\frac{\partial y}{\partial q_k} + F_z\frac{\partial z}{\partial q_k}\right)\delta q_k = \sum_{k=1}^{n} Q_k\,\delta q_k \qquad (12.7)$$

where

$$Q_k = F_x\frac{\partial x}{\partial q_k} + F_y\frac{\partial y}{\partial q_k} + F_z\frac{\partial z}{\partial q_k} \qquad (12.8)$$

Q_k is called the *generalized force* associated wth the generalized coordinate q_k. The dimensions of Q_k depend on the dimensions of q_k. The dimensions of $Q_k\,\delta q_k$ are that of work. If the increment δq_k has the dimensions of distance, Q_k will have the dimensions of force; if δq_k has the dimensions of angle θ, Q_k will have dmensions of torque τ_θ. It may be pointed out that the quantity δq_k and the quantities δx, δy, and δz are called *virtual displacements* of the system because it is not necessary that such displacements represent any actual displacements.

A System of Particles

Let us apply the preceding ideas to a general case of a system consisting of N particles acted on by forces $\mathbf{F}_i$ $(i = 1, 2, \ldots, N)$. The total work done δW for a virtual displacement $\delta\mathbf{r}_i$ of the system is

$$\delta W = \sum_{i=1}^{N}\mathbf{F}_i \cdot \delta\mathbf{r}_i = \sum_{i=1}^{N} F_{x_i}\,\delta x_i + F_{y_i}\,\delta y_i + F_{z_i}\,\delta z_i \qquad (12.9)$$

... displacements in terms of the generalized coordinates ...
Eq. (12.5), we get

$$\delta W = \sum_{i=1}^{N}\left[\sum_{k=1}^{n}\left(F_{x_i}\frac{\partial x_i}{\partial q_k} + F_{y_i}\frac{\partial y_i}{\partial q_k} + F_{z_i}\frac{\partial z_i}{\partial q_k}\right)\delta q_k\right] \tag{12.10a}$$

Interchanging the order of summation, we get

$$\delta W = \sum_{k=1}^{n}\left[\sum_{i=1}^{N}\left(F_{x_i}\frac{\partial x_i}{\partial q_k} + F_{y_i}\frac{\partial y_i}{\partial q_k} + F_{z_i}\frac{\partial z_i}{\partial q_k}\right)\delta q_k\right] \tag{12.10b}$$

or

$$\delta W = \sum_{k=1}^{n} Q_k\,\delta q_k \tag{12.11}$$

where

$$Q_k = \sum_{i=1}^{N}\left(F_{x_i}\frac{\partial x_i}{\partial q_k} + F_{y_i}\frac{\partial y_i}{\partial q_k} + F_{z_i}\frac{\partial z_i}{\partial q_k}\right) \tag{12.12}$$

Q_k is called the *generalized force* associated with the generalized coordinate q_k. Once again, the dimensions of the generalized force Q_k depend on the dimensions of q_k but the product $Q_k q_k$ is always work.

Conservative Systems

Let us write an expression for the generalized forces that are conservative. Suppose a conservative force field is represented by a potential function $V = V(x, y, z)$. The rectangular components of a force acting on a particle are given by

$$F_x = -\frac{\partial V}{\partial x}, \qquad F_y = -\frac{\partial V}{\partial y}, \qquad F_z = -\frac{\partial V}{\partial z} \tag{12.13}$$

Expression Q_k for a generalized force given by Eq. (12.8) becomes

$$Q_k = F_x\frac{\partial x}{\partial q_k} + F_y\frac{\partial y}{\partial q_k} + F_z\frac{\partial z}{\partial q_k}$$

$$= -\left(\frac{\partial V}{\partial x}\frac{\partial x}{\partial q_k} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial q_k} + \frac{\partial V}{\partial z}\frac{\partial z}{\partial q_k}\right) \tag{12.8}$$

The expression in the parentheses is the partial derivative of the function V with respect to q_k. That is,

$$Q_k = -\frac{\partial V}{\partial q_k} \tag{12.14}$$

This expresses the relation between a generalized force and the potential representing a conservative system.