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R-K Method of order-4 for simultaneous equations:

Consider the system of equations

$$\frac{dy}{dx} = f(x, y, z) \quad ; \quad y(x_0) = y_0$$

$$\frac{dz}{dx} = g(x, y, z) \quad , \quad z(x_0) = z_0$$

where, x is independent variable and y, z are dependent variables.

Runge-Kutta for simultaneous equations is defined as:

$$k_1 = hf(x_0, y_0, z_0) \quad ; \quad l_1 = hg(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) \quad ; \quad l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \quad ; \quad l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_2, z_0 + l_3) \quad ; \quad l_4 = hg(x_0 + h, y_0 + k_2, z_0 + l_3)$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad ; \quad l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$y_1 = y_0 + k \quad ; \quad z_1 = z_0 + l$$

Adam's Predictor-Corrector Method (Adam Bashforth Method) :-

Consider, $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$

Adam's Predictor Formula:

$$y_{n+1} = y_n + \frac{h}{24}\left[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}\right]$$

$n = 3, 4, 5, \ldots$

$$y_{4,P} = y_3 + \frac{h}{24}\left[55y'_3 - 59y'_2 + 37y'_1 - 9y'_0\right]$$

Adam's Corrector Formula:

$$y_{n+1} = y_n + \frac{h}{24}\left[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}\right]$$

$n = 3, 4, 5, \ldots$

$$y_{4,C} = y_3 + \frac{h}{24}\left[9y'_4 + 19y'_3 - 5y'_2 + y'_1\right]$$

Q: Compute $y(1.4)$ given $y' = \frac{1}{x^2} - \frac{y}{x}$;

$y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$ by using Adam's Bashforth method.

$$y' = \frac{1}{x^2} - \frac{y}{x}$$

$$f(x,y) = \frac{dy}{dx} = \frac{1}{x^2} - \frac{y}{x}$$

	x_0	x_1	x_2	x_3
x	1	1.1	1.2	1.3
y	1	0.996	0.986	0.972
	y_0	y_1	y_2	y_3

By using Adam's Bashforth method:

$$y_{4,p} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right] \quad (1)$$

$$y' = \frac{1}{x^2} - \frac{y}{x}$$

$$y_0' = \frac{1}{x_0^2} - \frac{y_0}{x_0} = \frac{1}{(1)^2} - \frac{1}{1} = 0$$

$$y_1' = \frac{1}{x_1^2} - \frac{y_1}{x_1} = \frac{1}{(1.1)^2} - \frac{0.996}{1.1} = -0.079$$

$$y_2' = \frac{1}{x_2^2} - \frac{y_2}{x_2} = \frac{1}{(1.2)^2} - \frac{0.986}{1.2} = -0.1272$$

$$y_3' = \frac{1}{x_3^2} - \frac{y_3}{x_3} = \frac{1}{(1.3)^2} - \frac{0.972}{1.3} = -0.1560$$

Putting above values in (1)

$$y_{4,p} = 0.972 + \frac{(0.1)}{24}\left[55(-0.1560) - 59(-0.1272) + 37(-0.079) - 9(0)\right]$$

$$\boxed{y_{4,p} = 0.9553}$$

$$y_{4,c} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right] \quad — ②$$

$$x_n = x_0 + nh$$

$$x_4 = x_0 + 4h$$

$$= 1 + 4(0.1)$$

$$x_4 = 1.4$$

$$y_4' = \frac{1}{x_4^2} - \frac{y_4}{x_4} = \frac{1}{(1.4)^2} - \frac{0.9553}{1.4} = -0.1722$$

Putting values in ② we have:

$$y_{4,c} = 0.972 + \frac{(0.1)}{24}\left[9(-0.1722) + 19(-0.1560) - 5(-0.1272) + (-0.079)\right]$$

$$\boxed{y_{4,c} = 0.9555}$$

Q: Using Adam's Bashforth method find $y(4.4)$ given $5xy' + y^2 = 2$; $y(4) = 1$, $y(4.1) = 1.0049$; $y(4.2) = 1.0097$, $y(4.3) = 1.0143$

$$5xy' + y^2 = 2$$

$$y' = \frac{1}{5x}(2 - y^2)$$

$$f(x,y) = \frac{1}{5x}(2 - y^2)$$

	x_0	x_1	x_2	x_3
x	4	4.1	4.2	4.3
y	1	1.0049	1.0097	1.0143
	y_0	y_1	y_2	y_3

By using Adam's Bashforth method:

$$y_{4,p} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right] \quad — ①$$

$$y' = \frac{1}{5x}(2 - y^2)$$

$$y_0' = \frac{1}{5x_0}(2 - y_0^2) = \frac{2 - (1)^2}{5(4)} = 0.05$$

$$y_1' = \frac{2-y_1^2}{5x_1} = \frac{2-(1.0049)^2}{5(4.1)} = 0.0483$$

$$y_2' = \frac{2-y_2^2}{5x_2} = \frac{2-(1.0097)^2}{5(4.2)} = 0.0467$$

$$y_1' = \frac{2-y_3^2}{5x_3} = \frac{2-(1.0143)^2}{5(4.3)} = 0.0452$$

Putting above values in ①

$$y_{4,p} = 1.0143 + \frac{(0.1)}{24}\left[55(0.0452)-59(0.0467)+37(0.0483)-9(0.05)\right]$$

$$\boxed{y_{4,p} = 1.0187}$$

$$y_{4,c} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right] \quad ②$$

$x_{n} = x_0 + nh$

$x_4 = x_0 + 4h$

$= 4 + 4(0.1)$

$x_4 = 4.4$

$$y_4' = \frac{2-y_4^2}{5x_4} = \frac{2-(1.0187)^2}{5(4.4)} = 0.0437$$

Putting values in ② we have:

$$y_{4,c} = 1.0143 + \frac{0.1}{24}\left[9(0.0437)+19(0.0452)-5(0.0467)+0.0483\right]$$

$$\boxed{y_{4,c} = 1.0187}$$

Q: Solve numerically the differential equation $y' = \frac{dy}{dx} = \frac{1}{x+y}$; $y(0) = 2$; taking starting values $y(0.2) = 2.0933$; $y(0.4) = 2.1755$; $y(0.6) = 2.2493$. Find the values of $y(0.8)$ using Milne's Predictor Corrector Method.

Sol: The given differential equation is:

$$y' = \frac{dy}{dx} = \frac{1}{x+y}$$

$$f(x,y) = \frac{1}{x+y} \quad ; \quad h = 0.2$$

	x_0	x_1	x_2	x_3	
x	0	0.2	0.4	0.6	
y	2	2.0933	2.1755	2.2493	
	y_0	y_1	y_2	y_3	

Milne's Predictor Method:

$$y_4 = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right] \quad —①$$

Now,

$$y_1' = \frac{1}{x_1+y_1} = \frac{1}{0.2+2.0933} = 0.4361$$

$$y_2' = \frac{1}{x_2+y_2} = \frac{1}{0.4+2.1755} = 0.3883$$

$$y_3' = \frac{1}{x_3+y_3} = \frac{1}{0.6+2.2493} = 0.3509$$

Putting above values in ①

$$y_{4,p} = 2 + \frac{4(0.2)}{3}\left[2(0.4361) - 0.3883 + 2(0.3509)\right]$$

$$y_{4,p} = 2.3162$$

$$\boxed{y_4 = y(x_4) = y(0.8) = 2.3162}$$

Milne's Corrector Method:

$$y_4 = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right] \quad (2)$$

$$x_n = x_0 + nh$$

$$x_4 = x_0 + 4h$$

$$= 0 + 4(0.2)$$

$$x_4 = 0.8$$

$$y_4' = \frac{1}{x_4 + y_4} = \frac{1}{0.8 + 2.3162} = 0.3209$$

Putting values in (2) we have:

$$y_4 = 2.1755 + \frac{(0.2)}{3}\left[0.3883 + 4(0.3509) + 0.3209\right]$$

$$y_4 = 2.3164$$

$$\boxed{y_4 = y(x_4) = y(0.8) = 2.3164}$$

Q: Compute the first three steps of initial value problem $\frac{dy}{dx} = \frac{x-y}{2}$, $y(0) = 1$

by Taylor Series Method and next step by Milne's Predictor Corrector Method with step length $h = 0.1$

Sol: The differential equation is:

$$y' = \frac{x-y}{2} \quad ; \quad y(0) = 1$$

$$x_0 = 0 \quad , \quad y_0 = 1 \quad , \quad h = 0.1$$

Taylor Series Formula:

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \cdots \quad (1)$$

Now,

$$y' = \frac{x-y}{2}$$

$$y'' = \frac{1-y'}{2}$$

$$y''' = \frac{-y''}{2}$$

Putting $n=0$ in (1)

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots \quad (2)$$ (52)

$$y_0 = y(x_0) = y(0) = 1$$
$$y_0' = \frac{x_0 - y_0}{2} = \frac{0-1}{2} = -0.5$$
$$y_0'' = \frac{1 - y_0'}{2} = \frac{1-(-0.5)}{2} = 0.75$$
$$y_0''' = \frac{-y_0''}{2} = \frac{-0.75}{2} = -0.375$$

Then (2) becomes:

$$y_1 = 1 + (0.1)(-0.5) + \frac{(0.1)^2}{2!}(0.75) + \frac{(0.1)^3}{3!}(-0.375)$$

$$y_1 = 0.9537$$

$$\boxed{y_1 = y(x_1) = y(0.1) = 0.9537}$$

Putting $n=1$ in (1)

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \dots \quad (3)$$

$$y_1 = y(x_1) = y(0.1) = 0.9537$$
$$y_1' = \frac{x_1 - y_1}{2} = \frac{0.1 - 0.9537}{2} = -0.4269$$
$$y_1'' = \frac{1 - y_1'}{2} = \frac{1-(-0.4269)}{2} = 0.7135$$
$$y_1''' = \frac{-y_1''}{2} = \frac{-0.7135}{2} = -0.3568$$

(3) becomes:

$$y_2 = 0.9537 + (0.1)(-0.4269) + \frac{(0.1)^2}{2!}(0.7135) + \frac{(0.1)^3}{3!}(-0.3568)$$

$$y_2 = 0.9145$$

$$\boxed{y_2 = y(x_2) = y(0.2) = 0.9145}$$

Putting $n=2$ in (1)

$$y_3 = y_2 + hy_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \cdots \quad (4)$$

$$y_2 = y(x_2) = y(0.2) = 0.9145$$

$$y_2' = \frac{x_2 - y_2}{2} = \frac{0.2 - 0.9145}{2} = -0.3573$$

$$y_2'' = \frac{1 - y_2'}{2} = \frac{1-(-0.3573)}{2} = 0.6787$$

$$y_2''' = \frac{-y_2''}{2} = -\frac{0.6787}{2} = -0.3394$$

(4) becomes:

$$y_3 = 0.9145 + (0.1)(-0.3573) + \frac{(0.1)^2}{2!}(0.6787) + \frac{(0.1)^3}{3!}(-0.3394)$$

$$y_3 = 0.8821$$

$$\boxed{y_3 = y(x_3) = y(0.3) = 0.8821}$$

	x_0	x_1	x_2	x_3
x	0	0.1	0.2	0.3
y	1	0.9537	0.9145	0.8821
	y_0	y_1	y_2	y_3

Milne's Predictor Method:

$$y_4 = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right] \quad (5)$$

Now,

~~Putting above values in (5)~~ ~~$x_n = x_0 + nh$~~ ~~$x_3 = 0 + 3(0.1)$~~ ~~$x_3 = 0.3$~~

$$\cancel{y_4 = 1 + \frac{4(0.1)}{3}\left[2(-0.4264)\right.}$$

$$y_3' = \frac{x_3 - y_3}{2} = \frac{0.3 - 0.8821}{2} = -0.2911$$

Putting above values in (5)

$$y_4 = 1 + \frac{4(0.1)}{3}\left[2(-0.4269) - (-0.3573) + 2(-0.2911)\right]$$

$$y_4 = 0.8562$$

$$\boxed{y_4 = y(x_4) = y(0.4) = 0.8562}$$

Milne's Corrector Method:

$$y_4 = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right] \quad (6)$$

$$y_4' = \frac{x_4 - y_4}{2} = \frac{0.4 - 0.8562}{2} = -0.2281$$

Putting values in (6) we have:

$$y_4 = 0.9145 + \frac{(0.1)}{3}\left[-0.3573 + 4(-0.2911) + (-0.2281)\right]$$

$$\boxed{y_4 = 0.8562}$$

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11^{th} March, 2015 Wednesday

Picard's Method of successive Approximations:

Consider the first order differential equation

$$\frac{dy}{dx} = f(x,y) \quad ; \quad y(x_0) = y_0$$

Now,

$$\frac{dy}{dx} = f(x,y)$$

or

$$dy = f(x,y)\, dx$$

Integrating from $x_0 \to x$, corresponding y values are

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\, dx$$

$$y - y_0 = \int_{x_0}^{x} f(x,y)\, dx$$

$$y = y_0 + \int_{x_0}^{x} f(x,y)\, dx \quad \text{——} \; (1)$$

This type of equation is called an integral equation.

It is solved by successive approximations or iterations

1^{st} Approximation:

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0^{(0)})\, dx$$

2^{nd} Approximation:

2^{nd} approximation is obtained by putting $y^{(1)}$ in equ. 1

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx$$

3^{rd} Approximation:

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})\, dx$$

In general, n^{th} approximation

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx$$

This is known as Picard's method of successive - approximations.

Q: Given $\frac{dy}{dx} = x+y$; $y(0)=1$. Find the value of y when $x=0.1$, $x=0.2$ by Picard's method. Checking the result with the exact value.

Sol: Here, $\frac{dy}{dx} = f(x,y) = x+y$; $y(0)=1$

$x_0 = 0$, $y_0 = 1$

Picard's Method:

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx$$

1st approximation:

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\, dx$$

$$= y_0 + \int_{0}^{x} (x + y^{0})\, dx$$

$$= 1 + \int_{0}^{x} (x+1)\, dx$$

$$= 1 + \frac{x^2}{2}\Big|_0^x + x\Big|_0^x$$

$$= 1 + \frac{x^2}{2} + x$$

$$y^{(1)} = \frac{x^2}{2} + x + 1$$

2nd approximation:

$$y^{(2)} = y_0 + \int_{x_0}^{x} (x + y^{(1)})\, dx$$

$$= y_0 + \int_{0}^{x} \left(x + \frac{x^2}{2} + x + 1\right) dx$$

$$= 1 + \int_{0}^{x} \left(2x + \frac{x^2}{2} + 1\right) dx$$

$$= 1 + \left| \frac{2x^2}{2} + \frac{x^3}{6} + x \right|_0^x$$

$$= 1 + x^2 + \frac{x^3}{6} + x$$

$$y^{(2)} = \frac{x^3}{6} + x^2 + x + 1$$

3rd Approximation:

$$y^{(3)} = y_0 + \int_{x_0}^{x} (x + y^{(2)})\, dx$$

$$= y_0 + \int_{x_0}^{x} \left(x + \frac{x^3}{6} + x^2 + x + 1\right) dx$$

$$= 1 + \int_0^x \left(2x + \frac{x^3}{6} + x^2 + 1\right) dx$$

$$1 + \left| \frac{2x^2}{2} + \frac{x^4}{24} + \frac{x^3}{3} + x \right|_0^x$$

$$= 1 + x^2 + \frac{x^4}{24} + \frac{x^3}{3} + x$$

$$y^{(3)} = \frac{x^4}{24} + \frac{x^3}{3} + x^2 + x + 1$$

4th Approximation:

$$y^{(4)} = y_0 + \int_{x_0}^{x} (x + y^{(3)})\, dx$$

$$= y_0 + \int_{x_0}^{x} \left(x + \frac{x^4}{24} + \frac{x^3}{3} + x^2 + x + 1\right) dx$$

$$= 1 + \int_0^x \left(x^2 + 2x + \frac{x^4}{24} + \frac{x^3}{3} + 1\right) dx$$

$$= 1 + \frac{x^3}{3} + \frac{2x^2}{2} + \frac{x^5}{120} + \frac{x^4}{12} + x$$

$$y^{(4)} = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

5^{th} Approximation:

$$y^{(5)} = y_0 + \int_{x_0}^{x} (x + y^{(4)})\,dx$$

$$= y_0 + \int_{x_0}^{x} \left(x + 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right) dx$$

$$= 1 + \int_{0}^{x} \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right) dx$$

$$= 1 + \left| x + \frac{2x^2}{2} + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720} \right|_0^x$$

$$y^{(5)} = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}$$

When $x = 0.1$

$$y^{(5)} = 1 + (0.1) + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{12} + \frac{(0.1)^5}{60} + \frac{(0.1)^6}{720}$$

$$y^{(5)} = 1.11034$$

When $x = 0.2$

$$y^{(5)} = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{12} + \frac{(0.2)^5}{60} + \frac{(0.2)^6}{720}$$

$$y^{(5)} = 1.24281$$

Exact Solution:-

$$\frac{dy}{dx} = x + y$$

$$\frac{dy}{dx} - y = x$$

$$(D-1)y = x$$

i) For complementary function:

$$(D-1)y = 0$$

The characteristic equation is:

$D - 1 = 0$

$D = 1$

$y_c = c_1$

(ii) For particular solution.

$$y_p = \frac{x}{D-1}$$

$$= \frac{-x}{1-D}$$

$$= -x(1-D)^{-1}$$

$$= -x\left[1 + (-1)(-D) + \frac{(-1)(-1-1)}{2!}(-D)^2 + \dots\right]$$

$$= -x(1+D)$$ Neglecting higher terms of D

$$y_p = -x - Dx$$

$$= -x - (1)x$$

$$= -2x$$

Q: Solve $\frac{dy}{dx} = \frac{y-x}{y+x}$; $y(0)=1$ by using Picard's method of successive approximations.

Sol: Here, $\frac{dy}{dx} = \frac{y-x}{y+x}$; $y(0)=1$

$x_0 = 0$, $y_0 = 1$

Picard's method:

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx$$

1st approximation:

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\, dx$$

$$= y_0 + \int_{x_0}^{x} \frac{y^{(0)}-x}{y^{(0)}+x}\, dx$$

$$= \cancel{y}\, 1 + \int_{0}^{x} \frac{1-x}{1+x}\, dx$$

$y^{(1)} = 1 +$